On certain new inequalities and limits for the Smarandache function

József Sándor

Department of Mathematics, Babes-Bolyai University,
3400 Cluj - Napoca, Romania

I. Inequalities

1) If \( n > 4 \) is an even number, then \( S(n) \leq \frac{n}{2} \).

—Indeed, \( \frac{n}{2} \) is integer, \( \frac{n}{2} > 2 \), so in \( (\frac{n}{2})! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot \frac{n}{2} \) we can simplify with 2, so \( n \mid (\frac{n}{2})! \).

This simplifies clearly that \( S(n) \leq \frac{n}{2} \).

2) If \( n > 4 \) is an even number, then \( S(n^2) \leq n \).

—By \( n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot \frac{n}{2} \cdot \cdots \cdot n \), since we can simplify with 2, for \( n > 4 \) we get that \( n^2 \mid n! \). This clearly implies the above stated inequality. For factorials, the above inequality can be much improved, namely one has:

3) \( S\left(\left(\frac{m!}{n!}\right)^2\right) \leq 2m \) and more generally, \( S\left(\left(\frac{m!}{n!}\right)^n\right) \leq n \cdot m \) for all positive integers \( m \) and \( n \).

—First remark that \( \frac{(mn)!}{(m!)^n} = \frac{(mn)!}{m!(mn-m)!} \cdot \frac{(mn-m)!}{m!(mn-2m)!} \cdots \frac{(2m)!}{m! \cdot m!} = \)

\( = C_{2m}^m \cdot C_{3m}^m \cdot \cdots C_{nm}^m \), where \( C_k^n = \binom{n}{k} \) denotes a binomial coefficient. Thus \( (m!)^n \) divides \( (mn)! \), implying the stated inequality. For \( n = 2 \) one obtains the first part.

4) Let \( n > 1 \). Then \( S\left(\left(\frac{m!}{n!}\right)^{(n-1)!}\right) \leq n! \).

—We will use the well-known result that the product of \( n \) consecutive integers is divisible by \( n! \). By \( (n!)! = 1 \cdot 2 \cdot 3 \cdots n \cdot ((n+1) \cdot (n+2) \cdots 2n) \cdots ((n-1)!-1) \cdots (n-1)! \) \( n! \)

each group is divisible by \( n! \), and there are \( (n-1)! \) groups, so \( (n!)^{(n-1)!} \) divides \( (n!)! \). This gives the stated inequality.

5) For all \( m \) and \( n \) one has \( [S(m), S(n)] \leq S(m \cdot S(n)) \leq [m, n] \). where \([a, b]\) denotes the
\[ \ell \cdot c \cdot m \text{ of } a \text{ and } b. \]

- If \( m = \prod_{i} a_{i} \), \( n = \prod_{j} b_{j} \) are the canonical representations of \( m \), resp. \( n \), then it is well-known that \( S(m) = S\left( \frac{a_{i}}{p_{i}} \right) \) and \( S(n) = S\left( \frac{b_{j}}{q_{j}} \right) \), where \( S\left( \frac{a_{i}}{p_{i}} \right) = \max \left\{ S\left( \frac{a_{i}}{p_{i}} \right) : i = 1, \cdots, t \right\} \); \( S\left( \frac{b_{j}}{q_{j}} \right) = \max \left\{ S\left( \frac{b_{j}}{q_{j}} \right) : j = 1, \cdots, h \right\} \), with \( r \) and \( h \) the number of prime divisors of \( m \), resp. \( n \). Then clearly \([S(m), S(n)] \leq S(m) \cdot S(n) \leq \prod_{i} a_{i} \cdot \prod_{j} b_{j} \leq [m, n]\)

6) \( \left( S(m), S(n) \right) \geq \frac{S(m) \cdot S(n)}{\frac{m}{n}} \cdot (m, n) \) for all \( m \) and \( n \)

- Since \( \left( S(m), S(n) \right) = \frac{S(m) \cdot S(n)}{\frac{m}{n}} \), \( S(m) = \frac{S(m) \cdot S(n)}{\frac{m}{n}} \cdot (m, n) \)

by 5) and the known formula \([m, n] = \frac{mn}{(m, n)} \).

