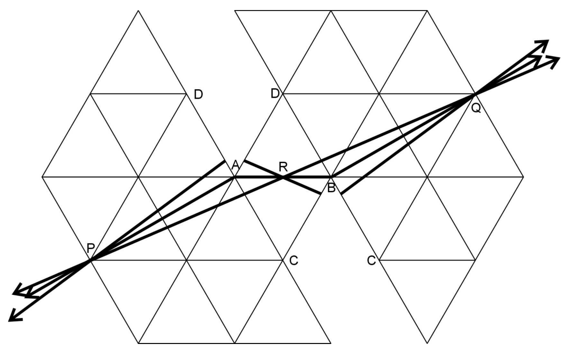
SMARANDACHE MANIFOLDS

Howard Iseri



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The picture on the cover is a representation of an s-manifold illustrating some of the behavior of lines in an s-manifold.

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An international club on Smarandache geometries can be found at
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Paper abstracts can be submitted online to the First International Conference on Smarandache Geometries, that will be held between 3-5 May, 2003, at the Griffith University, Gold Coast Campus, Queensland, Australia, organized by Dr. Jack Allen, at http://at.yorku.ca/cgi-bin/amca-calendar/public/display/conference_info/fabz54.

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Introduction

A complete understanding of what something is must include an understanding of what it is not. In his paper, "Paradoxist Mathematics" **[19]**, Florentin Smarandache proposed a number of ways in which we could explore "new math concepts and theories, especially if they run counter to the classical ones." In a manner consistent with his unique point of view, he defined several types of geometry that are purposefully not Euclidean and that focus on structures that the rest of us can use to enhance our understanding of geometry in general.

To most of us, Euclidean geometry seems self-evident and natural. This feeling is so strong that it took thousands of years for anyone to even consider an alternative to Euclid's teachings. These non-Euclidean ideas started, for the most part, with Gauss, Bolyai, and Lobachevski, and continued with Riemann, when they found counterexamples to the notion that geometry is precisely Euclidean geometry. This opened a whole universe of possibilities for what geometry could be, and many years later, Smarandache's imagination has wandered off into this universe.

The geometry associated with Gauss, Bolyai, and Lobachevski is now generally called hyperbolic geometry. Compared to Euclidean geometry, the lines in hyperbolic geometry are less prone to intersecting one another. Whereas even the slightest change upsets the delicate balance of parallelism for Euclidean lines, parallelism of hyperbolic lines is distinctly more robust. On the other hand, it is impossible for lines to be parallel in Riemann's geometry. It is not clear which Riemann had in mind (see **[3]**), but today we would call it either elliptic or spherical geometry. All of these geometries (Euclidean, hyperbolic, elliptic, and spherical) are homogeneous and isotropic. This is to say that each of these geometries looks the same at any point and in any direction within the space. Most of the study of geometry at the undergraduate level concerns these "modern" geometries (see **[3, 12, 10]**).

Although the term *Riemannian geometry* sometimes refers specifically to one of the geometries just mentioned (elliptic or spherical), it is now most likely to be associated with a class of differential geometric spaces called Riemannian manifolds. Here, geometry is studied through curvature, and the basic Euclidean, hyperbolic, elliptic, and spherical geometries are particular constant curvature examples. Riemannian geometry eventually evolved into the geometry of general relativity, and it is currently a very active area of mathematical research (see **[21, 16, 4, 8]**).

Riemannian manifolds could be described as those possible universes that inhabitants might mistake as being Euclidean, elliptic, or hyperbolic. Great insight comes from the realization that the geometries of Euclid, Gauss, Bolyai, et al, are particular examples of one kind of space, and extending attention to non-uniform spaces brings much generality, applicability (e.g. general relativity), and much more to understand.

Smarandache continues in the spirit of Riemann by wanting to explore non-uniformity, but he does this from another, and perhaps more classical, point of view. While much of the current study of geometry continues the work of Riemann and the transformational approach of Klein (see **[13]**), Smarandache challenges the axiomatic approach inspired by Euclid, and now closely associated with Hilbert. This axiomatic approach is generally referred to as synthetic geometry (see **[9, 14, 12]**).

By its nature, the axiomatic approach promotes uniformity. If we require that through any two points there is exactly one line, for example, then all points share this property. Each axiom of a geometry, therefore, tends to force the space to be more uniform. If an axiom holding true in a geometry creates uniformity, then Smarandache asks, what if it is false? Simply being false, however, does not necessarily counter uniformity. With Hilbert's axioms, for example, replacing the Euclidean parallel axiom with its negation, the hyperbolic parallel axiom, only results in transforming Euclidean uniformity into hyperbolic uniformity.

In Smarandache geometry, the intent is to study non-uniformity, so we require it in a very general way. A **Smarandache geometry** (1969) is a geometric space (i.e., one with points and lines) such that some "axiom" is false in at least two different ways, or is false and also sometimes true. Such an axiom is said to be **Smarandachely denied** (or **S-denied** for short).

As first mentioned, Smarandache defined several specific types of Smarandache geometries: paradoxist geometry, non-geometry, counter-projective geometry, and anti-geometry (see **[19]**). For the paradoxist geometry, he gives an example and poses the question, "Now, the problem is to find a nice model (on manifolds) for this Paradoxist Geometry, and study some of its characteristics." This particular study of Smarandache manifolds began with an attempt to find a solution to this problem.

A paradoxist geometry focuses attention on the parallel postulate, the same postulate of Euclid that Gauss, Bolyai, Lobachevski, and Riemann sought to contradict. In fact, Riemann began the study of geometric spaces that are non-uniform with respect to the parallel postulate, since in a Riemannian manifold, the curvature may change from point to point. This corresponds roughly with what we will call semi-paradoxist. It would seem, therefore, that a study of Smarandache geometry should start with Riemannian manifolds, and inadvertently, it has. Unfortunately, describing and manipulating Riemannian manifolds is far from trivial, and many Smarandache-type structures probably cannot exist in a Riemannian manifold.

In discussions within the Smarandache Geometry Club **[2]**, a special type of manifold, similar in many ways to a Riemannian manifold, showed promise as a tool to easily construct paradoxist geometries. This led to the paper, "Partially paradoxist geometries" **[15]**. It quickly became apparent that almost all of the properties that Smarandache proposed in **[19]** could be found in manifolds of this type.

These s-manifolds, which is what we will call them, follow a long tradition of piecewise linear approaches to, and avoidances of, the problems of the differential and the continuous. As we will define them, s-manifolds are a very restricted subclass of the polyhedral surfaces. The relationship between polyhedral surfaces and Riemannian manifolds is as old as Riemannian geometry itself, and the all-important notion of curvature can be viewed as an extension of the angle defect in a polyhedral surface due to Descartes (see **[17, 11]**). In addition, some concept of a line, or a geodesic, is a natural part of the study of polyhedral surfaces, but we will use a particular definition of a line that may have appeared as recently as 1998 in **[18]**. So while the basic ideas studied in this book are not new, the particular formulations and the focus on plane figures seems to be unique, and therefore, the potential exists for original research at all levels.

The purpose of this book is to lay out basic definitions and terminology, rephrase the most obvious applicable results from existing areas of geometry and topology, and to show that s-manifolds can be a useful tool in studying Smarandache geometry.

In Chapter 1, we define what an s-manifold is. Smarandache geometry is quite general, so it is difficult to see any basic structures that exist widely. Our definition for an s-manifold, therefore, is purposefully restrictive, so that we may have a reasonable opportunity to find general results. We will probably have to focus attention even more tightly before making significant progress.

In Chapter 2, we analyze the axioms of Hilbert in an s-manifold context. These cover most of the basic concepts of 2-dimensional geometry, and of course, all of the theorems of Euclidean geometry are based on them.

Some s-manifold examples of Smarandache geometries are presented in Chapter 3, and some of the basic issues surrounding closed s-manifolds, in particular the topology of 2-manifolds, are discussed in Chapter 4. The book ends with some notes on continued study.

Throughout this book, questions and conjectures are posed. You are invited to post answers to these questions to the Smarandache Geometry Club **[2]**. You may also pose questions of your own and participate in discussions about Smarandache geometry here. The publisher of this book is interested in publishing papers generated out of these explorations as a collection of papers or in the *Smarandache Notions Journal*.

Members of the Smarandache Geometry Club **[2]** were involved in the discussions that generated the basic idea of an s-manifold and many of the concepts explored in this book. These include mikeantholy (Mike Antholy), m_l_perez (Minh Perez), noneuclid (M. Downly), johnkamla2000 (Kamla), dacosta_teresinha (Dacosta), jeanmariecharrier (Jean Marie), marcelleparis (Marcelle), ken5prasad (Ken Prasad), zimolson (Zim Olson), duncan4320001 (Joan), charlestle (Charlie), ghniculescu, bsaucer (Ben Saucer), and klaus1997de. Most of these discussions can be viewed at the club website.

I would like to thank the reviewers Joel Hass, Marcus Marsh, and Catherine D'Ortona. Prof. Marsh was the teacher most responsible for my turning to mathematics, and Prof. Hass, my thesis advisor, introduced me to the real world of geometry and topology. Virtually all of my thinking in mathematics can be traced back in some way to these two

mathematicians. Prof. D'Ortona, a valued colleague, and my wife Linda are currently the most active forces on my professional ideas. I am, of course, responsible for the correctness of the material presented here and how I chose to implement the suggestions of the reviewers.

This book is dedicated to my two huns, Linda and Zoe.

Chapter 1. Smarandache Manifolds

We present here a definition for a special type of Smarandache manifold, which we will call an **s-manifold**. Since at present, these s-manifolds are the only manifolds presented in the context of Smarandache geometry, we will leave a more general definition to the future. We will see that an s-manifold is general enough to display almost all of the properties of a Smarandache geometry, but is restrictive enough so that we can start to make general statements about them.

For the purposes of this book, an uppercase "S," as in "S-denial", will be short for "Smarandache." A lowercase "s," as in "s-manifold", will refer to the special type of Smarandache manifold that is the focus here.

s-Manifolds

The idea of an s-manifold was based on the hyperbolic paper described in **[21]** and credited to W. Thurston. Essentially the same idea in a more general setting can be found in the *straightest geodesics* of **[18]** (see also **[1]**). In **[21]**, the structure of the hyperbolic plane is visualized by taping together equilateral triangles made of paper so that each vertex is surrounded by seven triangles. Squeezing seven equilateral triangles around a single vertex, as opposed to the six triangles we would see in a tiling of the plane, forces the paper into a kind of saddle shape (see Figure 1).

