The Smarandache functions of the second kind are defined in [1] thus:

\[ S^k : \mathbb{N}^* \to \mathbb{N}^*, \quad S^k(n) = S_n(k) \quad \text{for } n \in \mathbb{N}^*, \]

where \( S_n \) are the Smarandache functions of the first kind (see [3]).

We remark that the function \( S^1 \) has been defined in [4] by F. Smarandache because \( S^1 = S \).

Let, for example, the following table with the values of \( S^2 \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S^2(n) )</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>6</td>
<td>14</td>
<td>12</td>
<td>12</td>
<td>10</td>
<td>22</td>
<td>8</td>
<td>26</td>
<td>14</td>
</tr>
</tbody>
</table>

Obviously, these functions \( S^k \) aren't monotony, aren't periodical and they have fixed points.

1. **Theorem.** For \( k, n \in \mathbb{N}^* \) is true \( S^k(n) \leq n \cdot k \).

**Proof.** Let \( n = p_1^{a_1} p_2^{a_2} \ldots p_i^{a_i} \) and \( S(n) = \max_{\alpha \leq n} \left\{ S_p(\alpha) \right\} = S(p_j^{a_j}) \).

Because \( S^k(n) = S(n^k) = \max_{\alpha \leq n} \left\{ S_{p_j}(\alpha, k) \right\} = S(p_i^{a_i, k}) \leq kS(p_i^{a_i}) \leq kS(p_j^{a_j}) = kS(n) \) and \( S(n) \leq n \), [see [3]], it results:

\[ (1) \quad S^k(n) \leq n \cdot k \quad \text{for every } n, k \in \mathbb{N}^*. \]

2. **Theorem.** All prime numbers \( p \geq 5 \) are maximal points for \( S^k \), and

\[ S^k(p) = p[k - i_p(k)], \quad \text{where} \quad 0 \leq i_p(k) \leq \left\lfloor \frac{k - 1}{p} \right\rfloor \]

**Proof.** Let \( p \geq 5 \) be a prime number. Because \( S_{p-1}(k) < S_p(k), \quad S_{p+1}(k) < S_p(k) \) [see [2]] it results that \( S^k(p-1) < S^k(p) \) and \( S^k(p+1) < S^k(p) \), so that \( S^k(p) \) is a relative maximum value.
Obviously,

(2) \[ S^k(p) = S_p(k) = p[k - i_p(k)] \quad \text{with} \quad 0 \leq i_p(k) \leq \left[ \frac{k - 1}{p} \right]. \]

(3) \[ S^k(p) = pk \quad \text{for} \quad p \geq k. \]

3. **Theorem.** The numbers \( kp \) for \( p \) prime and \( p > k \) are the fixed points of \( S^k \).

**Proof.** Let \( p \) be a prime number, \( m = p_1^{\alpha_1} \cdots p_t^{\alpha_t} \) be the prime factorization of \( m \) and \( p > \max\{m, k\} \). Then \( p, \alpha_i \leq p_i^{\alpha_i} < p \quad \text{for} \quad i \in 1, t \), therefore we have:

\[ S^k(m \cdot p) = S[(mp)^k] = \max_{1 \leq r \leq m} \left\{ S_{p_i^{\alpha_i}, S_p(k)} \right\} = S_p(k) = kp. \]

For \( m = k \) we obtain:

\[ S^k(kp) = kp \quad \text{so that} \quad kp \quad \text{is a fixed point.} \]

4. **Theorem.** The functions \( S^k \) have the following properties:

\[ S^k = 0 \quad (n^{1+\varepsilon}), \quad \text{for} \quad \varepsilon > 0 \]

\[ \lim_{n \to \infty} \sup \frac{S^k(n)}{n} = k. \]

**Proof.** Obviously,

\[ 0 \leq \lim_{n \to \infty} \frac{S^k(n)}{n^{1+\varepsilon}} = \lim_{n \to \infty} \frac{S(n^k)}{n^{1+\varepsilon}} \leq \lim_{n \to \infty} \frac{KS(n)}{n^{1+\varepsilon}} = k \lim_{n \to \infty} \frac{S(n)}{n^{1+\varepsilon}} = 0 \quad \text{for} \]

\[ S = 0 \quad (n^{1+\varepsilon}), \quad \text{[see[4]].} \]

