THE MONOTONY OF SMARANDACHE FUNCTIONS OF FIRST KIND

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Smarandache functions of first kind are defined in [1] thus:

\[ S_n : \mathbb{N}^* \rightarrow \mathbb{N}^*, \quad S_n(k) = 1 \text{ and } S_n(k) = \max_{1 \leq j \leq r} \{ S_p(j)^{i} \}, \]

where \( n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r} \) and \( S_p(j) \) are functions defined in [4].

They \( \Sigma_1 \)-standardize \((\mathbb{N}^*, +)\) in \((\mathbb{N}^*, \leq, +)\) in the sense that

\[ \Sigma_1 : \quad \max \{ S_n(a), S_n(b) \} \leq S_n(a + b) \leq S_n(a) + S_n(b) \]

for every \( a, b \in \mathbb{N}^* \) and \( \Sigma_2 \)-standardize \((\mathbb{N}^*, +)\) in \((\mathbb{N}^*, \leq, \cdot)\) by

\[ \Sigma_2 : \quad \max \{ S_n(a), S_n(b) \} \leq S_n(a \cdot b) \leq S_n(a) \cdot S_n(b), \text{ for every } a, b \in \mathbb{N}^* \]

In [2] it is proved that the functions \( S_n \) are increasing and the sequence \( \{ S_p \}_{p \in \mathbb{N}^*} \) is also increasing. It is also proved that if \( p, q \) are prime numbers, then

\[ p \cdot i < q \Rightarrow S_p(i) < S_q \text{ and } i < q \Rightarrow S_i < S_q, \]

where \( i \in \mathbb{N}^* \).

It would be used in this paper the formula

\[ S_p(k) = p(k - i_k), \text{ for same } i_k \text{ satisfying } 0 \leq i_k \leq \left\lfloor \frac{k - 1}{p} \right\rfloor, \text{ (see [3])} \] (1)

1. Proposition. Let \( p \) be a prime number and \( k_1, k_2 \in \mathbb{N}^* \). If \( k_1 < k_2 \) then \( i_{k_1} \leq i_{k_2} \), where \( i_{k_1}, i_{k_2} \) are defined by (1).

Proof. It is known that \( S_p : \mathbb{N}^* \rightarrow \mathbb{N}^* \) and \( S_p(k) = pk \) for \( k \leq p \). If \( S_p(k) = mp^a \) with \( m, a \in \mathbb{N}^*, (m, p) = 1 \), there exist \( a \) consecutive numbers:

\[ n, n + 1, \ldots, n + a - 1 \]

so that

\[ k \in \{ n, n + 1, \ldots, n + a - 1 \} \]

and

\[ S_p(n) = S_p(n + 1) = \cdots = S(n + a - 1), \]
this means that $S_p$ is stationed the $\alpha - 1$ steps ($k \rightarrow k + 1$).

If $k_1 < k_2$ and $S_p(k_1) = S_p(k_2)$, because $S_p(k_1) = p(k_1 - ik_1)$, $S_p(k_2) = p(k_2 - ik_2)$
it results $i_{k_1} < i_{k_2}$.

If $k_1 < k_2$ and $S_p(k_1) < S_p(k_2)$, it is easy to see that we can write:

$$i_{k_1} = \beta_1 + \sum_{\alpha}(\alpha - 1)$$

where $\beta_1 = 0$ for $S_p(k_1) = mp^a$, if $S_p(k_1) = mp^a$

then $\beta_1 \in \{0, 1, 2, ..., \alpha - 1\}$

and

$$i_{k_2} = \beta_2 + \sum_{\alpha}(\alpha - 1)$$

where $\beta_2 = 0$ for $S_p(k_2) = mp^a$, if $S_p(k_2) = mp^a$ then

$\beta_2 \in \{0, 1, 2, ..., \alpha - 1\}$.