7) \( \frac{S(m, S(n))}{(m, n)} \geq \left( \frac{S(mn)}{m, n} \right)^{2} \) for all \( m \) and \( n \)

- Since \( S(mn) \leq m \cdot S(n) \) and \( S(mn) \leq n \cdot S(m) \) (See [1]), we have \( \left( \frac{S(mn)}{m, n} \right)^{2} \leq \frac{S(m) \cdot S(n)}{\frac{m}{n}} \),

and the result follows by 6).

8) We have \( \left( \frac{S(mn)}{m, n} \right)^{2} \leq \frac{S(m) \cdot S(n)}{m, n} \leq \frac{1}{(m, n)} \)

- This follows by 7) and the stronger inequality from 6), namely \( S(m) \cdot S(n) \leq [m, n] = \frac{mn}{(m, n)} \).

Corollary \( S(mn) \leq \frac{mn}{\sqrt{mn}} \).

9) Max \( \left\{ S(m), S(n) \right\} \geq \frac{S(mn)}{(m, n)} \) for all \( m, n \); where \( (m, n) \) denotes the \( g \cdot c \cdot d \) of \( m \) and \( n \).

- We apply the known result: \( \max \left\{ S(m), S(n) \right\} = S(\left[ m, n \right]) \) On the other hand, since \( [m, n] \mid m \cdot n \), by Corollary 1 from our paper [1] we get \( \frac{S(mn)}{m, n} \leq \frac{S(mn, n)}{(m, n)} \).

Since \( [m, n] = \frac{mn}{(m, n)} \),

The result follows:

Remark. Inequality 6) compliments Theorem 3 from [1], namely that \( \max \left\{ S(m), S(n) \right\} \leq S(mn) \).

64
10) Let $d(n)$ be the number of divisors of $n$. Then \[ \frac{S(n!)}{n!} \leq \frac{S(n^{d(n)/2})}{n^{d(n)/2}}. \]

We will use the known relation \( \prod_{k | n} k = n^{d(n)/2} \), where the product is extended over all divisors $k$ of $n$. Since this product divides \( \prod_{k \leq n} k = n! \), by Corollary 1 from [1] we can write
\[ \frac{S(n!)}{n!} \leq \frac{S(\prod_{k \leq n} k)}{\prod_{k | n} k^{1/n}} \], which gives the desired result.

**Remark** If $n$ is of the form $m^2$, then $d(n)$ is odd, but otherwise $d(n)$ is even. So, in each case $n^{d(n)/2}$ is a positive integer.

11) For infinitely many $n$ we have $S(n + 1) < S(n)$, but for infinitely many $m$ one has $S(m + 1) > S(m)$.

This is a simple application of 1). Indeed, let $n = p - 1$, where $p \geq 5$ is a prime. Then, by 1) we have $S(n) = S(p - 1) \leq \frac{p-1}{2} < p$. Since $p = S(p)$, we have $S(p - 1) < S(p)$.

Let in the same manner $n = p + 1$. Then, as above, $S(p + 1) \leq \frac{p+1}{2} < p = S(p)$.

12) Let $p$ be a prime. Then $S(p! + 1) > S(p !)$ and $S(p! - 1) > S(p!)$

Clearly, $S(p!) = p$. Let $p! + 1 = \prod q_j^{a_j}$ be the prime factorization of $p! + 1$. Here each $q_j > p$, thus $S(p! + 1) = S(\prod q_j^{a_j})$ (for certain $j \geq S(p^{a_j}) \geq S(p) = p$. The same proof applies to the case $p! - 1$.

**Remark:** This offers a new proof for $(\text{M})$.

13) Let $P_k$ be the $k$th prime number. Then $S(P_1 P_2 \cdots P_k + 1) > S(P_1 P_2 \cdots P_k)$ and $S(P_1 P_2 \cdots P_k - 1) > S(P_1 P_2 \cdots P_k)$

Almost the same proof as in 12) is valid, by remarking that $S(P_1 P_2 \cdots P_k) = P_k$ (since $P_1 < P_2 < \cdots < P_k$).

14) For infinitely many $n$ one has \( (S(n))^2 < S(n - 1) \cdot S(n + 1) \) and for infinitely many $m$, \( (S(m))^2 > S(m - 1) \cdot S(m + 1) \).
—By $S(p + 1) < p$ and $S(p - 1) < p$ (See the proof in 11) we have

$$\frac{S(p+1)}{S(p)} < \frac{S(p)}{S(p+1)} < \frac{S(p)}{S(p-1)}.$$  Thus $\left(\frac{S(p)}{S(p+1)}\right)^2 > S(p-1) \cdot S(p+1)$.