Figure 1. A paper model with seven equilateral triangles around one vertex.

We will extend this idea to elliptic geometry by putting five triangles around a vertex, and of course, to Euclidean geometry by using six (see Figures 2 and 3). The basic concept of an s-manifold is contained in these paper models made of equilateral triangles taped together edge to edge with five, six, or seven triangles around any particular vertex.

In these paper models, the paper will bend, but will not be stretched. Because of this, and the fact that paper is inherently Euclidean, we can assume that the geometry within any single triangle is Euclidean. Since two triangles taped together can lie flat, the geometry within any pair of adjacent triangles must also be Euclidean. Any non-Euclidean geometry comes from the curvature that is concentrated in the vertices.

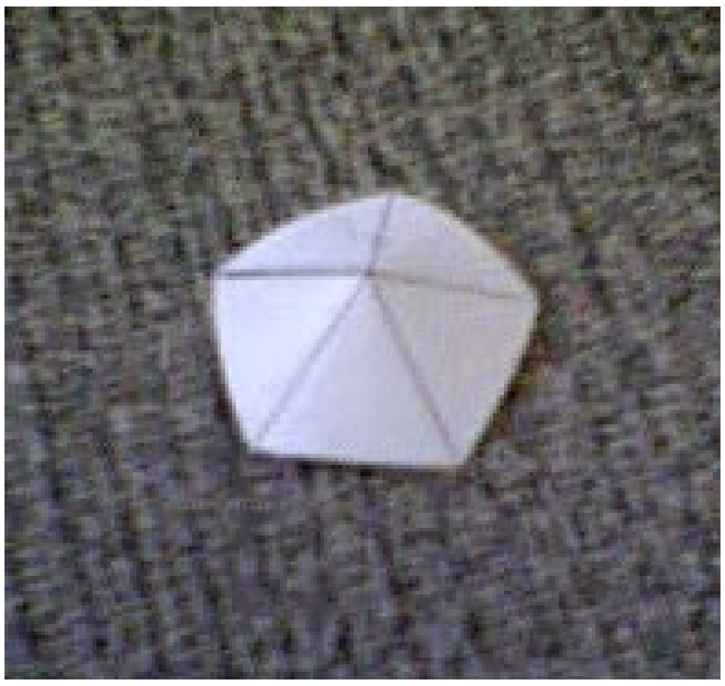


Figure 2. A paper model with five equilateral triangles around one vertex.

Figure 3. A paper model with six equilateral triangles around one vertex.

Instead of paper triangles, an s-manifold is constructed from triangular disks. These disks are subspaces of the Euclidean plane formed by equilateral triangles with sides of length one and consisting of the vertices, the edges, and the interiors of the triangles. The geometry within a single triangular disk is Euclidean, and concepts such as line segments, length, and angle measure will remain intact.

We will join pairs of triangles by identifying edges. For example, if edge AB of triangle ABC is to be identified with edge EF of triangle DEF (see Figure 4), then we will consider A and E to be the same point, and any point P on AB a distance x from A will be identified with the point Q on EF that is a distance x from E. In particular, B is identified with F. We will say that these disks **share the edge** AB (or EF).

After an identification of this type, we will assume that the two adjacent triangles lie next to each other, and that they lie in a plane. The geometry, therefore, within any pair of adjacent triangular disks is Euclidean. We will not be able to think of all pairs of adjacent triangular disks as lying in a plane simultaneously, but we will always assume this for any particular pair.

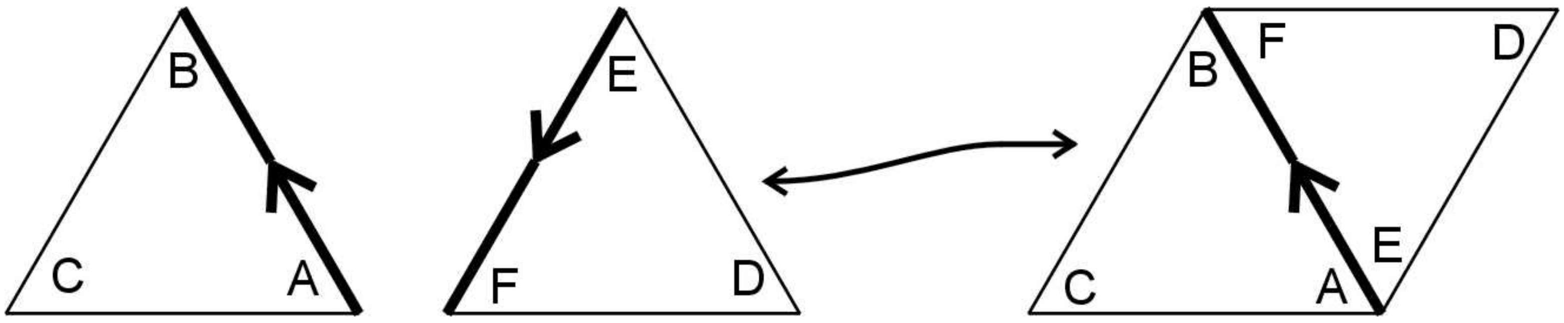


Figure 4. Triangles ABC and DEF share the edge AB.

An **s-manifold** will be any collection of these (equilateral) triangular disks joined together such that each edge is the identification of one edge each from two distinct disks and each vertex is the identification of one vertex from each of five, six, or seven distinct disks.

There is no requirement that an s-manifold must "exist" in $\mathbf{R}^3$ or any other Euclidean space. For example, a Klein bottle can have an s-manifold structure.

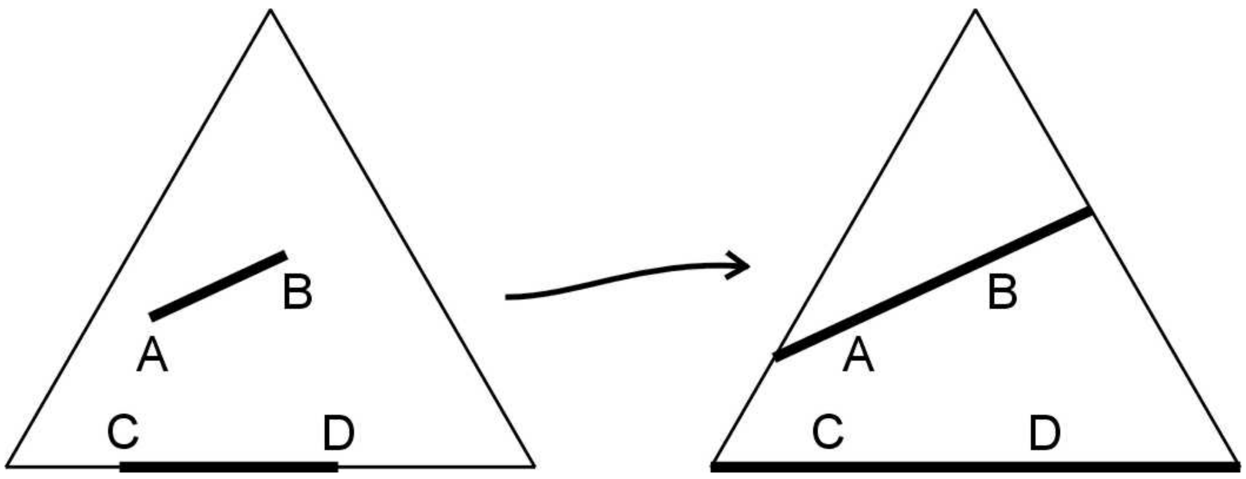


Figure 5. Segments inside a disk are extended to the boundaries of the disk.

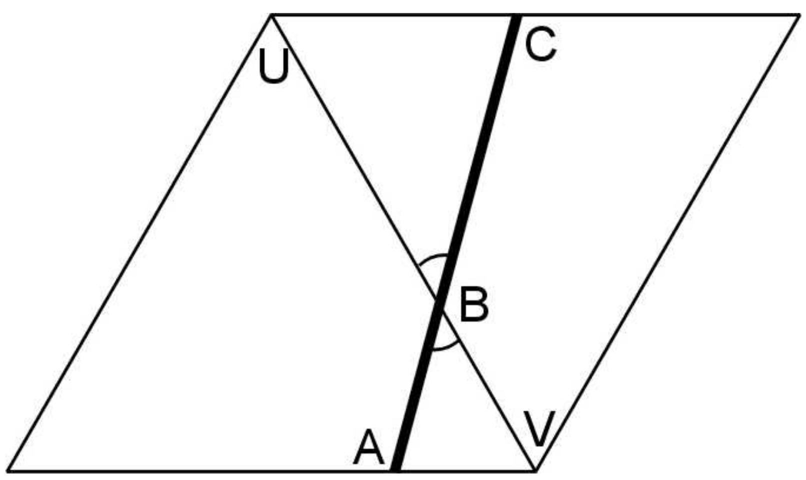


Figure 6. Extending s-lines across edges forms two segments that make congruent vertical angles with the edge.

A **geodesic** in a manifold is a curve that is as straight as possible. Lines in an s-manifold will be the natural geodesics, and we will call them **s-lines** to differentiate them from the lines in the Euclidean plane. An s-line will be any piecewise linear curve that can be constructed from a line segment lying within one of the triangular disks and extended as follows. Since the triangular disks are subspaces of the Euclidean plane, any segment in a disk can be extended to the boundaries along a straight line in the Euclidean sense, as in Figure 5. Here, both segment AB and segment CD are extended to the boundaries of the disk.

From an endpoint that lies on the interior of an edge, the s-line extends across the adjacent triangle as a straight line segment in the Euclidean sense, so that vertical angles formed by the edge and the s-line are congruent, as in Figure 6. Here, ∠ABV is congruent to ∠CBU. If we think of these two adjacent triangular disks as lying in the plane, then s-lines are straight in the Euclidean sense as they cross edges.

From an endpoint that is a vertex, the s-line extends across a triangular disk sharing that vertex as a straight line segment in the Euclidean sense so that these two segments form two equal angles. An s-line passing through a vertex will sometimes be referred to as **singular**. The measure of the two equal angles depends on the number of triangular disks around that vertex.

