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Adjacency Matrices of Some Directional Paths and Stars

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Abstract: Graph labeling where vertices and edges are assigned values subject to certain conditions have been motivated by various applied mathematical fields. Most of the problems in graph labeling are discussed on undirected graphs. Bloom G.S. and Hsu D.F. defined the labeling on directed graphs. In this paper, we discuss the adjacency matrices of graceful digraphs such as unidirectional paths,alternating paths,many orientations of directed star and a class of directed bistar. We also discuss the adjacency matrices of unidirectional paths and alternating paths if they are odd digraceful.

Key Words: Digraceful labeling, digraceful graph, Smarandachely H-digraceful graph, adjacency matrix, unidirectional paths, alternating paths.

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§1. Introduction

Most of the applications in Engineering and Science have undirected graphs are in the problems.The graph labeling problems are widely focus on undirected graphs.There exists situations in which directed graphs are playing a key role in some problems. There we using graph labeling in directed graphs. In 1980,Bloom G.S. and Hsu D.F.([3],[4],[5])extend the concept of graceful labeling of undirected graphs to digraphs.They investigate the graceful labeling problems of digraphs. Graceful digraphs are related in a variety ways to other areas of Mathematics.

The underlying graph UG(D) of a digraph D is obtained from D by removing the direction of each arc in D. Here we consider UG(D) is connected and has no self loops or multiple edges. In otherwords the digraph D is simply connected. The vertex set and edge set of a simple connected digraph are denoted by V(D) and E(D) where $E(D) = \overrightarrow{uv}, uv\epsilon V(D)$. For an arc $\overrightarrow{uv}$ the first vertex u is its tail and second vertex v is its head.

For all terminology and notations in graph theory,we follow Harary[1] and for all terminology regarding graceful labeling,we follow [2]. A connected graph with p vertices and q edges is called graceful if it is possible to label the vertices of x with pairwise distinct integers f(x) in $\{0, 1, 2, 3, \cdots, q\}$ so that each edge, xy, is labeled $|f(x) - f(y)|$, the resulting edge labels are pairwise distinct (and thus from the entire set $\{1, 2, 3, \cdots, q\}$). A connected graph with p vertices and q edges is called odd graceful if it is possible to label the vertices of x with pairwise

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distinct integers f(x) in $\{0, 1, 2, 3, \cdots, 2q-1\}$ so that each edge, xy, is labeled $|f(x) - f(y)|$,the resulting edge labels are pairwise distinct (and thus from the entire set $\{1, 3, 5, \cdots, 2q-1\}$).

Definition 1.1([4]) *A digraph D with p vertices and q edges is said to be digraceful if there exists an injection $f : V(D) \to \{0, 1, 2, \cdots, q\}$ such that the induced function $f' : E(D) \to \{1, 2, 3, \cdots, q\}$ which is denoted by $f'(u, v) = [f(v) - f(u)](modq + 1)$ for every directed edge (u, v) is a bijection, where $[v](modn)$ denotes the least positive integer of v modulo n. If the edge values are all distinct then the labeling is called a digraceful labeling of a digraph. A digraph is called a graceful digraph if it has a digraceful labeling.*

Generally, a *Smarandachely H-digraceful graph* is such a digraceful graph G that each subgraph $H' \prec G$ is a graceful digraph if $H' \cong H$. Clearly, a Smarandachely G-digraceful graph is nothing else but a digraceful graph. Let f be a labeling of a digraph D from V(D) to $\{0, 1, 2, \cdots, q\}$ such that for each arc $\overrightarrow{uv} \epsilon D, f(\overrightarrow{uv}) = f(u) - f(v)$ if f(u)> f(v),otherwise $f(\overrightarrow{uv}) = q + 1 + f(u) - f(v)$.We call f a digraceful labeling of D if the arc label set $\{f(\overrightarrow{uv}) : \overrightarrow{uv} \epsilon E(D)\} = \{1, 2, 3, \cdots, q\}$.Therefore D is called a graceful digraph.

