THE SOLUTION OF THE DIOPHANTINE EQUATION $\sigma_n(n) = n$ ($\Omega$)

by Pál Gronás

This problem is closely connected to Problem 29916 in the first issue of the "Smarandache Function Journal" (see page 47 in [1]). The question is: "Are there an infinity of nonprimes $n$ such that $\sigma_n(n) = n$?". My calculations will show that the answer is negative.

Let us move on to the first step in deriving the solution of ($\Omega$). As the wording of Problem 29916 indicates, ($\Omega$) is satisfied if $n$ is a prime. This is not the case for $n = 1$ because $\sigma_n(1) = 0$.

Suppose $\prod_{i=1}^{k} p_i^{r_i}$ is the prime factorization of a composite number $n \geq 4$, where $p_1, \ldots, p_k$ are distinct primes, $r_i \in \mathbb{N}$ and $p_1 r_1 \geq p_i r_i$ for all $i \in \{1, \ldots, k\}$ and $p_i < p_{i+1}$ for all $i \in \{2, \ldots, k-1\}$ whenever $k \geq 3$.

First of all we consider the case where $k = 1$ and $r_1 \geq 2$. Using the fact that $\eta(p_1^{r_1}) \leq p_1 s_1$ we see that $p_1^{r_1} = n = \sigma_n(n) = \sigma_n(p_1^{r_1}) = \sum_{d=0}^{r_1} \eta(p_1^{d}) \leq \sum_{d=0}^{r_1} p_1 s_1 = \frac{p_1 r_1 (r_1 + 1)}{2}$. Therefore $2 p_1^{r_1-1} \leq r_1 (r_1 + 1)$ for some $r_1 \geq 2$. For $p_1 \geq 5$ this inequality ($\Omega_1$) is not satisfied for any $r_1 \geq 2$. So $p_1 < 5$, which means that $p_1 \in \{2, 3\}$. By the help of ($\Omega_1$) we can find a supremum for $r_1$ depending on the value of $p_1$. For $p_1 = 2$ the actual candidates for $r_1$ are 2, 3, 4 and for $p_1 = 3$ the only possible choice is $r_1 = 2$. Hence there are maximum 4 possible solution of ($\Omega$) in this case, namely $n = 4, 8, 9$ and 16. Calculating $\sigma_n(n)$ for each of these 4 values, we get $\sigma_n(4) = 6, \sigma_n(8) = 10, \sigma_n(9) = 9$ and $\sigma_n(16) = 16$. Consequently the only solutions of ($\Omega$) are $n = 9$ and $n = 16$.

Next we look at the case when $k \geq 2$:

$$n = \sigma_n(n)$$

Substituting $n$ with it’s prime factorization we get

$$\prod_{i=1}^{k} p_i^{r_i} = \sigma_n(\prod_{i=1}^{k} p_i^{r_i}) = \sum_{d \mid n} \eta(d) = \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \eta(\prod_{i=1}^{k} p_i^{s_i})$$

$$\leq \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\eta(p_1^{s_1}), \ldots, \eta(p_k^{s_k})\}$$

$$\leq \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \{p_1 s_1, \ldots, p_k s_k\} \text{ since } \eta(p_i^{s_i}) \leq p_i s_i$$

$$< \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \{p_1 r_1, \ldots, p_k r_k\} \text{ because } s_i \leq r_i$$

$$= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} p_1 r_1 \left( p_1 r_1 \geq p_i r_i \text{ for } i \geq 2 \right)$$

$$\leq p_1 r_1 \prod_{i=1}^{k} (r_i + 1).$$
which is equivalent to
\[ \prod_{i=2}^{k} \frac{p_i^{r_i}}{r_i + 1} < \frac{p_1 r_1(r_1 + 1)}{p_1^{r_1}} = \frac{r_1(r_1 + 1)}{p_1^{r_1 - 1}} \quad (\Omega_2) \]

This inequality motivates a closer study of the functions \( f(x) = \frac{ax^2}{x^2 + 1} \) and \( g(x) = \frac{b(x - 1)}{x^{p_1}} \) for \( x \in [1, \infty) \), where \( a \) and \( b \) are real constants \( \geq 2 \). The derivatives of these two functions are
\[ f'(x) = \frac{2ax(x + 1)\ln a - 1}{(x + 1)^2} \quad \text{and} \quad g'(x) = \frac{b(x - 1)(2 - \ln b)x - 1}{x^{p_1 + 1}}. \]
Hence \( f'(x) > 0 \) for \( x \geq 1 \) since \( (x + 1)\ln a - 1 \geq (1 + 1)\ln 2 - 1 = 2\ln 2 - 1 > 0 \). So \( f \) is increasing on \([1, \infty)\).