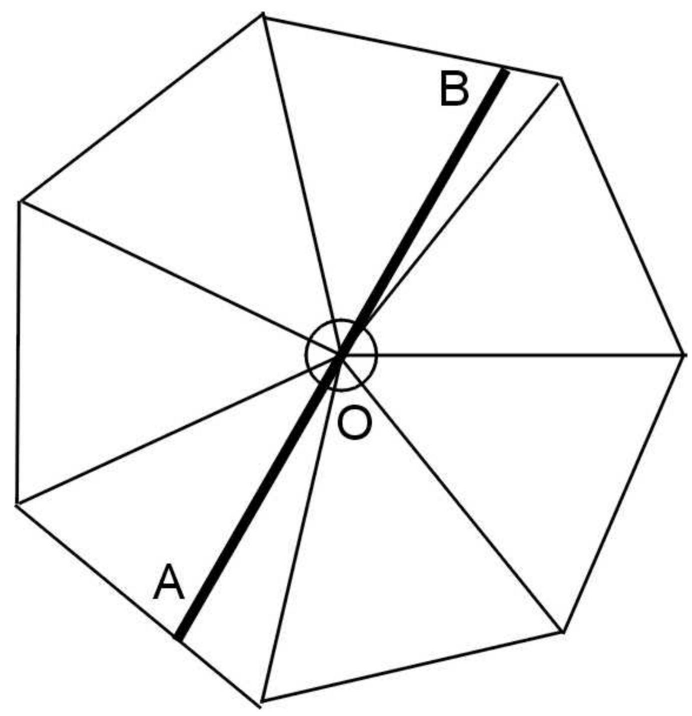


Figure 7. A deformed hyperbolic star: both angles ∠AOB have measure 210º.

A vertex with seven (equilateral) triangular disks around it will be called a **hyperbolic vertex**. The seven disks together will be called a **hyperbolic star**. There are seven 60º angles around a hyperbolic vertex for a total of 420º, and an s-line will form two 210º angles, as in Figure 7. Here, both angles designated as ∠AOB have measure 210º. This is how we will extend the concept of a **straight angle** to hyperbolic stars.

Around a **Euclidean vertex**, there are six (equilateral) triangular disks, which together we will call a **Euclidean star**. There is a total of 360º around a Euclidean vertex, so an s-line will form two 180º angles, as in Figure 8. Since a Euclidean star can lie flat in the plane, s-lines are straight in the Euclidean sense across a Euclidean vertex. The geometry within a Euclidean star is clearly Euclidean.

Around an **elliptic vertex**, there are five (equilateral) triangular disks, which form an **elliptic star**. There is a total of 300º around an elliptic vertex, so an s-line will form two

150º angles, as in Figure 9, and so both angles ∠AOB have measure 150º. As in hyperbolic stars, this is how we extend the concept of straight angles to elliptic stars.

In any s-manifold, s-lines extend indefinitely in this way. By this we mean that we can follow an s-line for any distance in either direction. It is possible that an s-line could be closed (like a circle) in an s-manifold, and in extending indefinitely, we may be traversing the same closed curve an infinite number of times. In addition, we will see that s-lines may have multiple self-intersections.

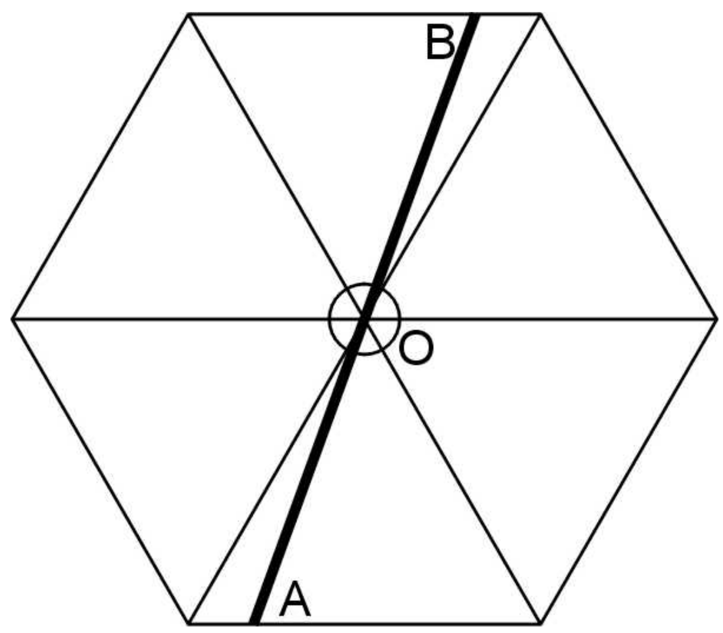


Figure 8. A Euclidean star: both angles ∠AOB have measure 180º.

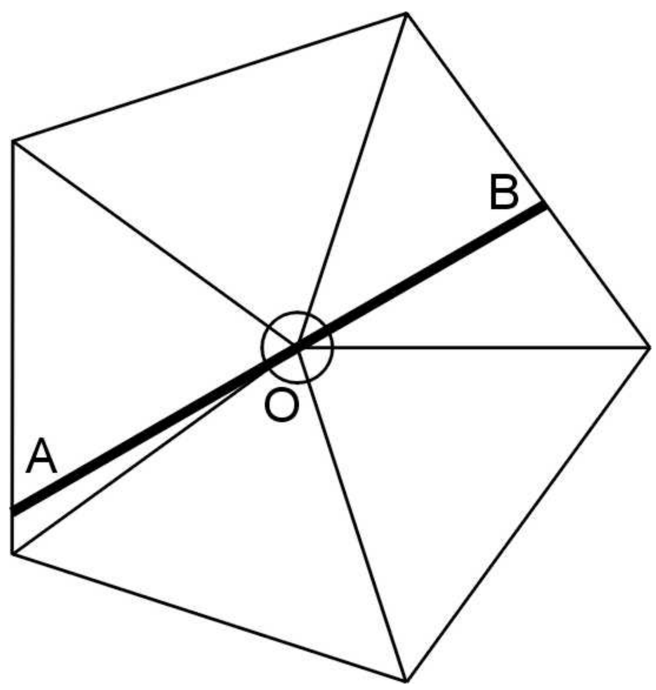


Figure 9. A deformed elliptic star: both angles ∠AOB have measure 150º.

Geometry in an s-manifold

Two things determine the geometric structure in a particular s-manifold, the configuration of the non-Euclidean vertices, and the global topology. The non-Euclidean vertices introduce a sort of curvature, and this affects the relationships between s-lines. The topology of an s-manifold can allow lines to wrap around the space, for example, and this allows for various types of interactions between s-lines beyond that caused by curvature. We will look first at the effects of the non-Euclidean vertices, and the effects of the topology of an s-manifold will be addressed throughout the book.

Geometry in elliptic stars

Instead of deforming an elliptic star, we can lay it flat by making a cut. In Figure 10, a cut has been made along the edge OA. Alternatively, we can think of Figure 10 as an identification scheme, and the two edges marked OA should be identified or glued. We will present most of the examples of s-manifolds in this book this way.

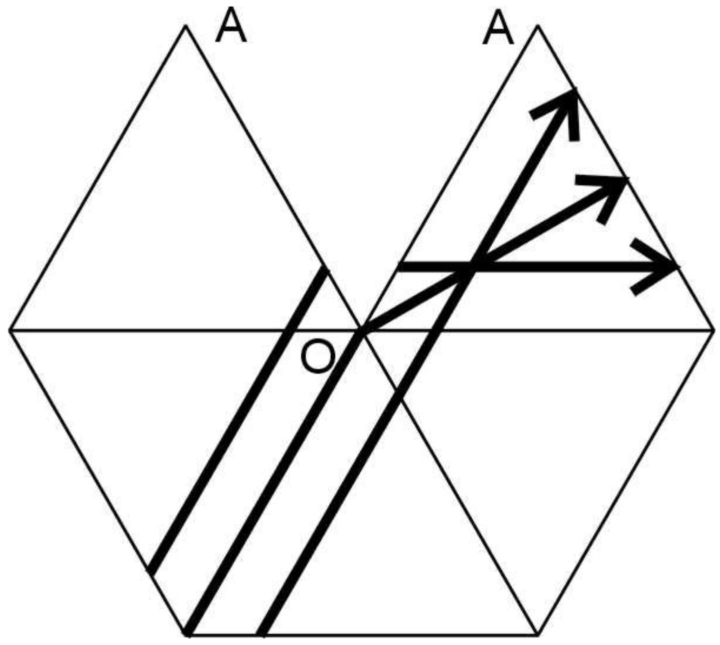


Figure 10. Some s-lines near an elliptic vertex.

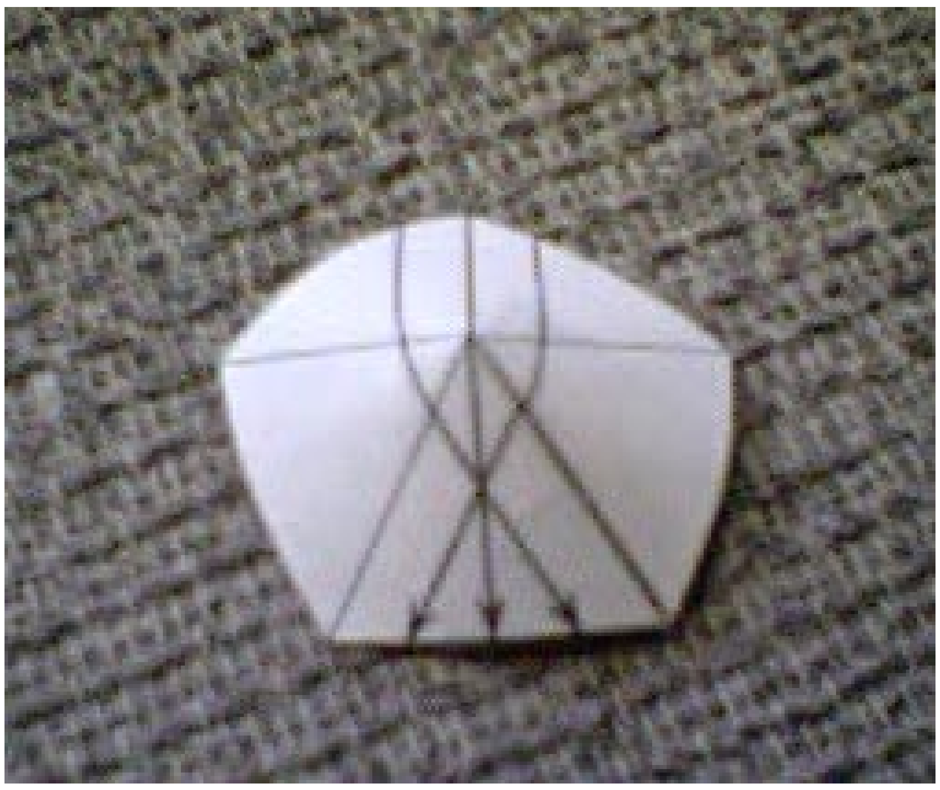


Figure 11. Paper model corresponding to Figure 10.