Now is obviously that $k_1 < k_2$ and $S_p(k_1) < S_p(k_2) \Rightarrow i_{k_1} \leq i_{k_2}$. We note that, for

$k_1 < k_2$, $i_{k_1} = i_{k_2}$ iff $S_p(k_1) < S_p(k_2)$ and

$$\{mp^a|\alpha > 1 \text{ and } mp^a \leq S_p(k_1)\} = \{mp^a|\alpha > 1 \text{ and } mp^a \leq S_p(k_2)\}$$

2. Proposition. If $p$ is a prime number and $p \geq 5$, then $S_p > S_{p-1}$ and $S_p > S_{p+1}$.

Proof. Because $p - 1 < p$ it results that $S_{p-1} < S_p$. Of course $p + 1$ is even and so:

(i) if $p + 1 = 2l$, then $i > 2$ and because $2i < 2^l - 1 = p$ we have $S_{p+1} < S_p$.

(ii) if $p + 1 = 2l$, let $p + 1 = p_1^{'1} \cdot p_2^{'2} \cdots p_l^{'l}$, then $S_{p+1}(k) = \max_{k \leq p_m} \{S_p^{'j}(k)\} = S_p^{\prime m}(k) = S_p^{\prime m}(i_m \cdot k)$.

Because $p_m \cdot i_m \leq p_m^\prime \leq \frac{p + 1}{2} < p$ it results that $S_p^{\prime m}(k) < S_p(k)$ for $k \in N^*$, so that $S_{p+1} < S_p$.

3. Proposition. Let $p, q$ be prime numbers and the sequences of functions

$$\{S_p\}_{i \in N^*}, \{S_q\}_{j \in N^*}$$

If $p < q$ and $i \leq j$, then $S_p^i < S_q^j$.

Proof. Evidently, if $p < q$ and $i \leq j$, then for every $k \in N^*$

$$S_p^i(k) \leq S_p^j(k) < S_q^j(k)$$

so,

$$S_p^i < S_q^j$$

4. Definition. Let $p, q$ be prime numbers. We consider a function $S_q^i$, a sequence of functions $\{S_p\}_{i \in N^*}$, and we note:

$$i_{(j)} = \max \left\{ i \mid S_p^i < S_q^j \right\}$$
\[ i^{(j)} = \min \{ i | S_{q,i} < S_{\nu,j} \} \]

then \( k \in N \mid i_{(j)} < k < i^{(j)} \) = \( \Delta_{\nu/q(i)} = \Delta_{(i)} \) defines the interference zone of the function \( S_{q,i} \) with the sequence \( \{S_{\nu,j}\}_{\nu \in N^*} \).

5. Remarque.
   a) If \( S_{q,i} < S_{\nu,j} \) for \( i \in N^* \), then now exists \( i_{(j)} \) and \( j_{(j)} = 1 \), and we say that \( S_{q,i} \) is separately of the sequence of functions \( \{S_{\nu,j}\}_{\nu \in N^*} \).
   b) If there exist \( k \in N^* \) so that \( S_{\nu,i} < S_{q,i} < S_{\nu,j} \), then \( \Delta_{\nu/q(i)} = \emptyset \) and say that the function \( S_{q,i} \) does not interfere with the sequence of functions \( \{S_{\nu,j}\}_{\nu \in N^*} \).

6. Definition. The sequence \( \{x_n\}_{n \in N^*} \) is generally increasing if

\[ \forall n \in N^* \exists m_0 \in N^* \text{ so that } x_m \geq x_n \text{ for } m \geq m_0. \]

7. Remarque. If the sequence \( \{x_n\}_{n \in N^*} \) with \( x_n \geq 0 \) is generally increasing and bounded, then every subsequence is generally increasing and bounded.

8. Proposition. The sequence \( \{S_n(k)\}_{n \in N^*} \), where \( k \in N^* \), is generally increasing and bounded.

   Proof. Because \( S_n(k) = S_n^{(1)}(1) \), it results that \( \{S_n(k)\}_{n \in N^*} \) is a subsequence of \( \{S_m(1)\}_{m \in N^*} \).