Moreover \( g(x) \) reaches its absolute maximum value for \( x = \max\{1, \frac{2 - \ln b + \sqrt{(\ln b)^2 - 4}}{2\ln b} \} \).

Now \( \sqrt{(\ln b)^2 + 4} < \ln b + 2 \) for \( b \geq 2 \), which implies that \( x < \frac{(2 - \ln b + (\ln b + 2)}{2\ln b} = \frac{2}{\ln b} \leq \frac{2}{2} < 3 \).

Furthermore it is worth mentioning that \( f(x) \to \infty \) and \( g(x) \to 0 \) as \( x \to \infty \).

Applying this to our situation means that \( \frac{p_i^{r_i}}{r_i + 1} \) strictly increasing from \( \frac{2}{\ln b} \) to \( \infty \). Besides \( \frac{r_1(r_1 + 1)}{p_1^{r_1 - 1}} \leq \max\{2, \frac{4}{p_1}, \frac{12}{p_1^2} \} = \max\{2, \frac{4}{p_1}, \frac{12}{p_1^2} \} \leq 3 \) because \( \frac{4}{p_1} \geq \frac{12}{p_1^2} \) whenever \( p_1 \geq 2 \).

Combining this knowledge with \( (\Omega_2) \) we get that \( \prod_{i=2}^{k} \frac{p_i^{r_i}}{r_i + 1} < \frac{r_1(r_1 + 1)}{p_1^{r_1 - 1}} \leq \frac{r_1(r_1 + 1)}{2^{r_1 - 1}} \leq 3 \) \( (\Omega_3) \) for all \( r_1 \in \mathbb{N} \). In other words, \( \prod_{i=2}^{k} \frac{p_i}{r_i + 1} < 3 \). Consequently, \( r_1 \leq 2 \) and \( r_2 \leq 2 \), a contradiction which implies that \( k \leq 3 \).

Let us assume \( k = 2 \). Then \( (\Omega_2) \) and \( (\Omega_3) \) state that \( \frac{p_2^{r_2}}{r_2 + 1} < \frac{r_1(r_1 + 1)}{p_1^{r_1 - 1}} \) and \( \frac{p_2^{r_2}}{r_2 + 1} < 3 \), i.e. \( p_2 < 6 \). Next we suppose \( r_2 \geq 3 \). It is obvious that \( p_2 \geq 2 \cdot 3 = 6 \), which is equivalent to \( p_2 \leq 3 \).

Using this fact we get \( \frac{p_2^{r_2}}{r_2 + 1} < \frac{r_1(r_1 + 1)}{p_1^{r_1 - 1}} \leq \max\{2, \frac{4}{p_1}, \frac{12}{p_1^2} \} = \max\{2, \frac{4}{p_1}, \frac{12}{p_1^2} \} \leq 3 \) because \( \frac{4}{p_1} \geq \frac{12}{p_1^2} \) whenever \( p_1 \geq 2 \).

Accordingly \( p_2 < 2 \), a contradiction which implies that \( r_2 \leq 2 \) and \( r_2 \in \{1, 2\} \).

Furthermore \( 1 \leq \frac{p_2^{r_2}}{r_2 + 1} \leq \frac{r_1(r_1 + 1)}{p_1^{r_1 - 1}} \), which implies that \( r_1 \leq 6 \). Consequently, by fixing the values of \( p_2 \) and \( r_2 \), the inequalities \( \frac{r_1(r_1 + 1)}{p_1^{r_1 - 1}} > \frac{p_2^{r_2}}{r_2 + 1} \) and \( p_1 r_1 \geq p_2 r_2 \) give us enough information to determine a supremum (less than 7) for \( r_1 \) for each value of \( p_1 \).