Three sample s-lines are indicated in Figures 10 and 11. These s-lines are drawn parallel to the edges for convenience, but s-lines can point in any direction. The two s-lines shown that do not pass through the vertex cross edges and makes congruent vertical angles with the edges. The singular s-line passing through the vertex makes two 150º angles (or two-and-a-half triangles). Note that these s-lines are straight within any pair of adjacent triangular disks and that the s-lines appear to bend at the vertex and across the cut. This is only because we have made a cut and flattened the surface. In the paper model shown in Figure 11, these s-lines curve, but only in a direction perpendicular to the surface. In other words, the s-lines are straight within the surface, and they bend only as the surface bends. An essential property of an elliptic star is that s-lines passing on opposite sides of the elliptic vertex turn towards each other. This is similar to the behavior of geodesics in a Riemannian manifold with positive curvature. A sphere, for example, has positive curvature, and its geodesics, the great circles, all turn towards each other.

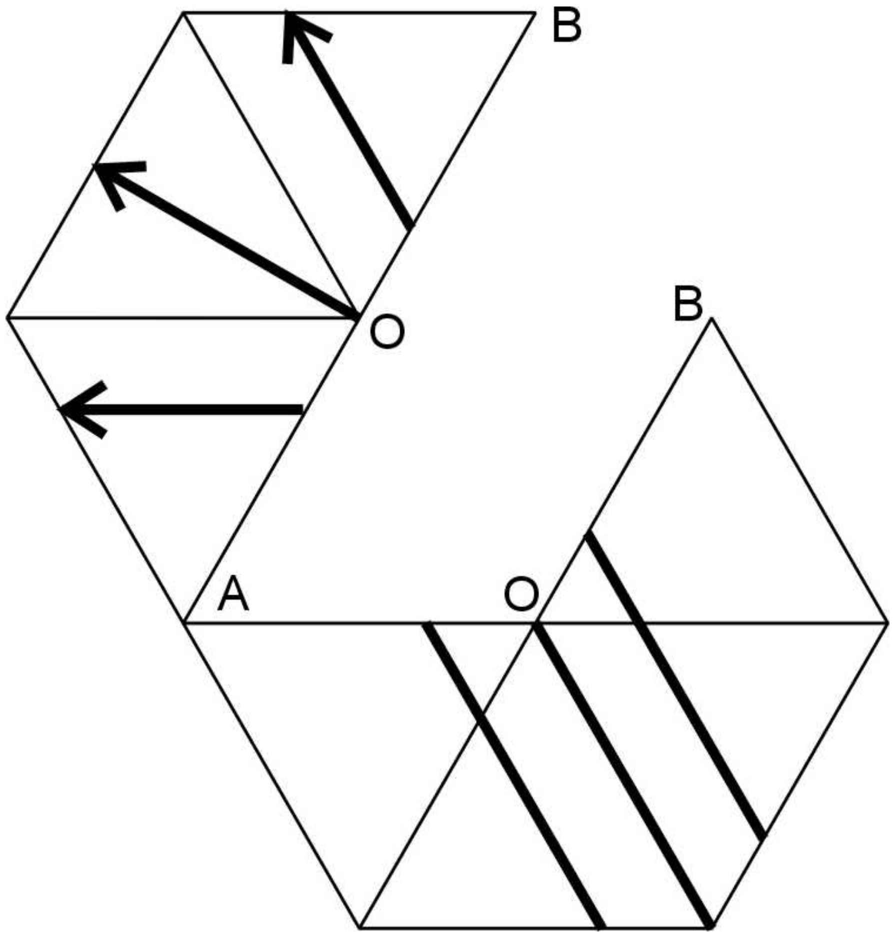
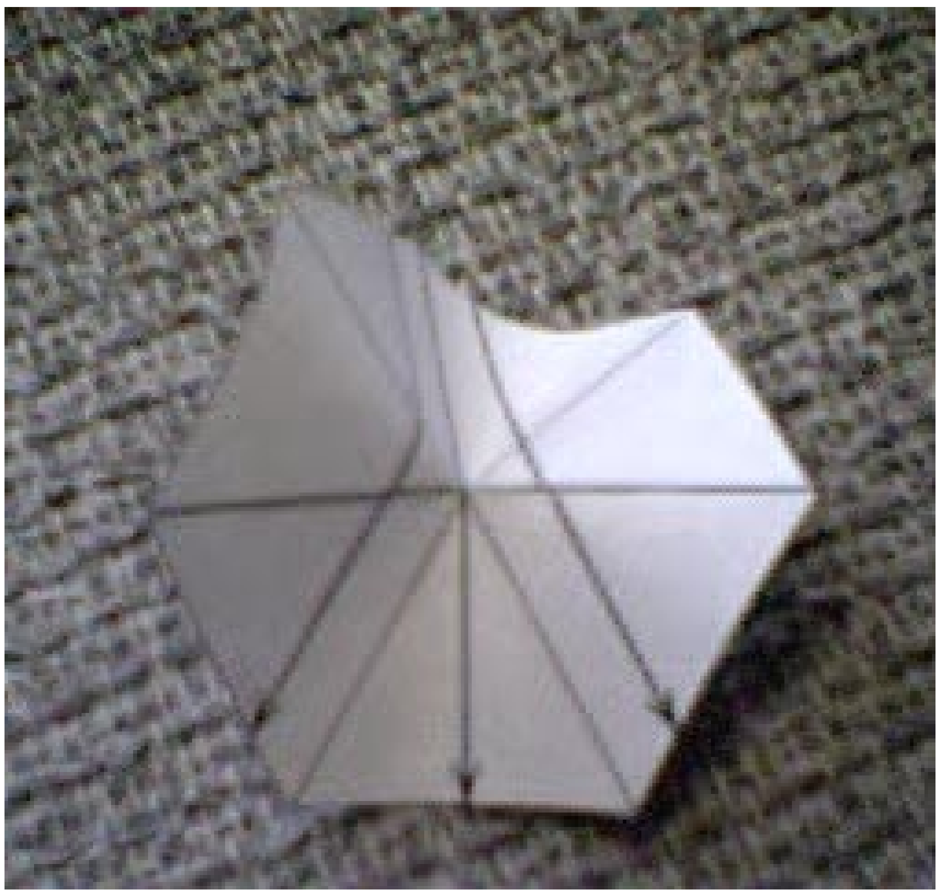


Figure 12. Some s-lines near a hyperbolic vertex.



Figures 13. Paper model corresponding to Figure 12.

Geometry in hyperbolic stars

We can lay a hyperbolic star flat by making a cut, as indicated in Figure 12. Here the segments OA are to be identified, as are the segments OB. The singular s-line shown passing through the vertex makes two 210º angles (or three-and-a-half triangles). In a hyperbolic star, s-lines turn away from each other. This is similar to the behavior of geodesics in a Riemannian manifold with negative curvature. Saddle shaped surfaces have negative curvature, and the paper model shown in Figure 13 exhibits a similar saddle-type shape.

Distance

Since the Euclidean geometry within the triangular disks is preserved, there is a natural notion of distance, and we will take the unit distance as the length of the edges of the triangular disks. This length concept extends to s-lines easily, but we will see that a pair of points may have no s-line joining them, or that the s-line joining them is not unique, so we must be a little careful about defining the distance between two points. A length can be associated naturally with any sequence of line segments joining two points, so the distance between the two points will be defined to be the infimum (i.e. the greatest lower bound) of all such lengths. If there is no such sequence, the distance will be ∞. This will occur if an s-manifold is not connected.

Basic Theorems

Many of the concepts related to Riemannian manifolds can be adapted to s-manifolds. The curvature in a Riemannian manifold, for example, can be replaced by something that we will call an impulse curvature that is concentrated at the vertices. Some of these basic concepts are discussed here.

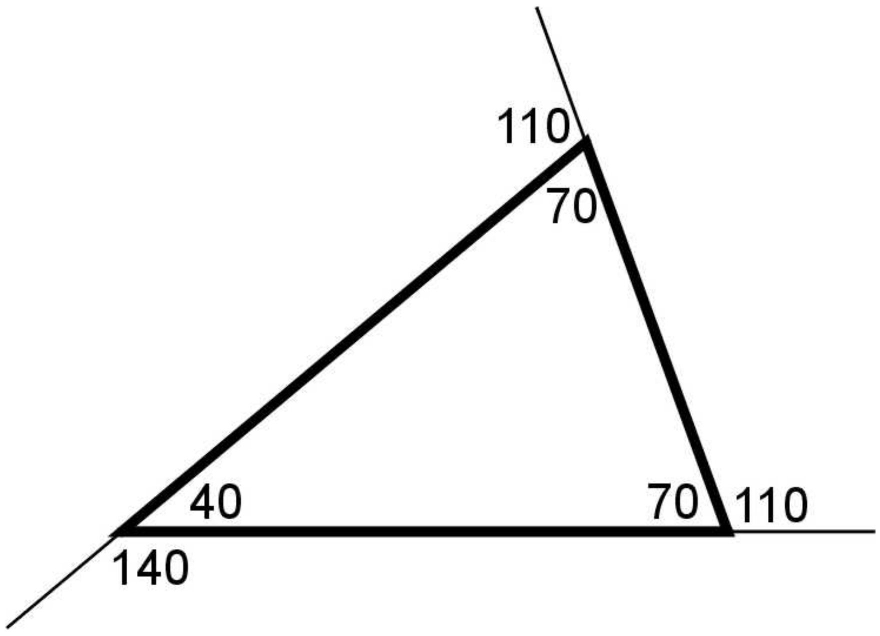


Figure 14. The angle sum of a triangle is 180º, and the sum of turning angles is 360º.