   The sequence \( \{S_m(1)\}_{m \in N^*} \) is generally increasing and bounded because:

\[ \forall m \in N^* \exists t_0 = m! \text{ so that } \forall t \geq t_0, S_t(1) \geq S_{t_0}(1) = m \geq S_m(1). \]

From the remarque 7 it results that the sequence \( \{S_n(k)\}_{n \in N^*} \) is generally increasing bounded.

9. Proposition. The sequence of functions \( \{S_n\}_{n \in N^*} \) is generally increasing bounded.

   Proof. Obviously, the zone of interference of the function \( S_m \) with \( \{S_n\}_{n \in N^*} \) is the set

\[ \Delta_{n(m)} = \{ k \in N^* | n_{(m)} < k < n^{(m)} \} \text{ where} \]

\[ n_{(m)} = \max \{ n \in N^* | S_n < S_m \} \]

\[ n^{(m)} = \min \{ n \in N^* | S_m < S_n \}. \]
The interference zone $\Delta_{m(m)}$ is nonempty because $S_m \in \Delta_{m(m)}$ and finite for $S_t \leq S_m \leq S_p$, where $p$ is one prime number greater than $m$.

Because $\{S_n(1)\}$ is generally increasing it results:

$$\forall m \in \mathbb{N}^* \exists t_0 \in \mathbb{N}^* \text{ so that } S_{t}(1) \geq S_m(1) \text{ for } \forall t \geq t_0.$$ 

For $t_0 = t_0 + n^{(m)}$ we have

$$S_r \geq S_m \geq S_m(1) \text{ for } \forall r \geq t_0,$$

so that $\{S_n\}_{n \in \mathbb{N}^*}$ is generally increasing bounded.

10. **Remarque.**

a) For $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$ are possible the following cases:

1) $\exists \ k \in \{1,2,\ldots,r\}$ so that

$$S_{p_k^{i_k}} \leq S_{p_k^{i_k}} \text{ for } j \in \{1,2,\ldots,r\},$$

then $S_n = S_{p_k^{i_k}}$ and $p_k^{i_k}$ is named the dominant factor for $n$.

2) $\exists \ k_1,k_2,\ldots,k_m \in \{1,2,\ldots,r\}$ so that :

$$\forall \ t \in \overline{1,m} \ \exists q_t \in \mathbb{N}^* \text{ so that } S_n(q_t) = S_{p_k^{i_k}}(q_t) \text{ and }$$

$$\forall \ t \in \mathbb{N}^* \ S_n(I) = \max_{1 \leq t \leq m} \left\{ S_{p_k^{i_k}}(I) \right\}.$$ 

We shall name $\{p_k^{i_k} | t \in \overline{1,m}\}$ the active factors, the others would be name passive factors for $n$.

b) We consider

$$N_{p_1,p_2} = \{n = p_1^{i_1} \cdot p_2^{i_2} | i_1,i_2 \in \mathbb{N}^*\}, \text{ where } p_1 < p_2 \text{ are prime numbers.}$$

For $n \in N_{p_1,p_2}$ appear the following situations:

1) $i_1 \in (0,i_1^{(p_1)})$, this means that $p_1^{i_1}$ is a passive factor and $p_2^{i_2}$ is an active factor.

2) $i_1 \in (i_1^{(p_1)},i_1^{(p_2)})$ this means that $p_1^{i_1}$ and $p_2^{i_2}$ are active factors.

3) $i_1 \in [i_1^{(p_2)},\infty)$ this means that $p_1^{i_1}$ is an active factor and $p_2^{i_2}$ is a passive factor.
For $P_1 < P_2$, the repartition of exponents is represented in the following scheme:

For numbers of type 2) $i_1 \in (i_{1(u_1)}, i_{1(l_1)})$ and $i_2 \in (i_{2(u_2)}, i_{2(l_2)})$

c) I consider that

$$N_{P_1P_2P_3} = \{ n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} \mid i_1, i_2, i_3 \in N^* \},$$

where $P_1 < P_2 < P_3$ are prime numbers.

