On the F. Smarandache function and its mean value

Zhongtian Lv

Department of Basic Courses, Xi'an Medical College
Xi'an, Shaanxi, P.R.China

Received December 19, 2006

Abstract For any positive integer \( n \), the famous F. Smarandache function \( S(n) \) is defined as the smallest positive integer \( m \) such that \( n \mid m! \). That is, \( S(n) = \min\{ m : n \mid m!, n \in \mathbb{N} \} \).

The main purpose of this paper is using the elementary methods to study a mean value problem involving the F. Smarandache function, and give a sharper asymptotic formula for it.

Keywords F. Smarandache function, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer \( n \), the famous F. Smarandache function \( S(n) \) is defined as the smallest positive integer \( m \) such that \( n \mid m! \). That is, \( S(n) = \min\{ m : n \mid m!, n \in \mathbb{N} \} \).

For example, the first few values of \( S(n) \) are \( S(1) = 1 \), \( S(2) = 2 \), \( S(3) = 3 \), \( S(4) = 4 \), \( S(5) = 5 \), \( S(6) = 3 \), \( S(7) = 7 \), \( S(8) = 4 \), \( S(9) = 6 \), \( S(10) = 5 \), \cdots. About the elementary properties of \( S(n) \), some authors had studied it, and obtained some interesting results, see reference [2], [3] and [4]. For example, Farris Mark and Mitchell Patrick [2] studied the elementary properties of \( S(n) \), and gave an estimates for the upper and lower bound of \( S(p^\alpha) \). That is, they showed that

\[
(p - 1)\alpha + 1 \leq S(p^\alpha) \leq (p - 1)[\alpha + 1 + \log_p \alpha] + 1.
\]

Murthy [3] proved that if \( n \) be a prime, then \( SL(n) = S(n) \), where \( SL(n) \) defined as the smallest positive integer \( k \) such that \( n \mid [1, 2, \cdots, k] \), and \([1, 2, \cdots, k]\) denotes the least common multiple of \( 1, 2, \cdots, k \). Simultaneously, Murthy [3] also proposed the following problem:

\[
SL(n) = S(n), \quad S(n) \neq n ?
\]

Le Maohua [4] completely solved this problem, and proved the following conclusion:

Every positive integer \( n \) satisfying (1) can be expressed as

\[
n = 12 \quad \text{or} \quad n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r},
\]

where \( p_1, p_2, \cdots, p_r, p \) are distinct primes, and \( \alpha_1, \alpha_2, \cdots, \alpha_r \) are positive integers satisfying \( p > p_i^{\alpha_i}, i = 1, 2, \cdots, r \).
Dr. Xu Zhefeng [5] studied the value distribution problem of $S(n)$, and proved the following conclusion: Let $P(n)$ denotes the largest prime factor of $n$, then for any real number $x > 1$, we have the asymptotic formula
\[
\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta(\frac{3}{2})}{3\ln x} x^\frac{3}{2} + O\left(\frac{x^\frac{3}{2}}{\ln^2 x}\right),
\]
where $\zeta(s)$ denotes the Riemann zeta-function.

On the other hand, Lu Yaming [6] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation
\[
S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)
\]
has infinite groups positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [7] proved for any positive integer $k \geq 2$, there exist infinite groups positive integer solutions $(m_1, m_2, \cdots, m_k)$ satisfied the following inequality:
\[
S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).
\]
Also, there exist infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$ such that
\[
S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).
\]

The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of $[S(n) - S(S(n))]^2$, and give an interesting mean value formula for it. That is, we shall prove the following conclusion:

**Theorem.** Let $k$ be any fixed positive integer. Then for any real number $x > 2$, we have the asymptotic formula
\[
\sum_{n \leq x} [S(n) - S(S(n))]^2 = \frac{2}{3} \cdot \zeta\left(\frac{3}{2}\right) \cdot x^{\frac{3}{2}} \cdot \sum_{i=1}^{k} \frac{c_i}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),
\]
where $\zeta(s)$ is the Riemann zeta-function, $c_i$ ($i = 1, 2, \cdots, k$) are computable constants and $c_1 = 1$.

§2. Proof of the Theorem

In this section, we shall prove our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of $n$ into prime powers, then from [3] we know that
\[
S(n) = \max\{S(p_1^{\alpha_1}), S(p_2^{\alpha_2}), \cdots, S(p_s^{\alpha_s})\} \equiv S(p^\alpha).
\]
(2)

Now we consider the summation
\[
\sum_{n \leq x} [S(n) - S(S(n))]^2 = \sum_{n \in A} [S(n) - S(S(n))]^2 + \sum_{n \in B} [S(n) - S(S(n))]^2,
\]
(3)
where $A$ and $B$ denote the subsets of all positive integer in the interval $[1, x]$. $A$ denotes the set involving all integers $n \in [1, x]$ such that $S(n) = S(p^2)$ for some prime $p$; $B$ denotes the set involving all integers $n \in [1, x]$ such that $S(n) = S(p^\alpha)$ with $\alpha = 1$ or $\alpha \geq 3$. If $n \in A$, then $n = p^2m$ with $P(m) < 2p$, where $P(m)$ denotes the largest prime factor of $m$. So from the definition of $S(n)$ we have $S(n) = S(mp^2) = S(p^2) = 2p$ and $S(S(n)) = S(2p) = p$ if $p > 2$.