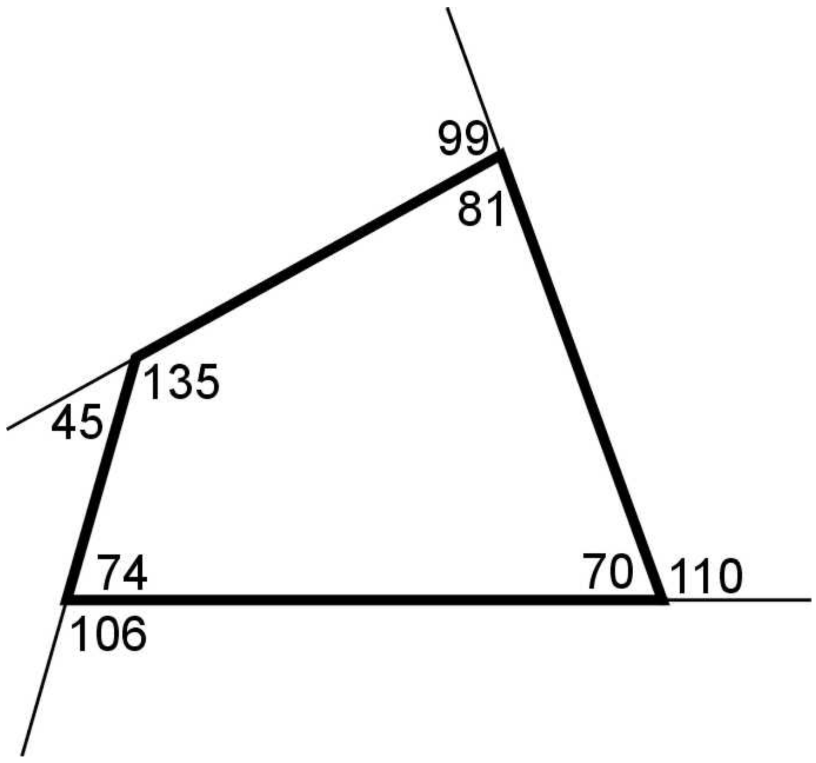


Figure 15. The angle sum of a quadrilateral is 360º, and the sum of the turning angles is 360º.

Impulse curvature on curves

The angle sum of a triangle is an invariant for triangles in Euclidean geometry, as is the angle sum of a quadrilateral. It is not an invariant for polygons in general, however, since the angle sum for polygons with different numbers of sides is different. A closely related quantity, the sum of the turning angles, is an invariant for polygons. In Figures 14 and 15, the turning angles are indicated outside of the triangle and quadrilateral as we traverse them in a counter-clockwise direction. Walking along the perimeter of these polygons, the **turning angle** is the angle required to change from one edge to start the next. The sum of the turning angles is 360º for any polygon in Euclidean geometry.

The turning angle can be interpreted in terms of a curvature singularity. The curvature for a smooth curve is a measure of how quickly the tangent vector changes direction with respect to arclength. Integrating this curvature along an arc, therefore, results in the net change in direction of the tangent vector as an angle measured in radians. For example, the curvature for the arc AC, shown in Figure 16, is $\kappa = 1/r$, since the radius of the circle is r. Integrating the curvature over this arc gives

$$\int \kappa \, ds = \kappa \int ds = \kappa \, \theta \, r = (1/r) \, \theta \, r = \theta,$$

and θ is exactly the angle between the tangent vector at A and the tangent vector at C.

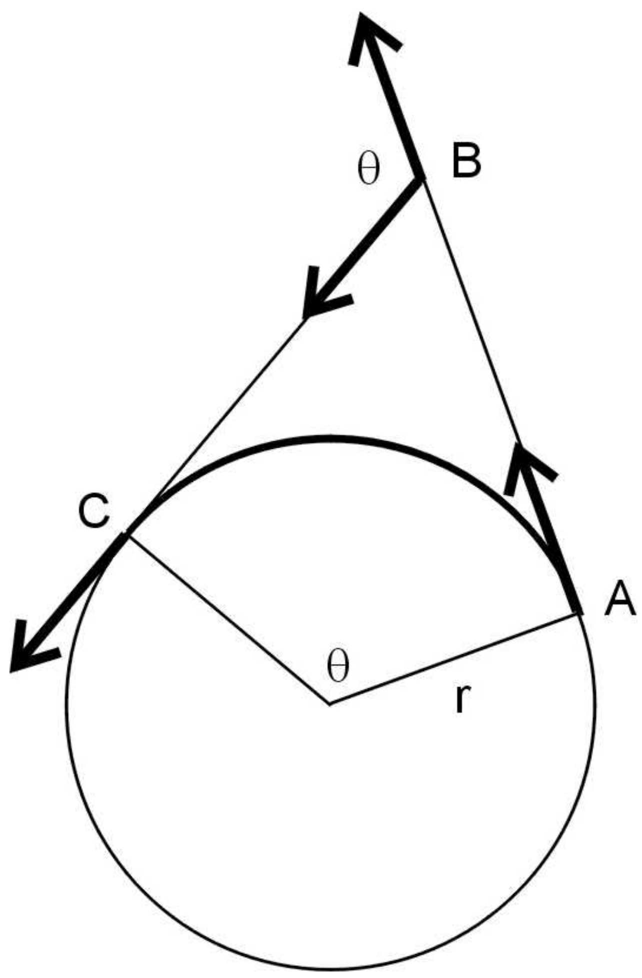


Figure 16. The change in direction of the tangent vector is the same for the arc AC and the path ABC.

The tangent vector on the path formed by the segments AB and BC has the same total change in direction θ, but all of this change occurs at the point B. In this case, we have a curve with zero curvature everywhere except at B, where the curvature, in some sense, is infinite. It would be convenient, however, to think of this “curvature singularity” in the same way as we do with smooth curves. This would require that the integral of the curvature over any part of the path containing B must always be θ, and over any part not containing B, the integral must be zero. These are the properties of an impulse function (see **[5]**), so we will call the measure of the turning angle the **impulse curvature**. Since we can substitute a sharp angle with a closely approximating smooth curve with the same curvature integral, we will assume that results from differential geometry regarding the integral of curvature extend to these impulse curvatures.

The Gauss-Bonnet theorem

There is a wonderful theorem from differential geometry that states that if a closed curve C bounds a region S on a surface, then the Gauss curvature K for the surface is related to the geodesic curvature κ for the curve by the following formula (see **[11, 16, 18]**).

$$2\pi - \int_S K\, dA = \int \kappa\, ds.$$

For example, the Gauss curvature for a sphere of radius r is $K = 1/r^2$. A spherical triangle formed by taking one quarter of the equator and connecting the endpoints of this segment with the north pole, like the spherical triangle ABN in Figure 17, has three right angles. The three turning angles are also right angles, so integrating these impulse curvatures is equivalent to adding them together, and this results in a total impulse curvature of $3\pi/2$. This triangle covers one eighth of the sphere, so integrating the constant K over the interior of this spherical triangle results in $(K)(4\pi r^2)/8 = (1/r^2)(4\pi r^2)/8 = \pi/2$. We have then, $2\pi - \pi/2 = 3\pi/2$.

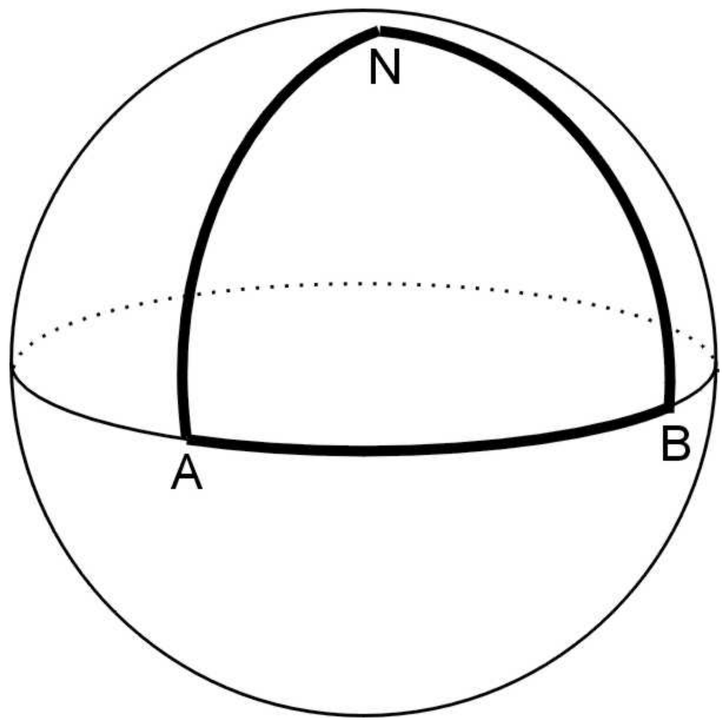


Figure 17. A spherical triangle with three right angles.

For a sphere of radius 1, and a triangle with area A, this generalizes to saying that the sum of the turning angles is $2\pi - A$. Since the sum of the turning angles plus the sum of the angles for the triangle is $3\pi/2$, we have that the angle sum of a spherical triangle is always $\pi/2 + A$. In particular, the angle sum of a spherical triangle is always greater than 180°.

Instead of having a smooth Gauss curvature like the sphere, the curvature on an s-manifold is concentrated at the elliptic and hyperbolic vertices. These curvature singularities can also be interpreted as **impulse Gauss curvatures**. Historically, we could even say that the Gauss curvature is a smooth version of this polyhedral curvature, which was originally developed by Descartes (see **[11, 18]**). Assuming that integrals of Gauss curvature can be extended to these impulse curvatures, we can compute what these should be.

Consider a small polygonal curve ABCDEA bounding a region consisting of five equilateral triangles around the elliptic vertex O, as in Figure 18. The turning angles at A, B, C, D, and E are all 60°, so the sum of the turning angles is 300°. For a larger polygonal path, such as the one through F, G H, I, and J in Figure 18, the contained region can be subdivided into the pentagon ABCDE and the region bounded by the polygonal curve FGHIJLKEDCBAKLF. This second region is Euclidean in its interior, so the sum of the turning angles is 360°. Taking angles measured in a counter-clockwise direction as being positive, the turning angles at E, D, C, B, and A are negative, but equal in magnitude to the corresponding turning angles on the pentagon. The two angles at L are positive and add up to 180°, as do the two angles at K. We have then

$360°$ = (turning angle sum of FGHIJF) + $(\angle L_1 + \angle L_2 + \angle K_1 + \angle K_2)$ + $(\angle A + \angle B + \angle C + \angle D + \angle E)$ = (turning angle sum of FGHIJF) + $360° - 300°$.