Definition 1.2 *A digraph D with p vertices and q edges is said to be odd digraceful if there exists an injection $f : V(D) \to \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f' : E(D) \to \{1, 2, 3, \cdots, 2q-1\}$ which is denoted by $f'(u, v) = [f(v) - f(u)](mod2q)$ for every directed edge (u, v) is a bijection,where [v](mod n) denotes the least positive integer of v modulo n.If the edge values are all distinct then the labeling is called a odd digraceful labeling of a digraph.A digraph is called a odd graceful digraph if it has a odd digraceful labeling.*

Let f be a labeling of a digraph D from V(D) to $\{0, 1, 2, \cdots, 2q-1\}$ such that for each arc $\overrightarrow{uv} \epsilon D, f(\overrightarrow{uv}) = f(u) - f(v)$ if f(u)> f(v),otherwise $f(\overrightarrow{uv}) = q + 1 + f(u) - f(v)$.We call f a odd digraceful labeling of D if the arc label set $\{f(\overrightarrow{uv}) : \overrightarrow{uv} \epsilon E(D)\} = \{1, 2, 3, \cdots, q\}$.
Therefore D is called a odd graceful digraph.

In next two sections, we develop a generalized adjacency matrix of unidirectional paths and alternating paths based on graceful labeling and odd graceful labeling.

§2. Adjacency Matrices of Unidirectional Paths

Definition 2.1 *Let D be a graph with $V(D) = \{v_1, v_2, v_3, \cdots, v_p\}$.Then the matrix $A_D = [x_{ij}]$ defined by*

$$x_{ij} = \begin{cases} +1, & \text{if node } v_i \text{ is connected to } v_j \text{ and directed from } v_i \text{ to } v_j \\ -1, & \text{if node } v_j \text{ is connected to } v_i \text{ and directed from } v_i \text{ to } v_j \\ 0, & \text{otherwise} \end{cases} \quad (0.1)$$

is called the adjacency matrix of D.

Definition 2.2 *Let D be a digraph with q edges and a labeling*

$f : V(D) \to \{0, 1, 2, \cdots, q\}$. Then the $(q+1)\times(q+1)$ matrix $M_D = [x_{ij}]$ defined by

$$x_{ij} = \begin{cases} +1, & \text{if } xy \in E(D) \text{ for } f(x) = i \text{ and } f(y) = j \text{ directed from } x \text{ to } y \\ -1, & \text{if } yx \in E(D) \text{ for } f(y) = i \text{ and } f(x) = j \text{ directed from } x \text{ to } y \\ 0, & \text{otherwise} \end{cases} \quad (0.2)$$

is called the generalized adjacency matrix of D induced by the labeling f.

The generalized adjacency matrix of the digraph is skew symmetric.The generalized adjacency matrix allows all zeros for rows and columns corresponding to missing labels, while in the adjacency matrix such rows corresponding to vertices of degree zero.If two graphs have the same adjacency matrix,then they are isomorphic, but if two graphs have the same generalized adjacency matrix they may not be isomorphic.

Definition 2.3 *Let A be a $p \times p$ matrix.Then the k^{th} diagonal line of A is the collection of entries $D_k = \{a_{ij}/j - i = k\}$,counting multiplicity.*

It is clear that $0 \le |k| \le p - 1$. When $M = M_D$ is a generalized adjacency matrix,M is symmetric and $D_k = D_{-k}$. The entry ± 1 corresponding to an edge between vertices v_i and v_j lies on the $(j-i)^{th}$ diagonal line.The main diagonal line is D_k for $k = 0$. The odd diagonal lines are D_k for $k = \pm 1, \pm 3, \pm 5, \cdots, \pm 2q - 1$ and the even diagonal line are D_k for $k = \pm 2, \pm 4, \pm 6, \cdots, \pm 2q - 2$.