On the other hand, by putting $x_n = \frac{S(n)}{S(n+1)}$, we shall see in part II, that $\lim \sup x_n = + \infty$. Thus $x_{n-1} < x_n$ for infinitely many $n$, giving

$$\left(\frac{S(n)}{S(n+1)}\right)^2 < S(n-1) \cdot S(n+1).$$

II. Limits:

1) $\lim \inf \frac{S(n)}{n} = 0$ and $\lim \sup \frac{S(n)}{n} = 1$

—Clearly, $\frac{S(n)}{n} > 0$. Let $n = 2^m$. Then, since $S(2^m) \leq 2m$, and $\lim_{m \to \infty} \frac{2m}{2^m} = 0$, we have

$$\lim_{m \to \infty} \frac{S(2^m)}{2^m} = 0,$$  proving the first part.  On the other hand, it is well known that $\frac{S(n)}{n} \leq 1$.

For $n = p_k$ (the $k$th prime), one has $\frac{S(p_k)}{p_k} = 1 \to 1$ as $k \to \infty$, proving the second part.

Remark: With the same proof, we can derive that $\lim \inf \frac{S(r)}{r} = 0$ for all integers $r$.

—As above $S(2^{kr}) \leq 2^{kr}$, and $\frac{2^{kr}}{2^r} \to 0$ as $k \to \infty$ ($r$ fixed), which gives the result.

2) $\lim \inf \frac{S(n+1)}{S(n)} = 0$ and $\lim \sup \frac{S(n+1)}{S(n)} = + \infty$

—Let $p_r$ denote the $r$th prime. Since $(p_1 \cdots p_r, 1) = 1$, Dirichlet's theorem on arithmetical progressions assures the existence of a prime $p$ of the form $p = a \cdot p_1 \cdots p_r - 1$.

Then $S(p + 1) = S(a p_1 \cdots p_r) \leq a \cdot S(p_1 \cdots p_r)$ by $S(mn) \leq m S(n)$ (see [1])

But $S(p_1 \cdots p_r) = \max \{p_1, \cdots, p_r\} = p_r$. Thus $\frac{S(p+1)}{S(p)} \leq \frac{ap_r}{ap_1 \cdots p_r - 1} \leq \frac{p_r}{p_1 \cdots p_r - 1} \to 0$ as $r \to \infty$. This gives the first part.

Let now $p$ be a prime of the form $p = bp_1 \cdots p_r + 1$. 66
Then $S(p - 1) = S(bp_1 \cdots p_r) \leq b \cdot S(p_1 \cdots p_r) = b \cdot p_r$.

and $\frac{S(p - 1)}{S(p)} \leq \frac{b p_1 \cdots p - 1}{b p_1 \cdots p_r} \to 0$ as $r \to \infty$.

3) $\liminf_{n \to \infty} [S(n + 1) - S(n)] = -\infty$ and $\limsup_{n \to \infty} [S(n + 1) - S(n)] = +\infty$.

We have $S(p + 1) - S/p) \leq \frac{p + 1}{2} - p = \frac{-p + 1}{2} \to -\infty$ for an odd prime $p$ (see 1 and 11). On the other hand, $S(p) - S(p - 1) \geq p - \frac{p - 1}{2} = \frac{p + 1}{2} \to \infty$.

(Here $S(p) = p$), where $p - 1$ is odd for $p \geq 5$. This finishes the proof.

4) Let $\sigma(n)$ denotes the sum of divisors of $n$. Then $\liminf_{n \to \infty} \frac{S(\sigma(n))}{n} = 0$.

This follows by the argument of 2) for $n = p$. Then $\sigma(p) = p + 1$ and $\frac{S(p - 1)}{p} \to 0$, where $\{p\}$ is the sequence constructed there.

5) Let $\varphi(n)$ be the Euler totient function. Then $\liminf_{n \to \infty} \frac{S(\varphi(n))}{n} = 0$.

Let the set of primes $\{p\}$ be defined as in 2). Since $\varphi(n) = p - 1$ and $\frac{S(p - 1)}{p} = \frac{S(p-1)}{S(p)} \to 0$, the assertion is proved. The same result could be obtained by taking $n = 2^k$. Then, since $\varphi(2^k) = 2^{k-1}$, and $\frac{S(2^{k+1})}{2^k} \leq \frac{2^{(k-1)}}{2^k} \to 0$ as $k \to \infty$, the assertion follows.