Therefore, (turning angle sum of FGHIJF) = 300°. A similar argument reveals that the turning angles will sum to 420°, if the polygonal path contains a hyperbolic vertex. This further extends to the following.

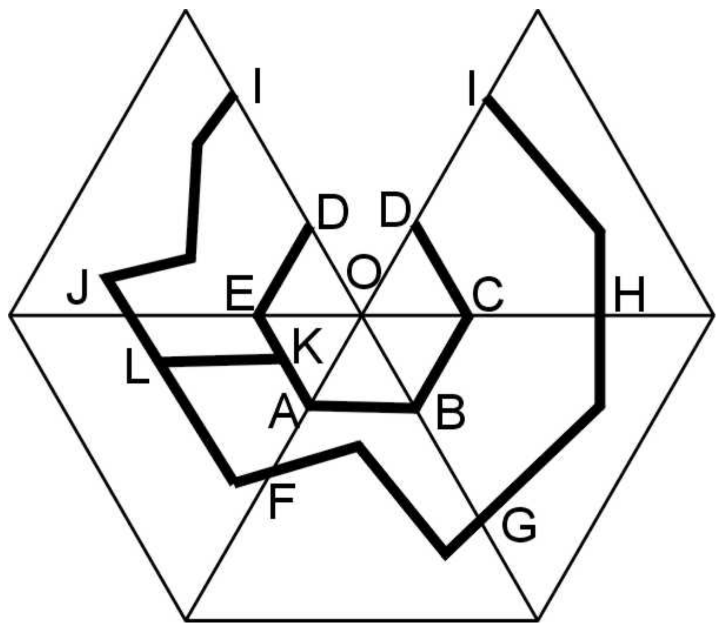


Figure 18. A curve around an elliptic vertex.

Theorem (s-manifold Gauss-Bonnet theorem). For any non-singular polygon (i.e., non-Euclidean vertices do not lie on the perimeter) in an s-manifold bounding a region that is simply connected and containing a total of h hyperbolic vertices and e elliptic vertices, the sum of the turning angles is 360º + 60º (h – e).

Angle sums of polygons can easily be computed from this theorem. For example, consider a regular pentagon with an elliptic vertex in its interior. We have a turning angle sum of 360º + 60º (0 – 1) = 300º. If x is the measure of each of the angles of this regular pentagon, then 5(180º - x) = 300º, and x = 120º. This compares to 108º for the angles of a regular Euclidean pentagon.

Relative angles

It will be convenient for us to talk, in a local sense, about s-lines being parallel or not parallel at different points along them, since this relationship between s-lines changes as we move from point to point along the s-line. In Figure 19, in the direction from right to left (from point B to point C), we will say that **the angle of the line *b*, at the point B, relative to the line *a* is ∠3**. At point C, the relative angle is ∠4. The relative angle is not always well-defined, but there should be little confusion in the contexts in which it will be used. If the relative angle is 90º at some point P of *b*, we will say that ***b* is parallel to *a* at P**. This is a term that we will use for convenience and does not imply that the s-lines are parallel.

For quadrilateral ABCD in Figure 19, we know that the sum of the turning angles depends on the number of elliptic and hyperbolic vertices inside of it. If there is one elliptic vertex, then the sum of the turning angles is 360º – 60º = 300º. In this case, ∠2 + ∠4 = 120º, and ∠2 + ∠3 = 180º, so ∠4 = ∠3 – 60º. A similar computation yields the fact that if there is a hyperbolic vertex in the interior of the quadrilateral, then ∠4 = ∠3 + 60º.

Fundamental principle. When an elliptic vertex lies between the two s-lines *a* and *b*, the angle of *b* relative to *a* decreases by 60º. When there is a hyperbolic vertex between the s-lines, the relative angle increases by 60º.

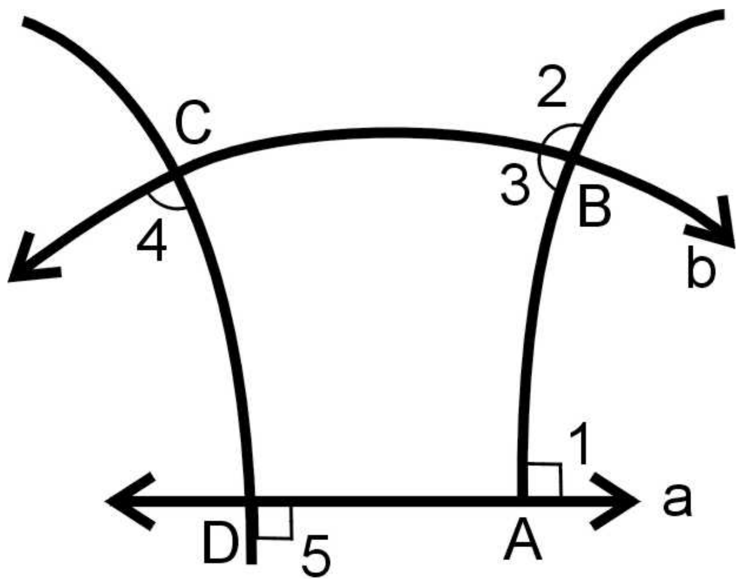


Figure 19. The angles ∠3 and ∠4 are the angles of the s-line *b* relative to s-line *a* at points B and C.

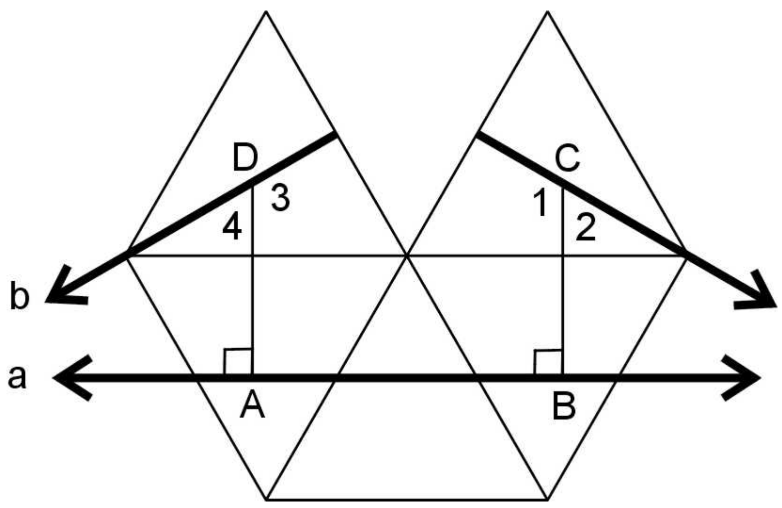


Figure 20. Relative angles around an elliptic vertex decrease by 60º.

Relative angles around elliptic and hyperbolic vertices

Around an elliptic vertex, the relative angle decreases by 60º. In Figure 20, from right to left, ∠4 is 60º less than ∠1. From left to right, ∠2 is 60º less than ∠3. The relative angle decreases by 60º in either direction.

Around a hyperbolic vertex, the relative angle increases by 60º in either direction. In Figure 21, from right to left, ∠2 is 60º greater than ∠1, and from left to right, ∠4 is 60º greater than ∠3.

The effects are additive. In Figure 22, we see the relative angle decreasing by 60º twice for a total of 120º. As drawn, the relative angle ∠1 is 150º, relative angle ∠2 is 90º, and relative angle ∠3 is 30º. In the other direction, the relative angles ∠4, ∠5, and ∠6 change in the same way.

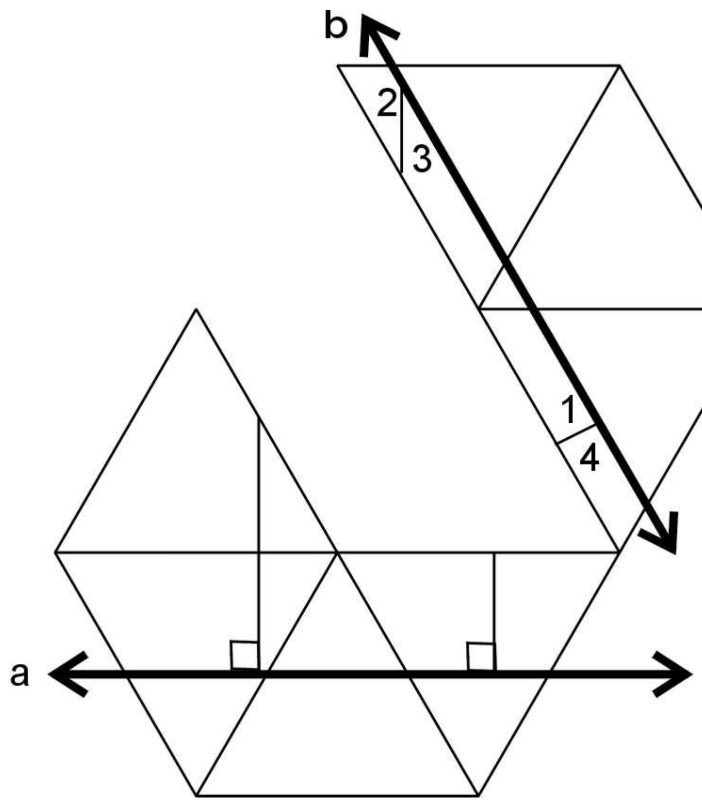


Figure 21. Relative angles around a hyperbolic vertex increase by 60º.

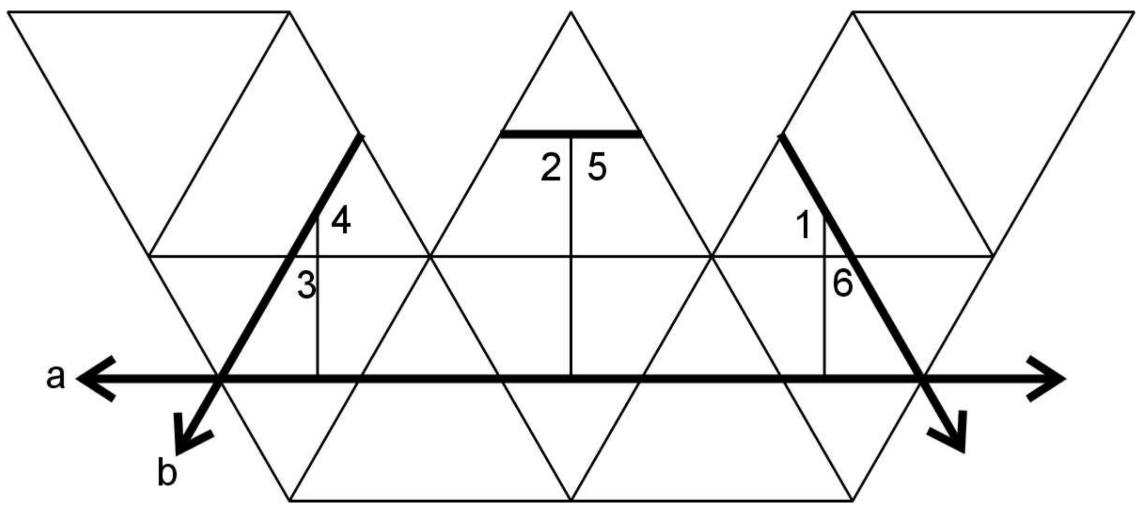


Figure 22. The relative angle after passing two elliptic vertices decreases by 120º.

