SOME PROPERTIES OF THE PSEUDO-SMARANDACHE FUNCTION

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Abstract. Charles Ashbacher [1] has posed a number of questions relating to the pseudo-Smarandache function $Z(n)$. In this note we show that the ratio of consecutive values $Z(n+1)/Z(n)$ and $Z(n)/Z(n-1)$ are unbounded; that $Z(2n)/Z(n)$ is unbounded; that $n/Z(n)$ takes every integer value infinitely often; and that the series $\sum_n 1/Z(n)^\alpha$ is convergent for any $\alpha > 1$.

1. Introduction

We define the $m$-th triangular number $T(m) = \frac{m(m+1)}{2}$. Kashihara [2] has defined the pseudo-Smarandache function $Z(n)$ by

$$Z(n) = \min\{m : n \mid T(m)\}.$$

Charles Ashbacher [1] has posed a number of questions relating to the pseudo-Smarandache function $Z(n)$. In this note we show that the ratio of consecutive values $Z(n)/Z(n-1)$ and $Z(n)/Z(n+1)$ are unbounded; that $Z(2n)/Z(n)$ is unbounded; and that $n/Z(n)$ takes every integer value infinitely often. He notes that the series $\sum_n 1/Z(n)^\alpha$ is convergent for any $\alpha = 1$ and asks whether it is convergent for $\alpha = 2$. He further suggests that the least value of $\alpha$ for which the series converges “may never be known”. We resolve this problem by showing that the series converges for all $\alpha > 1$.

2. Some properties of the pseudo-Smarandache function

We record some elementary properties of the function $Z$.

Lemma 1. (1) If $n \geq T(m)$ then $Z(n) \geq m$. $Z(T(m)) = m$.
               (2) For all $n$ we have $\sqrt{n} < Z(n)$.
               (3) $Z(n) \leq 2n - 1$, and if $n$ is odd then $Z(n) \leq n - 1$.
               (4) If $p$ is an odd prime dividing $n$ then $Z(n) \geq p - 1$.
               (5) $Z(2^k) = 2^{k+1} - 1$.
               (6) If $p$ is an odd prime then $Z(p^k) = p^k - 1$ and $Z(2p^k) = p^k - 1$ or $p^k$ according as $p^k \equiv 1$ or $3 \mod 4$.

We shall make use of Dirichlet’s Theorem on primes in arithmetic progression in the following form.

Lemma 2. Let $a, b$ be coprime integers. Then the arithmetic progression $a + bt$ is prime for infinitely many values of $t$.

Date: 2 April 2005.
1991 Mathematics Subject Classification. Primary 11A25; Secondary 11B83.
3. Successive values of the pseudo-Smarandache function

Using properties (3) and (5), Ashbacher observed that $|Z(2^k) - Z(2^k - 1)| > 2^k$ and so the difference between the consecutive values of $Z$ is unbounded. He asks about the ratio of consecutive values.

**Theorem 1.** For any given $L > 0$ there are infinitely many values of $n$ such that $Z(n + 1)/Z(n) > L$, and there are infinitely many values of $n$ such that $Z(n - 1)/Z(n) > L$.

**Proof.** Choose $k \equiv 3 \mod 4$, so that $T(k)$ is even and $k$ divides $T(k)$. We consider the conditions $k \mid m$ and $(k + 1) \mid (m + 1)$. These are satisfied if $m \equiv k \mod k(k + 1)$, that is, $m = k + k(k + 1)t$ for some $t$. We have $m(m+1) = k(1 + (k + 1)t) \cdot (k + 1)(1 + kt)$, so that if $n = k(k + 1)(1 + kt)/2$ we have $n \mid T(m)$. Now consider $n + 1 = T(k) + 1 + kT(k)t$. We have $k \mid T(k)$, so $T(k) + 1$ is coprime to both $k$ and $T(k)$. Thus the arithmetic progression $T(k) + 1 + kT(k)t$ has initial term coprime to its increment and by Dirichlet’s Theorem contains infinitely many primes. We find that there are thus infinitely many values of $t$ for which $n + 1$ is prime and so $Z(n) \leq m = k + k(k + 1)t$ and $Z(n + 1) = n = T(k)(1 + kt)$. Hence

$$\frac{Z(n + 1)}{Z(n)} \geq \frac{n}{m} = \frac{T(k) + kT(k)t}{k + 2T(k)t} > \frac{k}{3}.$$

A similar argument holds if we consider the arithmetic progression $T(k) + 1 + kT(k)t$. We then find infinitely many values of $t$ for which $n - 1$ is prime and

$$\frac{Z(n - 1)}{Z(n)} \geq \frac{n - 2}{m} = \frac{T(k) - 2 + kT(k)t}{k + 2T(k)t} > \frac{k}{4}.$$

The Theorem follows by taking $k > 4L$. □

We note that this Theorem, combined with Lemma 1(2), gives another proof of the result that the difference of consecutive values is unbounded.