Definition 2.4([6]) *A finite group (G,.) of order n is said to be sequenceable if its elements can be arranged in a sequence $a_0 = e, a_1, a_2, \cdots, a_{n-1}$ in such a way that the partial products $b_0 = a_0, b_1 = a_0a_1, b_2 = a_0a_1a_2, \cdots, b_{n-1} = a_0a_1a_2 \cdots a_{n-1}$ are all distinct.*

Definition 2.7 *If both the indegree and outdegree of all the internal vertices of a directed path are one,then it is called unidirectional path and is denoted by $\overrightarrow{P_n}$.*

Theorem 2.1([4]) *$\overrightarrow{P_n}$ on n vertices is graceful if and only if z_n is sequenceable.*

Theorem 2.2 *Let D be a labeled unidirectional path with q edges and let $[M_D]$ be the generalized adjacency matrix for D. Then D is digraceful if and only if $[M_D]$ has exactly one entry ± 1 in each diagonal lines,except the main diagonal line of zeros.*

Proof It is to be noted that the matrix $[M_D]$ is a staircase shaped matrix and skew symmetric.The main diagonal line is D_k for $k = 0$.

Suppose that $[M_D]$ has exactly one entry ± 1 in each diagonal lines except the main diagonal. Suppose to the contrary that the labeling of D that induces $[M_D]$ is not digraceful.Then there are distinct edges v_rv_s and v_tv_u with edge labels $s - r = t - u = k > 0$. This implies that the sum of all the elements in a_{ij} if $j - i = \pm k$ is either 2 or 0.In the upper triangular matrix the sum of a_{ij} is 0 and in lower triangular matrix the sum of a_{ij} is 2 contradicting the assumption that $[M_D]$ has exactly one entry ± 1 in each diagonal D_k. Therefore labeling of D is graceful.

Let f be an digraceful labeling on D and consider $[M_D]$. Then, for all $k = \pm 1, \pm 3, \pm 5, \cdots, \pm q$, there is exactly one nonzero entry a_{ij} =±1 for $j - i = k$, contributing to D_k(k≠ 0) since each edge has a unique label. Then $[M_D]$ has exactly one entry one in each diagonal line except the main diagonal line. This completes the proof. □

Theorem 2.3 *Unidirectional path $\overrightarrow{P_n}$ on n vertices is odd digraceful if n is even.*

Theorem 2.4 *Let D be a labeled unidirectional path with q edges and let $[M_D]$ be the generalized adjacency matrix for D . Then D is odd digraceful if and only if $[M_D]$ has exactly one entry ±1 in each odd diagonal lines and all the entries are 0 in the even diagonal lines including the main diagonal line of zeros.*

Proof It is to be noted that the matrix $[M_D]$ is a staircase shaped matrix and skew symmetric.The main diagonal line is D_k for $k = 0$. The odd diagonal lines are D_k for $k = \pm 1, \pm 3, \pm 5, \cdots, \pm 2q - 1$ and the even diagonal line are D_k for $k = \pm 2, \pm 4, \pm 6, \cdots, \pm 2q - 2$.

Suppose that $[M_D]$ has exactly one entry 1 in each odd diagonal lines and all the diagonal entries are zero in the even diagonal lines including the main diagonal. Suppose to the contrary that the labeling of G that induces $[M_D]$ is not odd dIgraceful Then there are distinct edges $v_r v_s$ and $v_t v_u$ with edge labels $s - r = t - u = k > 0$. This implies that there exists atleast one even diagonal which has a non zero entry and at least one odd diagonal which has all entries are zero contradicting the assumption that $[M_D]$ has exactly one entry 1 in each odd diagonal D_k for $k = \pm 1, \pm 3, \pm 5, \cdots, \pm 2q - 1$. Therefore labeling of is odd graceful.

Let f be an odd graceful labeling on G and consider $[M_D]$.Then for all $k = \pm 1, \pm 3, \pm 5, \cdots$, $\pm 2q - 1$, there is exactly one nonzero entry a_{ij} =±1 such that $j - i = k$,contributing to D_k since each edge has a unique label.Then $[M_D]$ has exactly one entry one in each odd diagonal line. Also in odd graceful labeling there is no even number edge labeling,so the diagonal line D_k are 0 for $k = \pm 2, \pm 4, \pm 6, \cdots, \pm 2q - 2$. This completes the proof. □