In Figure 23, the relative angle increases by 60º and decreases by 60º after passing a hyperbolic vertex and an elliptic vertex. As drawn, the relative angle ∠1 is 90º, the relative angle ∠2 is 150º, and the relative angle ∠3 is back to 90º. The change in relative angle from an elliptic and a hyperbolic vertex cancel out.

Lambert and Saccheri quadrilaterals

J. H. Lambert and G. Saccheri were prominent figures in the study of the parallel postulate. Special quadrilaterals, which were named after them, are natural objects to consider in this context. A **Saccheri quadrilateral** is a quadrilateral whose base angles are right angles and whose base adjacent sides are congruent. Clearly, in Euclidean geometry, a Saccheri quadrilateral must be a rectangle. In hyperbolic and elliptic geometry, a Saccheri quadrilateral is not a rectangle, but the summit angles must be congruent. In Figure 20, quadrilateral ABCD is a Saccheri quadrilateral. The upper angles are both 120º. In Figure 23, quadrilateral EGHJ has four right angles, but it is not a Saccheri quadrilateral, since the two base-adjacent sides are not congruent. This raises the following question.

Question. In an s-manifold, must the summit angles of a Saccheri quadrilateral be congruent?

A **Lambert quadrilateral** has three right angles. Here also, a Lambert quadrilateral in Euclidean geometry must be a rectangle. In Figure 23, the quadrilaterals EFIJ and EGHJ are Lambert quadrilaterals. The fourth angle of Lambert quadrilateral EFIJ is 150º. Lambert quadrilateral EGHJ actually has four right angles, but it is not a rectangle, since side GH is shorter than side EJ. On the other hand, an argument could be made that Lambert quadrilateral EGHJ is a rectangle, since opposite sides are parallel.

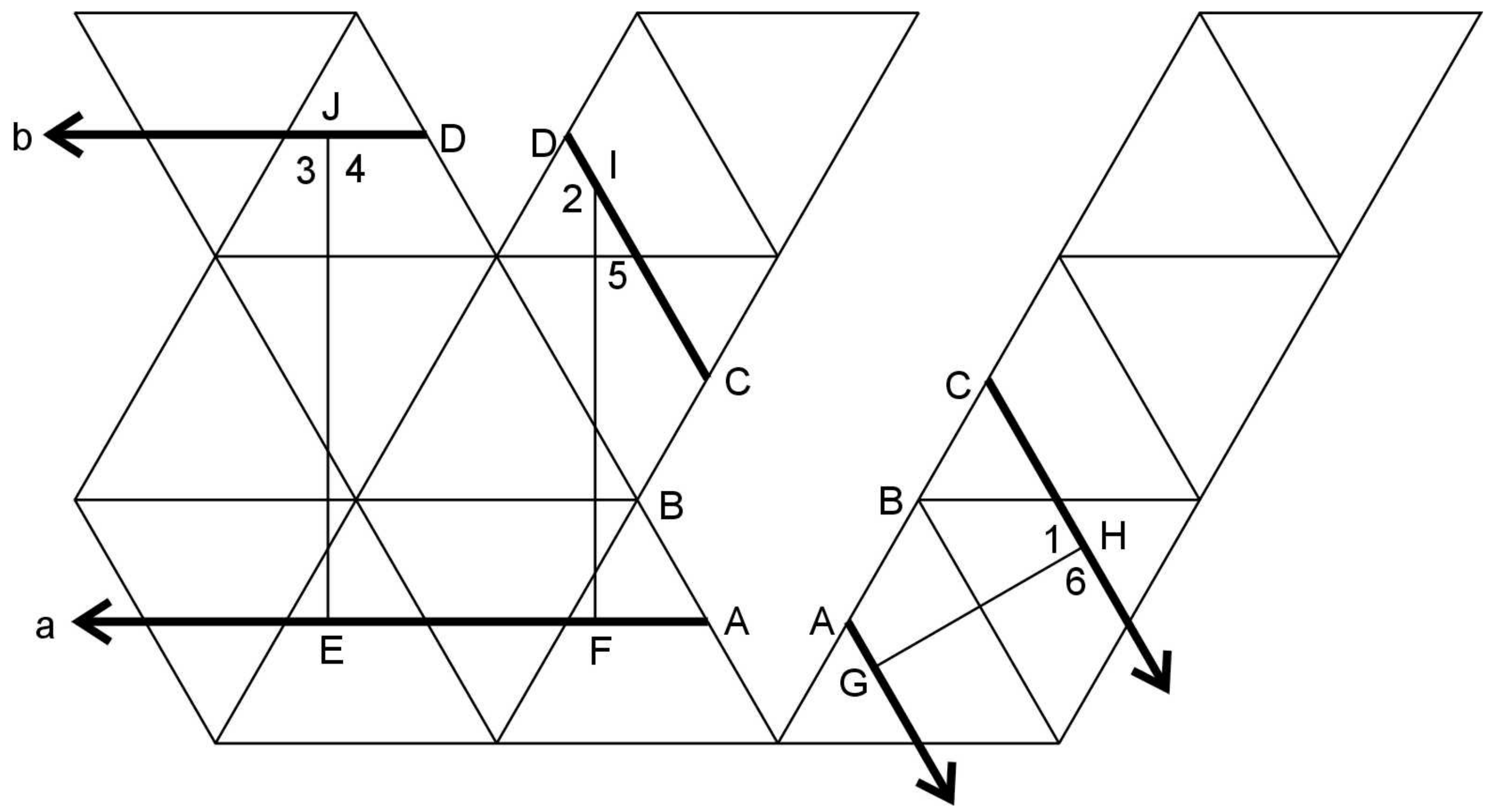


Figure 23. The relative angle after passing an elliptic and a hyperbolic vertex is the same.

Other Objects in an s-Manifold

Segments

Our s-lines correspond naturally to the real line, since they extend indefinitely and are continuous. Given a point P on an s-line *l* and a direction along *l*, we can define a mapping from **R** to *l* that satisfies the following conditions. The origin maps to P. For each x > 0, we can look a distance x along *l* in the given direction to find a point Q, and x maps to this point Q. The image for each x < 0 is found by traveling in the other direction. In terms of this **distance function**, the real numbers cover the s-line. In the case of a closed s-line, **R** covers the s-line an infinite number of times. We could, if we wanted, define an **s-segment** to be any part of an s-line that corresponds to a closed interval on the real line. If we were to choose this definition, a closed s-line, like a great circle on the sphere, could be covered by an s-segment more than once. This would be interesting from a Smarandache point of view, but we will use a more conservative concept by adding the requirement that an s-segment will never completely cover an s-line (self-intersections are OK). We should expect that this definition will provide us with s-segments that have properties we might not expect, but we should never have doubts that any of these should be called an s-segment.

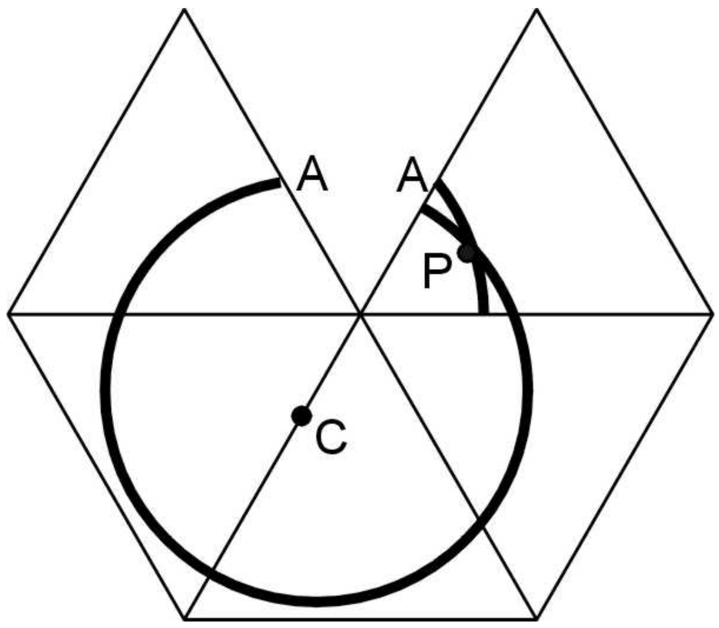


Figure 24. An s-proto-circle around an elliptic vertex.

Circles

We will do little with circles here, beyond considering whether they exist or not. In order to do that, we will need to define what will qualify as a circle. It is fairly standard to define a circle as the set of points a fixed distance from a given point. We could do this, but then existence would not be an issue, and we might have some odd things being called circles. Using Euclid as a standard, we will define circles as follows. An **s-proto-circle with center C and radius r** is the set of points P that have an s-segment CP with length r. Euclid defined a circle to be, “a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying with the figure are equal to one another; and the point is called the centre of the circle” **[9]**. With this in mind, we will say that an s-proto-circle is an **s-circle**, if it is a simple closed curve. Given a point C and a radius r, if the s-proto-circle is an s-circle, we will say that **the s-circle exists**.

In Figure 24, we have an s-proto-circle around an elliptic vertex. Since there are three ways for an s-line through C to get into the upper-right triangular disk, there are three parts to the s-proto-circle in the region. For those s-segments (s-radii) that pass to the right of the elliptic vertex, their endpoints lie on the continuation of the s-proto-circle from below. For those s-segments that pass to the left of the elliptic vertex, their endpoints lie on the continuation of the s-proto-circle through the point A. The point P is the endpoint of the s-segment that passes through the elliptic vertex, and it lies just inside of the two arcs. Since this s-proto-circle is not a simple closed curve, the s-circle with center C and this radius does not exist.