Exist the following situations:

1) $n \in N_{p_1}$, $j = 1, 2, 3$ this means that $p_1^{i_1}$ is an active factor.

2) $n \in N_{p_1P_2}$, $j = k$, $j, k \in \{1, 2, 3\}$, this means that $p_1^{i_1}, p_2^{i_2}$ are active factors.

3) $n \in N_{P_1P_2P_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N_{P_1P_2P_3}$ is named the S-active cone for $N_{P_1P_2P_3}$.

Obviously

$$N_{P_1P_2P_3} = \{ n = p_1^{i_1} p_2^{i_2} p_3^{i_3} \mid i_1, i_2, i_3 \in N^* \text{ and } j_k \in (i_{(u_1)}, i_{(l_1)}) \text{ where } j = k; j, k \in \{1, 2, 3\} \}.$$

The repartition of exponents is represented in the following scheme:
For $P_1 < P_2$ the repartion of exponents is representively in following scheme:

For numbers of type 2) $i_1 \in (i_{(r_2)}, i_1^{(r_2)})$ and $i_2 \in (i_{(r_3)}, i_2^{(r_3)})$

c) I consider that

$$N_{P_1 P_2 P_3} = \{ n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} | i_1, i_2, i_3 \in \mathbb{N}^* \},$$

where $P_1 < P_2 < P_3$ are prime numbers.

Exist the following situations:

1) $n \in N_{P_j}$, $j = 1, 2, 3$ this means that $p_j^{i_j}$ is active factor.

2) $n \in N_{P_j P_k}$, $j \neq k; j, k \in \{1, 2, 3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N_{P_1 P_2 P_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N_{P_1 P_2 P_3}$ is named the S-active cone for $N_{P_1 P_2 P_3}$.

Obviously

$$N_{P_1 P_2 P_3} = \{ n = p_1^{i_1} p_2^{i_2} p_3^{i_3} | i_1, i_2, i_3 \in \mathbb{N}^* \text{ and } i_k \in (i_{(r_k)}, i_k^{(r_k)}) \text{ where } j \neq k; j, k \in \{1, 2, 3\} \}.$$ 

The repartition of exponents is represented in the following scheme:
d) Generally, I consider $N_{p_1 p_2 \ldots p_r} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_r^{i_r} | i_1, i_2, \ldots, i_r \in \mathbb{N} \}$, where $p_1 < p_2 < \ldots < p_r$ are prime numbers.

On $N_{p_1 p_2 \ldots p_r}$ exist the following relation of equivalence:

$$n \rho m \Leftrightarrow n \text{ and } m \text{ have the same active factors.}$$

This have the following classes:

- $N_{p_1}^{p_2 \ldots p_r}$, where $j_1 \in \{1,2,\ldots,r\}$.

  $n \in N_{p_1}^{p_2 \ldots p_r} \Rightarrow n$ has only $p_1^{j_1}$ active factor

- $N_{p_1}^{p_2 \ldots p_r}$, where $j_1 \neq j_2$ and $j_1, j_2 \in \{1,2,\ldots,r\}$.

  $n \in N_{p_1}^{p_2 \ldots p_r} \Leftrightarrow n$ has only $p_1^{j_1}, p_2^{j_2}$ active factors.

$N_{p_1 p_2 \ldots p_r}$ which is named $S$-active cone.

$N_{p_1 p_2 \ldots p_r} = \{n \in N_{p_1 p_2 \ldots p_r} | n$ has $p_1^{i_1}, p_2^{i_2}, \ldots, p_r^{i_r}$ active factors$.}.$

Obviously, if $n \in N_{p_1 p_2 \ldots p_r}$, then $i_k \in (i_{k(j)}, i_{k(j)})$ with $k \neq j$ and $j, k \in \{1,2,\ldots,r\}$.

REFERENCES