4. Divisibility of the pseudo-Smarandache function

**Theorem 2.** For any integer $k \geq 2$, the equation $n/Z(n) = k$ has infinitely many solutions $n$.

**Proof.** Fix an integer $k \geq 2$. Let $p$ be a prime $\equiv -1 \mod 2k$ and put $p + 1 = 2kt$. Put $n = T(p)/t = p(p + 1)/2t = pk$. Then $n \mid T(p)$ so that $Z(n) \leq p$. We have $p \mid n$, so $Z(n) \geq p - 1$: that is, $Z(n)$ must be either $p$ or $p - 1$. Suppose, if possible, that it is the latter. In this case we have $2n \mid p(p + 1)$ and $2n \mid (p - 1)p$, so $2n$ divides $p(p + 1) - (p - 1)p = 2p$: but this is impossible since $k > 1$ and so $n > p$. We conclude that $Z(n) = p$ and $n/Z(n) = k$ as required. Further, for any given value of $k$ there are infinitely many prime values of $p$ satisfying the congruence condition and hence infinitely many values of $n = T(p)$ such $n/Z(n) = k$. □
5. Another divisibility question

**Theorem 3.** The ratio $Z(2n)/Z(n)$ is not bounded above.

**Proof.** Fix an integer $k$. Let $p \equiv -1 \mod 2^k$ be prime and put $n = T(p)$. Then $Z(n) = p$. Consider $Z(2n) = m$. We have $2^k p \mid p(p+1) = 2n$ and this divides $m(m+1)/2$. We have $m \equiv \epsilon \mod p$ and $m \equiv \delta \mod 2^{k+1}$ where each of $\epsilon, \delta$ can be either 0 or $-1$.

Let $m = pt + \epsilon$. Then $m \equiv \epsilon - t \equiv \delta \mod 2^k$: that is, $t \equiv \epsilon - \delta \mod 2^k$. This implies that either $t = 1$ or $t \geq 2^k - 1$. Now if $t = 1$ then $m \leq p$ and $T(m) \leq T(p) = n$, which is impossible since $2n \leq T(m)$. Hence $t \geq 2^k - 1$. Since $Z(2n)/Z(n) = m/p > t/2$, we see that the ratio $Z(2n)/Z(n)$ can be made as large as desired. \hfill \Box

6. Convergence of a series

Ashbacher observes that the series $\sum n^{1/Z(n)^\alpha}$ diverges for $\alpha = 1$ and asks whether it converges for $\alpha = 2$.

In this section we prove convergence for all $\alpha > 1$.

**Lemma 3.**

$$\log n \leq \sum_{m=1}^{n} \frac{1}{m} \leq 1 + \log n;$$

$$\frac{1}{2} (\log n)^2 - 0.257 \leq \sum_{m=1}^{n} \frac{\log m}{m} \leq \frac{1}{2} (\log n)^2 + 0.110 \text{ for } n \geq 4.$$

**Proof.** For the first part, we have $1/m \leq 1/t \leq 1/(m - 1)$ for $t \in [m - 1, m]$. Integrating,

$$\frac{1}{m} \leq \int_{m-1}^{m} \frac{1}{t} \, dt \leq \frac{1}{m-1}.$$

Summing,

$$\sum_{2}^{n} \frac{1}{m} \leq \int_{1}^{n} \frac{1}{t} \, dt \leq \sum_{2}^{n} \frac{1}{m-1},$$

that is,

$$\sum_{1}^{n} \frac{1}{m} \leq 1 + \log n \text{ and } \log n \leq \sum_{1}^{n-1} \frac{1}{m}.$$ 

The result follows.