From (2) and the definition of $A$ we have

$$
\sum_{n \in A} [S(n) - S(S(n))]^2 = \sum_{n \leq x, \sqrt{\pi}n \leq p^2} [S(p^2) - S(S(p^2))]^2 + \sum_{n \leq x, \sqrt{\pi} < p^2} [S(p^2) - S(S(p^2))]^2
$$

$$
= \sum_{p^2n \leq x, n < p^2, (p, n) = 1} [S(p^2) - S(S(p^2))]^2 + \sum_{p^2n \leq x, p^2n < x, (p, n) = 1} [S(p^2) - S(S(p^2))]^2
$$

$$
= \sum_{p^2n \leq x, n < p^2, (p, n) = 1} p^2 + \sum_{p^2n \leq x, n < p^2, (p, n) = 1} p^2 + O(1)
$$

$$
= \sum_{n \leq \sqrt{\pi}} \sum_{p \leq \sqrt{x}} p^2 + O\left(\sum_{n \leq \sqrt{\pi}} \sum_{p \leq \frac{x}{\ln x}} p^2\right) + O\left(\sum_{p \leq \frac{x}{\ln x}} \sum_{p^2n \leq x} p^2\right)
$$

$$
= \sum_{n \leq \sqrt{\pi}} \sum_{p \leq \sqrt{x}} p^2 + O\left(\frac{x}{\ln x}\right),
$$

(4)

where $p^2n$ denotes $p^2n$ and $p^3 \mid n$.

By the Abel’s summation formula (See Theorem 4.2 of [8]) and the Prime Theorem (See Theorem 3.2 of [9]):

$$
\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),
$$

where $a_i (i = 1, 2, \ldots, k)$ are computable constants and $a_1 = 1$.

We have

$$
\sum_{p \leq \sqrt{x}} p^2 = \frac{x}{\pi} \cdot \pi\left(\frac{\sqrt{x}}{\pi}\right) = \int_{\frac{1}{2}}^{\frac{x}{\pi}} 2y \cdot \pi(y)dy
$$

$$
= \frac{1}{3} \cdot \frac{x^2}{\pi^2} \sum_{i=1}^{k} \frac{b_i}{\ln^i \sqrt{\pi}} + O\left(\frac{x^2}{\pi^2 \cdot \ln^{k+1} x}\right),
$$

(5)

where we have used the estimate $n \leq \sqrt{x}$, and all $b_i$ are computable constants and $b_1 = 1$.

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta\left(\frac{3}{2}\right)$, and $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent for all $i = 1, 2, 3, \ldots, k$. So from
(4) and (5) we have
\[
\sum_{n \in A} [S(n) - S(S(n))]^2
\]
\[= \sum_{n \leq \sqrt{x}} \left[ \frac{1}{3} \cdot \frac{x^{\frac{3}{2}}}{n^2} \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i \sqrt{n}} + O \left( \frac{x^{\frac{3}{2}}}{n^{\frac{3}{2} \cdot \ln^{k+1} x}} \right) \right] + O \left( \frac{x^{\frac{3}{2}}}{\ln x} \right)
\]
\[= \frac{2}{3} \cdot \zeta \left( \frac{3}{2} \right) \cdot x^{\frac{3}{2}} \cdot \sum_{i=1}^{k} \frac{c_i}{\ln^i x} + O \left( \frac{x^{\frac{3}{2}}}{\ln^{k+1} x} \right),
\]
(6)
where \(c_i \) (\(i = 1, 2, 3, \ldots, k\)) are computable constants and \(c_1 = 1\).

Now we estimate the summation in set \(B\). For any positive integer \(n \in B\), if \(S(n) = S(p) = p\), then
\[
[S(n) - S(S(n))]^2 = [S(p) - S(S(p))]^2 = 0; \quad \text{If } S(n) = S(p^\alpha) \text{ with } \alpha \geq 3, \text{ then}
\]
\[
[S(n) - S(S(n))]^2 = [S(p^\alpha) - S(S(p^\alpha))]^2 \leq \alpha^2 p^2
\]
and \(\alpha \leq \ln x\). So that we have
\[
\sum_{n \in B} [S(n) - S(S(n))]^2 \ll \sum_{n \neq p^\alpha \leq x \atop \alpha \geq 3} \alpha^2 \cdot p^2 \ll x \cdot \ln^2 x.
\]
(7)
Combining (3), (6) and (7) we may immediately deduce the asymptotic formula
\[
\sum_{n \leq x} [S(n) - S(S(n))]^2 = \frac{2}{3} \cdot \zeta \left( \frac{3}{2} \right) \cdot x^{\frac{3}{2}} \cdot \sum_{i=1}^{k} \frac{c_i}{\ln^i x} + O \left( \frac{x^{\frac{3}{2}}}{\ln^{k+1} x} \right),
\]
where \(c_i \) (\(i = 1, 2, 3, \ldots, k\)) are computable constants and \(c_1 = 1\).

This completes the proof of Theorem.

References


