On an inequality for the Smarandache function

J. Sándor
Babeș-Bolyai University, 3400 Cluj-Napoca, Romania

1. In paper [2] the author proved among others the inequality $S(ab) \leq aS(b)$ for all $a, b$ positive integers. This was refined to

$$S(ab) \leq S(a) + S(b)$$

in [1]. Our aim is to show that certain results from our recent paper [3] can be obtained in a simpler way from a generalization of relation (1). On the other hand, by the method of Le [1] we can deduce similar, more complicated inequalities of type (1).

2. By mathematical induction we have from (1) immediately:

$$S(a_1a_2\ldots a_n) \leq S(a_1) + S(a_2) + \ldots + S(a_n)$$

for all integers $a_i \geq 1$ ($i = 1, \ldots, n$). When $a_1 = \ldots = a_n = n$ we obtain

$$S(a^n) \leq nS(a).$$

For three applications of this inequality, remark that

$$S((m!)^n) \leq nS(m!) = nm$$

since $S(m!) = m$. This is inequality 3) part 1. from [3]. By the same way, $S((n!)^{(n-1)!}) \leq (n-1)!S(n!) = (n-1)!n = n!$, i.e.

$$S((n!)^{(n-1)!}) \leq n!$$
Inequality (5) has been obtained in [3] by other arguments (see 4) part 1.).

Finally, by \( S(n^2) \leq 2S(n) \leq n \) for \( n \) even (see [3], inequality 1), \( n > 4 \), we have obtained a refinement of \( S(n^2) \leq n \):

\[
S(n^2) \leq 2S(n) \leq n \quad (6)
\]

for \( n > 4 \), even.

3. Let \( m \) be a divisor of \( n \), i.e. \( n = km \). Then (1) gives \( S(n) = S(km) \leq S(m) + S(k) \), so we obtain:

If \( m \mid n \), then

\[
S(n) - S(m) \leq S \left( \frac{n}{m} \right). \quad (7)
\]

As an application of (7), let \( d(n) \) be the number of divisors of \( n \). Since \( \prod_{k \mid n} k = n^{d(n)/2} \), and \( \prod_{k \leq n} k = n! \) (see [3]), and by \( \prod_{k \mid n} k \prod_{k \leq n} k \), from (7) we can deduce that

\[
S(n^{d(n)/2}) + S(n! / n^{d(n)/2}) \geq n. \quad (8)
\]

This improves our relation (10) from [3].

4. Let \( S(a) = u \), \( S(b) = v \). Then \( b \mid v! \) and \( u! \mid x(x-1) \ldots (x-u+1) \) for all integers \( x \geq u \).

But from \( a \mid u! \) we have \( a \mid x(x-1) \ldots (x-u+1) \) for all \( x \geq u \). Let \( x = u + v + k \) (\( k \geq 1 \)). Then, clearly \( ab(v+1) \ldots (v+k) \mid (u+v+k)! \), so we have \( S[ab(v+1) \ldots (v+k)] \leq u + v + k \).

Here \( v = S(b) \), so we have obtained that

\[
S[ab(S(b) + 1) \ldots (S(b) + k)] \leq S(a) + S(b) + k. \quad (9)
\]

For example, for \( k = 1 \) one has

\[
S[ab(S(b) + 1)] \leq S(a) + S(b) + 1. \quad (10)
\]

This is not a consequence of (2) for \( n = 3 \), since \( S[S(b) + 1] \) may be much larger than 1.
References