For the second part, we similarly have $\log m/m \leq \log t/t \leq \log(m-1)/(m-1)$ for $t \in [m - 1, m]$ when $m \geq 4$, since $\log x/x$ is monotonic decreasing for $x > e$. Integrating,

$$\frac{\log m}{m} \leq \int_{m-1}^{m} \frac{\log t}{t} \, dt \leq \frac{\log(m-1)}{m-1}.$$

Summing,

$$\sum_{4}^{n} \frac{\log m}{m} \leq \int_{3}^{n} \frac{\log t}{t} \, dt \leq \sum_{4}^{n} \frac{\log(m-1)}{m-1}.$$
that is,

\[ \sum_{1}^{n} \frac{\log m}{m} - \frac{\log 2}{2} - \frac{\log 3}{3} \leq \frac{1}{2} (\log n)^2 - \frac{1}{2} (\log 3)^2 \]

\[ \leq \sum_{1}^{n} \frac{\log m}{m} - \frac{\log n}{n} - \frac{\log 2}{2} \]

We approximate the numerical values

\[ \frac{\log 2}{2} + \frac{\log 3}{3} - \frac{1}{2} (\log 3)^2 < 0.110 \]

and

\[ \frac{\log 2}{2} - \frac{1}{2} (\log 3)^2 > -0.257. \]

to obtain the result. \( \square \)

**Lemma 4.** Let \( d(m) \) be the function which counts the divisors of \( m \). For \( n \geq 2 \) we have

\[ \sum_{m=1}^{n} \frac{d(m)}{m} < 7(\log n)^2. \]

**Proof.** We verify the assertion numerically for \( n \leq 6 \). Now assume that \( n \geq 8 > e^2 \). We have

\[ \sum_{m=1}^{n} \frac{d(m)}{m} = \sum_{m=1}^{n} \sum_{d|m} \frac{1}{m} = \sum_{d \leq n} \sum_{d \leq n} \frac{1}{d} \]

\[ = \sum_{d \leq n} \frac{1}{d} \sum_{e \leq n/d} \frac{1}{e} \leq \sum_{d \leq n} \frac{1}{d} (1 + \log(n/d)) \]

\[ \leq (1 + \log n)^2 - \frac{1}{2} (\log n)^2 + 0.257 \]

\[ = 1.257 + 2 \log n + \frac{1}{2} (\log n)^2 \]

\[ < \frac{4}{3} \left( \frac{\log n}{2} \right)^2 + 2 \log n \left( \frac{\log n}{2} \right) + \frac{1}{2} (\log n)^2 \]

\[ < 2(\log n)^2. \]

\( \square \)

**Lemma 5.** Fix an integer \( t \geq 5 \). Let \( e^t > Y > e^{(t-1)/2} \). The number of integers \( n \) with \( e^{t-1} < n \leq e^t \) such that \( Z(n) \leq Y \) is at most \( 196Yt^2 \).

**Proof.** Consider such an \( n \) with \( m = Z(n) \leq Y \). Now \( n \mid m(m+1) \), say \( k_1 n_1 = m \) and \( k_2 n_2 = m + 1 \), with \( n = n_1n_2 \). Thus \( k = k_1k_2 = m(m + 1)/n \) and \( k_1n_1 \leq Y \). The value of \( k \) is bounded below by 2 and above by \( m(m + 1)/n \leq 2Y^2/e^{t-1} = K \), say. Given a pair \((k_1, k_2)\), the possible values of \( n_1 \) are bounded above by \( Y/k_1 \) and must satisfy the congruence condition \( k_1n_1 + 1 \equiv 0 \) modulo \( k_2 \); there are therefore at most \( Y/k_1k_2 + 1 \) such values.
Since \( Y/k \geq Y/K = e^{t-1}/2Y > 1/2e \), we have \( Y/k + 1 < (2e + 1)Y/k < 7Y/k \).

Given values for \( k, k_2 \) and \( n_1 \), the value of \( n_2 \) is fixed as \( n_2 = (k_1n_1 + 1)/k_2 \). There are thus at most \( \sum_{k \leq K} d(k) \) possible pairs \((k_1, k_2)\) and hence at most \( \sum_{k \leq K} 7Yd(k)/k \) possible quadruples \((k_1, k_2, n_1, n_2)\). We have \( K > 2 \) so that the previous Lemma applies and we can deduce that the number of values of \( n \) satisfying the given conditions is at most \( 49Y(\log K)^2 \). Now \( K = 2Y^2/e^{t-1} < 2e^{t+1} \) so that \( \log K < t + 1 + \log 2 < 2t \). This establishes the claimed upper bound of \( 196Yt^2 \).

**Theorem 4.** Fix \( \frac{1}{2} < \beta < 1 \) and an integer \( t \geq 5 \). The number of integers \( n \) with \( e^{t-1} < n \leq e^t \) such that \( Z(n) < n^\beta \) is at most \( 196e^{t/2} \).