§3. Adjacency Matrices of Alternating Paths

Now we consider the adjacency matrix in digraceful labeling of alternating path $\overrightarrow{AP_p}$. An alternating path $\overrightarrow{AP_p}$ with p vertices is an oriented path in which any two consecutive arcs have opposite directions.Let $v_1, v_2, v_3, \cdots, v_p$ be the vertices of $\overrightarrow{AP_p}$ and the arcs of $\overrightarrow{AP_p}$ are $\overrightarrow{v_1v_2}, \overrightarrow{v_3v_2}, \overrightarrow{v_3v_4}, \cdots, \overrightarrow{v_{p-1}v_p}$ when p is even or $\overrightarrow{v_1v_2}, \overrightarrow{v_3v_2}, \overrightarrow{v_3v_4}, \cdots, \overrightarrow{v_pv_{p-1}}$ when p is odd. An alternating path $\overrightarrow{AP_p}$ is digraceful based on the following labeling
$f : V(\overrightarrow{AP_p}) \to \{0, 1, 2, 3, \cdots, p - 1\}$ defined by

$f(v_1) = 0;$
$f(v_{2i}) = p - i$ for $i = 1, 2, 3 \cdots, \left\lfloor \frac{p}{2} \right\rfloor$;
$f(v_{2i+1}) = i$ for $i = 1, 2, 3, \cdots, \left\lfloor \frac{p-1}{2} \right\rfloor$.

We have the following theorem.

Theorem 3.1 *Let $D = \overrightarrow{AP_p}$ be a labeled alternating path with q edges and let $[M_D]$ be the* *generalized adjacency matrix for D. Then $[M_D]$ has exactly one entry +1 in each diagonals of the upper triangular matrix and exactly one entry -1 in each diagonals of the lower triangular matrix if and only if D is digraceful.*

Proof It is to be noted that the matrix is skew symmetric. First assume that $[M_D]$ has exactly one entry +1 in each diagonals of upper triangular matrix and exactly one entry -1 in each diagonals of lower triangular matrix.Suppose to the contrary that the D is not digraceful.Then there are distinct edge labels $h-g=f-q=k>0$. This implies that the sum of all the elements in a_{ij} is 0 if j-i = $\pm k$, contradicting the assumption.So D is digraceful.

Let f be a digraceful labeling on D and consider $[M_D]$.consider the arc $\overrightarrow{v_iv_j}$ on D.Since D is digraceful the labeling on the vertex v_j is greater than labeling on v_i. Then for all diagonals D_k for k = $\pm1, \pm2, \pm3, \cdots, \pm q$ there is exactly one nonzero entry $a_{ij}=1$ for $j>i$ and $a_{ij}=-1$ for $j<i$. □

Now we consider the adjacency matrix in odd digraceful labeling of alternating path $\overrightarrow{AP_p}$. An alternating path $\overrightarrow{AP_p}$ is odd digraceful based on the following labeling $f: V(\overrightarrow{AP_p}) \rightarrow \{0,1,2,3,\cdots,2p-3\}$ defined by

$f(v_1)=0;$
$f(v_{2i})=2p-2i-1, for i=1,2,3,\cdots,\lfloor \frac{p}{2}\rfloor;$
$f(v_{2i+1})=2i, for i=1,2,3,\cdots,\lfloor \frac{p-1}{2}\rfloor.$

Corollary 3.1 *Let $D=\overrightarrow{AP_p}$ be a labeled alternating path with q edges and let $[M_D]$ be the generalized adjacency matrix for D. Then $[M_D]$ has exactly one entry +1 in each odd diagonals of the upper triangular matrix and exactly one entry -1 in each odd diagonals of the lower triangular matrix if and only if D is odd digraceful.Also all the entries are zero for even diagonal lines.*

§4. Adjacency Matrices of Many Orientations of Directed Star

In this section we consider the adjacency matrices of many orientations of directed star.Let $K_{1,m}$ be an orientations of a star $K_{1,m}$ on $(m+1)$ vertices.In ([7]) Bing Yao, Ming Yao and Hui Cheng proved the following conditions on gracefulness of directed star.

(1) A directed star $\overrightarrow{K_{1,m}}$ is digraceful if m is odd;

(2) A directed star $\overrightarrow{K_{1,m}}$ is digraceful if m is even and one of out-degree and in-degree of the center w of $\overrightarrow{K_{1,m}}$ must be even.