6) $\liminf_{n \to \infty} \frac{S(S(n))}{n} = 0$ and $\max_{n \in \mathbb{N}} \frac{S(S(n))}{n} = 1$.

Let $n = p!$ ($p$ prime). Then, since $S(p)! = p$ and $S(p) = p$, from $\frac{p}{p!} \to 0$ ($p \to \infty$) we get the first result. Now, clearly $\frac{S(S(n))}{n} \leq \frac{S(n)}{n} \leq 1$. By letting $n = p$ (prime), clearly one has $\frac{S(S(p))}{p} = 1$, which shows the second relation.

7) $\liminf_{n \to \infty} \frac{\sigma(S(n))}{S(n)} = 1.$
—Clearly, $\frac{\sigma(k)}{k} > 1$. On the other hand, for $n = p$ (prime), $\frac{\sigma(S(p))}{S(p)} = \frac{p+1}{p} \rightarrow 1$ as $p \rightarrow \infty$.

8) Let $Q(n)$ denote the greatest prime power divisor of $n$. Then $\liminf_{n \rightarrow \infty} \frac{\varphi(S(n))}{\vartheta(n)} = 0$.

—Let $n = p_1^k \cdots p_r^k$ ($k > 1$, fixed). Then, clearly $\vartheta(n) = p_r^k$.

By $S(n) = S(p_r^k)$ (since $S(p_i^k) > S(p_r^k)$ for $i < k$ and $S(p_r^k) = j \cdot p_r$, with $j \leq k$ (which is known) and by $\varphi(j \cdot p_r) \leq j \cdot \varphi(p_r) \leq k(p_r - 1)$, we get $\frac{\varphi(S(n))}{\vartheta(n)} \leq \frac{k(p_r - 1)}{p_r^k} \rightarrow 0$ as $r \rightarrow \infty$ ($k$ fixed).

9) $\lim_{m \rightarrow \infty} \frac{S(m^2)}{m^2} = 0$

—By 2) we have $\frac{S(m^2)}{m^2} \leq \frac{1}{m}$ for $m > 4$, even. This clearly implies the above remark.

Remark. It is known that $\frac{S(m)}{m} \leq \frac{2}{3}$ if $m \neq 4$ is composite. From $\frac{S(m^2)}{m^2} \leq \frac{1}{m} < \frac{2}{3}$ for $m > 4$, for the composite numbers of the perfect squares we have a very strong improvement.

10) $\liminf_{n \rightarrow \infty} \frac{\sigma(S(n))}{n} = 0$

—By $\sigma(n) = \sum_{d \mid n} d = n \sum_{d \mid n} \frac{1}{d} < n \sum_{d \mid n} \frac{1}{d} \leq n \cdot (2 \log n)$, we get $\sigma(n) < 2n \log n$ for $n > 1$. Thus $\frac{\sigma(S(n))}{n} < \frac{2S(n) \log S(n)}{S(n)}$. For $n = 2^k$ we have $S(2^k) \leq 2k$, and since $\frac{4k \log 2k}{2^k} \rightarrow 0$ ($k \rightarrow \infty$), the result follows.

11) $\lim_{n \rightarrow \infty} \sqrt{S(n)} = 1$

—This simple relation follows by $1 \leq S(n) \leq n$, so $1 \leq \sqrt{S(n)} \leq \sqrt{n}$; and by $\sqrt{n} \rightarrow 1$ as $n \rightarrow \infty$. However, 11) is one of a (few) limits, which exists for the Smarandache function.

Finally, we shall prove that:

12) $\limsup_{n \rightarrow \infty} \frac{\sigma(nS(n))}{nS(n)} = +\infty$. 

68
We will use the facts that \( S(p!) = p \), \( \frac{\sigma(p!)}{p!} = \prod_{d | p} \left(1 + \frac{1}{d} + \cdots + \frac{1}{p} \right) \to \infty \) as \( p \to \infty \), and the inequality \( \sigma(ab) \geq a \sigma(b) \) (see [2]).

Thus \( \frac{\sigma(p!)}{p!} \geq \frac{S(p!)}{p!} \), \( \frac{\sigma(p!)}{p!} \to \infty \). Thus, for the sequence \( \{n\} = \{p!\} \), the results follows.

References