**Proof.** We apply the previous result with \( Y = e^{t/2} \). The conditions of \( \beta \) ensure that the previous lemma is applicable and the upper bound on the number of such \( n \) is \( 196e^{t/2} \) as claimed.

**Theorem 5.** The series

\[
\sum_{n=1}^{\infty} \frac{1}{Z(n)^\alpha}
\]

is convergent for any \( \alpha > \sqrt{2} \).

**Proof.** We note that if \( \alpha > 2 \) then \( 1/Z(n)^\alpha < 1/n^{\alpha/2} \) and the series is convergent. So we may assume \( \sqrt{2} < \alpha \leq 2 \). Fix \( \beta \) with \( 1/\alpha < \beta < \alpha/2 \). We have \( \frac{1}{2} < \beta < \sqrt{1/2} < \alpha/2 \).

We split the positive integers \( n > e^4 \) into two classes \( A \) and \( B \). We let class \( A \) be the union of the \( A_t \) where, for positive integer \( t \geq 5 \) we put into class \( A_t \) those integers \( n \) such that \( e^{t-1} < n \leq e^t \) for integer \( t \) and \( Z(n) \leq n^\beta \). All values of \( n \) with \( Z(n) > n^\beta \) we put into class \( B \). We consider the sum of \( 1/Z(n)^\alpha \) over each of the two classes. Since all terms are positive, it is sufficient to prove that each series separately is convergent.

Firstly we observe that for \( n \in B \), we have \( 1/Z(n)^\alpha < 1/n^{\alpha\beta} \) and since \( \alpha\beta > 1 \) the series summed over the class \( B \) is convergent.

Consider the elements \( n \) of \( A_t \): so for such \( n \) we have \( e^{t-1} < n \leq e^t \) and \( Z(n) < e^t \). By the previous result, the number of values of \( n \) satisfying these conditions is at most \( 196t^2e^{3t} \). For \( n \in A_t \), we have \( Z(n) \geq \sqrt{n} \), so \( 1/Z(n)^\alpha \leq 1/n^{\alpha/2} < 1/e^{\alpha(t-1)/2} \). Hence the sum of the subseries \( \sum_{n \in A_t} 1/Z(n)^\alpha \) is at most \( 196e^{\alpha/2}e^{(\beta-\alpha/2)t} \). Since \( \beta < \alpha/2 \) for \( \alpha > \sqrt{2} \), the sum over all \( t \) of these terms is finite.

We conclude that \( \sum_{n=1}^{\infty} 1/Z(n)^\alpha \) is convergent for \( \alpha > \sqrt{2} \).

**Theorem 6.** The series

\[
\sum_{n=1}^{\infty} \frac{1}{Z(n)^\alpha}
\]

is convergent for any \( \alpha > 1 \).

**Proof.** We fix \( \beta_0 = 1 > \beta_1 > \cdots > \beta_r = \frac{1}{2} \) with \( \beta_j < \alpha \beta_{j+1} \) for \( 0 \leq j \leq r-1 \). We define a partition of the integers \( e^{t-1} < n < e^t \) into classes \( B_t \) and \( C_t(j) \), \( 1 \leq j \leq r-1 \). Into \( B_t \) place those \( n \) with \( Z(n) > n^{\beta_1} \). Into \( C_t(j) \) place those \( n \)
with \( n^{\beta_{j+1}} < Z(n) < n^{\beta_j} \). Since \( \beta_r = \frac{1}{2} \) we see that every \( n \) with \( e^{t-1} < n < e^t \) is placed into one of the classes.

The number of elements in \( C_t(j) \) is at most \( 196t^2e^{\beta_j t} \) and so

\[
\sum_{n \in C_t(j)} \frac{1}{Z(n)^\alpha} < 196t^2e^{\beta_j t}e^{-\beta_{j+1} \alpha (t-1)} = 196e^{\beta_{j+1} \alpha t}t^2e^{(\beta_j - \alpha \beta_{j+1})t}.
\]

For each \( j \) we have \( \beta_j < \alpha \beta_{j+1} \) so each sum over \( t \) converges.

The sum over the union of the \( B_t \) is bounded above by

\[
\sum_n \frac{1}{n^{\alpha \beta_1}},
\]

which is convergent since \( \alpha \beta_1 > \beta_0 = 1 \).

We conclude that \( \sum_{n=1}^{\infty} 1/Z(n)^\alpha \) is convergent. \( \square \)

References


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