We have the following theorem based on adjacency matrices on graceful directed stars.

Theorem 4.1 *The directed star $\overrightarrow{K_{1,m}}$ is digraceful if and only if the adjacency matrices are any one of the following two forms:*

(1) *There exists exactly one entry 1 or -1 in each diagonal lines except the main diagonal;*

(2) *There exists both ±1 in the diagonals $D_{\pm1}$ and the remaining diagonal lines have exactly one entry ±1. The main and last diagonal lines have all entries are zeros.*

Proof The adjacency matrix is skew symmetric.The directed star have no self loops,the entries in the main diagonal line are all zeros. First suppose that the directed star $\overrightarrow{K_{1,m}}$ is digraceful. The center vertex in the directed star is connected to all the other vertices.The labeling on the center vertex has either zero or any one of the numbers from 1 to m-1.If the labeling on the center vertex is zero,then it is connected to all the m labeled vertices.since each edge have a unique label,in each edge of the adjacency matrix there must exists an entry ±1.Suppose the labeling on the center vertex is other than zero.The labeling on the center vertex is any one of the numbers from 1 to $m-1$.If r is the labeling on the center vertex,then there exists at least two vertices connected to the center vertex have labelings $r-1$ and $r+1$.The diagonal lines in the adjacency matrix corresponding to these vertices are $D_{\pm1}$ and they have both ±1. Also there exists a labeling from center vertex to all other remaining vertices.So the remaining diagonal lines have exactly one entry ±1.Since there is no connection between the labeling zero and m,the last diagonal is zero.Conversly suppose that the adjacency matrix satisfies any one of the given conditions.The directed star $\overrightarrow{K_{1,m}}$ is digraceful followed by Theorem 2.1. □

§5. Adjacency Matrices of a Class of Directed Bistar

A ditree H with diameter three is called a directed bistar. In ([7]), we have the following description about a class of directed bistars $T(s,t)$ Dibistar(I): The vertex set and arc set of a directed bistar T(s,t) are defined as $V(T(s,t)) = u_i, u, v, v_j : i\varepsilon[1, s-1], j\varepsilon[1, t]$ and $A(T(s,t)) = \overrightarrow{uu_i}, \overrightarrow{uv}, \overrightarrow{vv_j} : i\varepsilon[1, s-1], j\varepsilon[1, t]$,respectively,where u is the root of T(s,t).Clearly,the in-degrees $d_T^-(s,t)(u) = 0$ and $d_T^-(s,t)(v) = 1$ and the out degrees $d_T^+(s,t)(u) = s$ and $d_T^+(s,t)(v) = t$.In ([7]) it is to be proved that every directed bistar T(2l+1,2k-2l) defined by Dibistar(I) is digraceful for integers $k > l \geq 0$.

Theorem 5.1 *The directed bistar T(2l+1,2k-2l) defined by dibistar(I) is digraceful if and only if the adjacency matrices have at least* ±1 *in each diagonal lines except 4l diagonal lines in which*

(1) *2l diagonal entries contains both 1 and -1;*

(2) *2l diagonal entries are all zero except the main diagonal.*

Proof If $l = 0$, the directed bistar T(1,2k)is a directed star $K_{1,2k+1}$. The result follows from theorem(6). Let v_1 and v_2 be the centers of the directed bistar $T(2l+1, 2k-2l)$. Suppose the center v_1 is connected to $2l$ vertices and the center v_2 is connected to 2k-2l+1 vertices.Since the bistar is digraceful,each edge has a unique labeling.Let the labeling on v_2 is zero which is not connected to 2l vertices.The labeling contributed a non zero entry ±1 in each diagonal lines of the adjacency matrix except the $2l$ diagonal lines and the main diagonal. Also the labeling on v_1 is any integer from 1 to $2k+1$ and it contributed a non zero entry ±1 in $2l$ diagonal lines of the adjacency matrix. So the $2l$ diagonal entries contains both 1 and -1. The converse follows from Theorem 2.1. □

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