This is just what we have done, and the result is as follows:

<table>
<thead>
<tr>
<th>( p_1 )</th>
<th>( r_2 )</th>
<th>( r_1 )</th>
<th>( n = p_1^{r_1} p_2^{r_2} )</th>
<th>( \sigma_n(n) )</th>
<th>( \text{IF } \sigma_n(n) = n ) \ THEN</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>( 1 \leq r_1 \leq 3 )</td>
<td>( 2 \cdot 3^1 )</td>
<td>( 2 + 3r_1(r_1 + 1) )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>( 1 \leq r_1 \leq 2 )</td>
<td>( 2 \cdot 5^1 )</td>
<td>( 2 + 5r_1(r_1 + 1) )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( p_1 \geq 7 )</td>
<td>1 ( 2p_1 )</td>
<td>( 2 + 2p_1 )</td>
<td>0 | 2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>( 2p_1 )</td>
<td>( 36 )</td>
<td>34 | 36</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( p_1 \geq 5 )</td>
<td>( 3p_1 )</td>
<td>( 3p_1 + 6 )</td>
<td>( p_1 ) | 6</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>( 2 \leq r_1 \leq 5 )</td>
<td>( 3 \cdot 2^1 )</td>
<td>( 2r_1^2 - 2r_1 + 12 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( p_1 \geq 5 )</td>
<td>1</td>
<td>( 3p_1 )</td>
<td>( 2p_1 + 3 )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>( 3 )</td>
<td>( 40 )</td>
<td>30 | 40</td>
</tr>
</tbody>
</table>

By looking at the rightmost column in the table above, we see that there are only contradictions except in the case where \( n = 3 \cdot 2^3 \) and \( r_1 = 3 \). So \( n = 3 \cdot 2^3 = 24 \) and \( \sigma_n(24) = 24 \). In other words, \( n = 24 \) is the only solution of \( (\Omega) \) when \( k = 2 \).
Finally, suppose \( k = 3 \). Then we know that \( \frac{2^3}{2} \cdot \frac{2^3}{2} < 3 \), i.e. \( p_2 p_3 < 12 \). Hence \( p_2 = 2 \) and \( p_3 \geq 3 \). Therefore \( \frac{\sigma(p_{i+1})}{p_i} \leq \frac{\sigma(p_{i+1})}{p_i} \leq 2 \) (\( \Omega_4 \)) and by applying (\( \Omega_3 \)) we find that
\[
\prod_{i=2}^{\infty} \frac{p_i}{p_i} = \frac{p_2}{p_2} < 2,
\]
giving \( p_3 = 3 \).

Combining the two inequalities (\( \Omega_3 \)) and (\( \Omega_4 \)) we get that \( \frac{2^3}{r_{2+1}} \cdot \frac{3^2}{r_{3+1}} < 2 \). Knowing that the left side of this inequality is a product of two strictly increasing functions on \([1, \infty)\), we see that the only possible choices for \( r_2 \) and \( r_3 \) are \( r_2 = r_3 = 1 \). Inserting these values in (\( \Omega_3 \)), we get \( \frac{2^3}{1+1} \cdot \frac{3^2}{1+1} = \frac{3}{2} < \frac{\sigma(p_{i+1})}{p_i} \leq \frac{\sigma(p_{i+1})}{p_i} \). This implies that \( r_1 = 1 \). Accordingly (\( \Omega \)) is satisfied only if \( n = 2 \cdot 3 \cdot p_1 = 6 p_1 \):

\[
6 p_1 = \sigma_n(6 p_1)
\]
\[
= \eta(1) + \eta(2) + \eta(3) + \eta(6) + \sum_{i=0}^{1} \sum_{j=0}^{1} \eta(2^i 3^j p_1)
\]
\[
= 0 + 2 + 3 + 3 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{ \eta(p_1), \eta(2^i 3^j) \}
\]
\[
= 8 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{ p_1, \eta(2^i 3^j) \}
\]
\[
= 8 + 4 p_1 \text{ because } \eta(2^i 3^j) \leq 3 < p_1 \text{ for all } i, j \in \{0, 1\}
\]
\[
p_1 = 4
\]

which contradicts the fact that \( p_1 \geq 5 \). Therefore (\( \Omega \)) has no solution for \( k = 3 \).

**Conclusion:** \( \sigma_n(n) = n \) if and only if \( n \) is a prime, \( n = 9, n = 16 \) or \( n = 24 \).

**REMARK:** A consequence of this work is the solution of the inequality \( \sigma_n(n) > n \) (\( \ast \)). This solution is based on the fact that (\( \ast \)) implies (\( \Omega_2 \)).

So \( \sigma_n(n) > n \) if and only if \( n = 8, 12, 18, 20 \) or \( n = 2p \) where \( p \) is a prime. Hence \( \sigma_n(n) \leq n + 4 \) for all \( n \in \mathbb{N} \).

Moreover, since we have solved the inequality \( \sigma_n(n) \geq n \), we also have the solution of \( \sigma_n(n) < n \).

**References**


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