Therefore we have \( S^k = 0 \quad (n^{1+\varepsilon}) \), and:

\[ \lim_{n \to \infty} \sup \frac{S^k(n)}{n} = \lim_{n \to \infty} \frac{S(n^k)}{n} = \lim_{p \to \infty} \frac{S(p^k)}{p} = k \]

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5. Theorem. [see[1]]. The Smarandache functions of the second kind standardize $(\mathbb{N}^*, \cdot)$ in $(\mathbb{N}^*, \leq, +)$ by:

$$\sigma_3: \max\{S^k(a), S^k(b)\} \leq S^k(ab) \leq S^k(a) + S^k(b)$$

and $(\mathbb{N}^*, \cdot)$ in $(\mathbb{N}^*, \leq, \cdot)$ by:

$$\sigma_4: \max\{S^k(a), S^k(b)\} \leq S^k(ab) \leq S^k(a) \cdot S^k(b) \text{ for every } a, b \in \mathbb{N}^*$$

6. Theorem. The functions $S^k$ are, generally speaking, increasing. It means that:

$$\forall n \in \mathbb{N}^* \exists m_0 \in \mathbb{N}^* \text{ so that } \forall m \geq m_0 \Rightarrow S^k(m) \geq S^k(n)$$

Proof. The Smarandache function is generally increasing, [see [4]], it means that:

$$(3) \quad \forall t \in \mathbb{N}^* \exists r_0(t) \in \mathbb{N}^* \text{ so that } \forall r \geq r_0 \Rightarrow S(r) \geq S(t)$$

Let $t = n^k$ and $r_0 = r_0(t)$ so that $\forall r \geq r_0 \Rightarrow S(r) \geq S(n^k)$.

Let $m_0 = \left[\sqrt[k]{r_0}\right] + 1$. Obviously $m_0 \geq \sqrt[k]{r_0} = m_0 \geq r_0$ and $m \geq m_0 \iff m^k \geq m_0^k$.

Because $m^k \geq m_0^k \geq r_0$ it results $S(m^k) \geq S(n^k)$ or $S^k(m) \geq S^k(n)$.

Therefore

$$\forall n \in \mathbb{N}^* \exists m_0 = \left[\sqrt[k]{r_0}\right] - 1 \text{ so that } \forall m \geq m_0 \Rightarrow S^k(m) \geq S^k(n) \text{ where } r_0 = r_0(n^k)$$

is given from (3).

7. Theorem. The function $S^k$ has its relative minimum values for every $n = p!$, where $p$ is a prime number and $p \geq \max\{3, k\}$.

Proof. Let $p! = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m} \cdot p$ be the canonical decomposition of $p!$, where $2 = p_1 < 3 = p_2 < \cdots < p_m < p$. Because $p!$ is divisible by $p_j^j$ it results $S(p_j^j) \leq p = S(p)$ for every $j = 1, m$.

Obviously,

$$S^k(p!) = S[(p!)^k] = \max\left\{S\left(p_j^{k_j^j}\right), S(p^k)\right\}$$

Because $S\left(p_j^{k_j^j}\right) \leq kS(p_j^j) < kS(p) = kp = S(p^k) \text{ for } k \leq p$, it results that we have

$$(4) \quad S^k(p!) = S(p^k) = kp, \text{ for } k \leq p$$
Let $p!-1 = q_1 \cdot q_2 \cdots q_i$ be the canonical decomposition for $p!-1$, then $q_j > p$ for $j \in \{1, \ldots, i\}$.

It follows $S(p!-1) = \max_{1 \leq j \leq r} \left[ S(q_i^{j/j}) \right] = S(q_m^*)$ with $q_m > p$.

Because $S(q^*) > S(p) = S(p!)$ it results $S(p!-1) > S(p!)$.

Analogous it results $S(p!+1) > S(p!)$.

Obviously

$$S^k(p!-1) = S[(p!-1)^k] \geq S(q_m^{k \cdot m}) \geq S(q_m^k) > S(p^k) = kp$$

$$S^k(p!+1) = S[(p!+1)^k] > k \cdot p$$

For $p \geq \max \{3, k\}$ out of (4), (5), (6) it results that $p!$ are the relative minimum points of the functions $S^k$.

REFERENCES