Around a hyperbolic vertex, s-proto-circles will have an open region instead of overlapping (an example is shown in the non-geometry section of Chapter 3), and away from non-Euclidean vertices, s-proto-circles will look like Euclidean circles in the Euclidean plane. Clearly then, s-circles exist away from the non-Euclidean vertices, and medium sized s-circles do not exist near non-Euclidean vertices. If the center of an s-circle lies on a vertex, then the s-circle also exists (although its circumference may be a bit larger or smaller than $2\pi r$).

Question. Are there s-circles other than these? In particular, are there s-circles that contain non-Euclidean vertices? Also, how wild can an s-proto-circle be? For example, can an s-proto-circle be shaped like a figure-8?

Angles

When two lines meet, there will always be at least a short (and straight) line segment corresponding to each side, so no great leap is needed to define what an angle is. The only thing that is unusual is that angles around an elliptic vertex or hyperbolic vertex will sum to 300º or 420º. It should be clear why this is the case.

Parallel lines

In the study of manifolds, the notion of parallel lines is of secondary interest. It is of primary interest in the study of synthetic (axiomatic) geometry. Here, the concept of parallel lines defines the differences between Euclidean, elliptic, and hyperbolic geometry, and whether two lines intersect, or not, determines if they are parallel (at least in two dimensions, which is where our interests lie). On a manifold, it is curvature that differentiates Euclidean, elliptic, and hyperbolic geometry, and the important phenomena, the local ones, are determined by curvature, and whether lines intersect, or not, somewhere else in the space has little bearing. So while it is normal in a differentiable, or a Riemannian, manifold for the curvature, and therefore, the geometry, to change from region to region, this does not necessarily carry over to the relationships between lines, or geodesics. This is, in fact, one of the major issues that is before us here. We will take the synthetic definition of parallel, that is, two s-lines are **parallel**, if they do not intersect. We will look to see how this definition plays out in the world of manifolds.

Chapter 2. Hilbert's Axioms

In a Smarandache geometry, we want to look at how the Euclidean or non-Euclidean structure changes from place to place. Since Hilbert's axioms cover Euclidean geometry at an axiomatic level, this seems to be a reasonable place to start. Several of Hilbert's axioms will hold in any s-manifold, but most will be S-deniable in an s-manifold. The continuity of s-manifolds will make it impossible to S-deny all of Hilbert's axioms, and our choice to have congruence to be an equivalence relation further reduces our ability to S-deny Hilbert's axioms. Most of Hilbert's axioms will be S-deniable, however, and we will see a wide variety of geometric structures in intuitively manageable spaces.

Hilbert separated his set of axioms into groups, and we will break them up the same way.

Incidence

Hilbert's axioms of incidence for plane geometry are as follows **[14]**.

I-1. For every two points A and B, there exists a line *a* that contains each of the points A and B.

I-2. For every two points A and B, there exists no more than one line that contains each of the points A and B.

I-3. There are at least two points on a line. There are at least three points not on a line.

In Euclidean geometry, and in the elliptic geometry of Riemann and the hyperbolic geometry of Bolyai, Lobachevski, and Gauss, there is exactly one line through a pair of points, as is required in axioms I-1 and I-2. This is not quite the case in spherical geometry. Pairs of antipodal points, the north and south poles, for example, have an infinite number of lines (great circles) through them. There is, however, a unique line through any pair of non-antipodal points. On the sphere, therefore, the axiom requiring that pairs of points determine a unique line is S-denied (it is true for some pairs and false for others), and the sphere is a Smarandache geometry with respect to this axiom.

In an s-manifold, there are a number of ways in which a pair of points does not determine a unique s-line. We will say that if a pair of points has exactly n s-lines passing through them, then they are ***n*-aligned**. In Euclidean geometry, all pairs of points are 1-aligned. We will say that pairs of points with infinitely many s-lines through them, like antipodal points on the sphere, are ∞-aligned. Occasionally, we will also call pairs of points that are 0-aligned **remote**, n-aligned points with $n \geq 2$ **multiply aligned**, and pairs of 1-aligned points **uniquely aligned**.

Since s-lines in an s-manifold are extensions of line segments, every s-line will have at least two points. There will also be three points not on any given s-line. The only conceivable exception would be an s-line that completely covered an s-manifold. In this case, all the points of the s-manifold would be on the s-line, and none off of it. This cannot happen, however. If we had such an s-line, then each triangular disk would be covered by a countable number of segments. This could not happen for an s-line that followed an edge of a triangular disk, since such an s-line would only run along the edges of or bisect any particular disk. Therefore, each segment in any particular disk must intersect the boundary of the triangular disk at most twice. The s-line, therefore, could only cover a countable number of points on the perimeter of the disk. It follows that this s-line could not cover the entire s-manifold.

We see then that Hilbert's third incidence axiom will hold for every s-manifold, and cannot be S-denied.

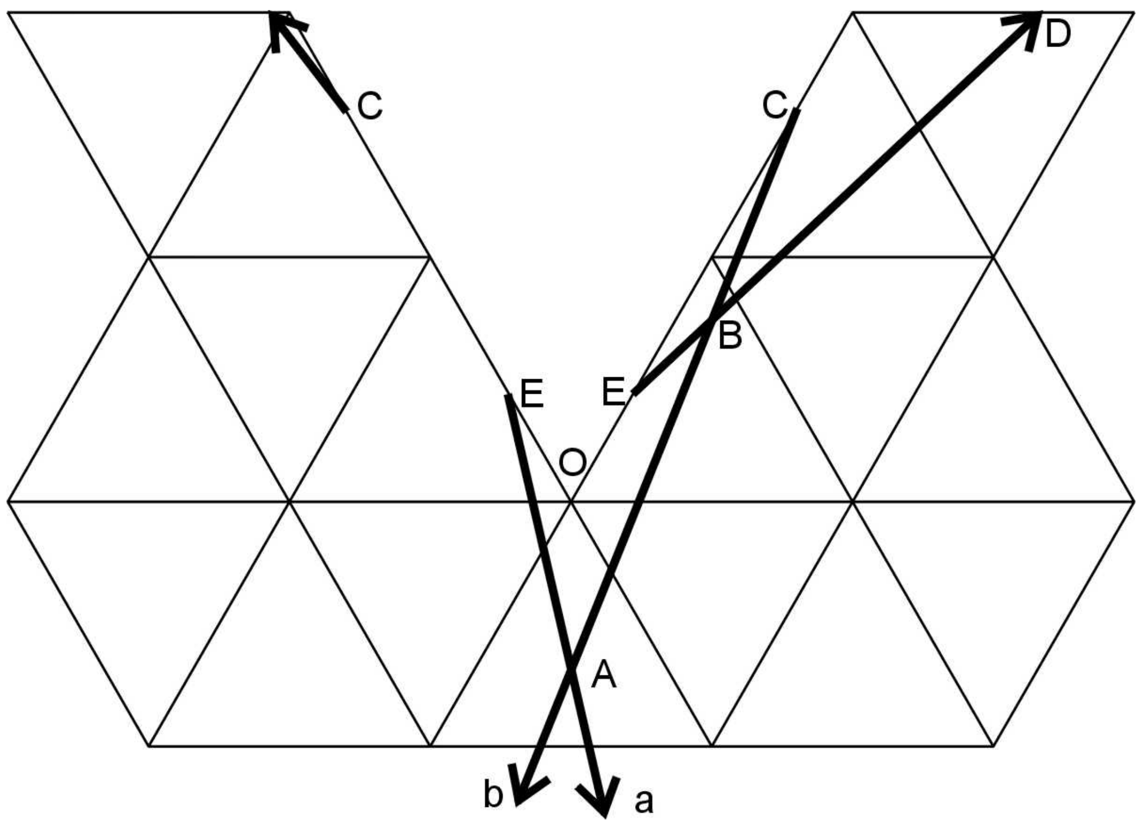


Figure 1. The elliptic cone-space.

Incidence around an elliptic vertex

The s-manifold shown in Figure 1 has a single elliptic vertex O. Here, the edges containing the points O, E, and C are identified, and the space extends to infinity with only Euclidean vertices. We will call this the **elliptic cone-space**, since a paper model of this s-manifold is cone-shaped (see the posts of Joan Duncan and Ken Prasad at **[2]**). The s-lines *a* and *b* illustrate that the points A and B are at least 2-aligned. Further consideration makes it clear that the s-line *b* is the only s-line that passes to the right of the point O and through A and B, and that the s-line *a* is the only s-line that passes to the left of O. Therefore, the points A and B are, in fact, 2-aligned. There are three s-lines passing through the pair of points A and E, one to the left of O, one to the right, and the singular s-line through O. The points A and E are therefore 3-aligned. The points A and O satisfy a third category of alignment. They are uniquely aligned. Hilbert's axiom I-2, therefore, is S-denied, and the elliptic cone-space is a Smarandache geometry relative to this axiom.

Two-sided polygons

If two points are multiply aligned there is a 2-sided figure with these points as vertices. These are called **2-gons**. It is interesting to note that 2-gons in the elliptic cone-space have an angle sum that is almost always 60º. If the two sides of a 2-gon do not pass through the elliptic vertex (a singularity), we will call it a non-singular 2-gon. We then have the following elliptic cone-space theorem.

Theorem. The angle sum of a non-singular 2-gon in the elliptic cone-space is 60º. (Singular 2-gons have an angle sum of 30º.)

Consider the 2-gon AB in Figures 1 and 2. The triangles AOC and AOE are Euclidean triangles (singular 2-gons), and the angles $\angle$AOC and $\angle$AOE both measure 150º. Therefore, $\angle$OAE + $\angle$OEA = 30º and $\angle$OAC + $\angle$OCA = 30º. The angles $\angle$OEA and $\angle$BEC are equal, since they are vertical angles. Since the triangle BCE is a Euclidean triangle, it has an angle sum of 180º, and $\angle$BCE + $\angle$BEC = $\angle$EBA (an exterior angle). The angle sum of the 2-gon AB is $\angle$EAB + $\angle$EBA = ($\angle$OAE + $\angle$OAB) + ($\angle$BCE + $\angle$BEC) = ($\angle$OAE + $\angle$BEC) + ($\angle$OAB + $\angle$BCE) = ($\angle$OAE + $\angle$OEA) + ($\angle$OAC + $\angle$OCA) = 30º + 30º = 60º. This is not completely general, but a trivial general proof can be obtained from the s-manifold Gauss-Bonnet theorem.

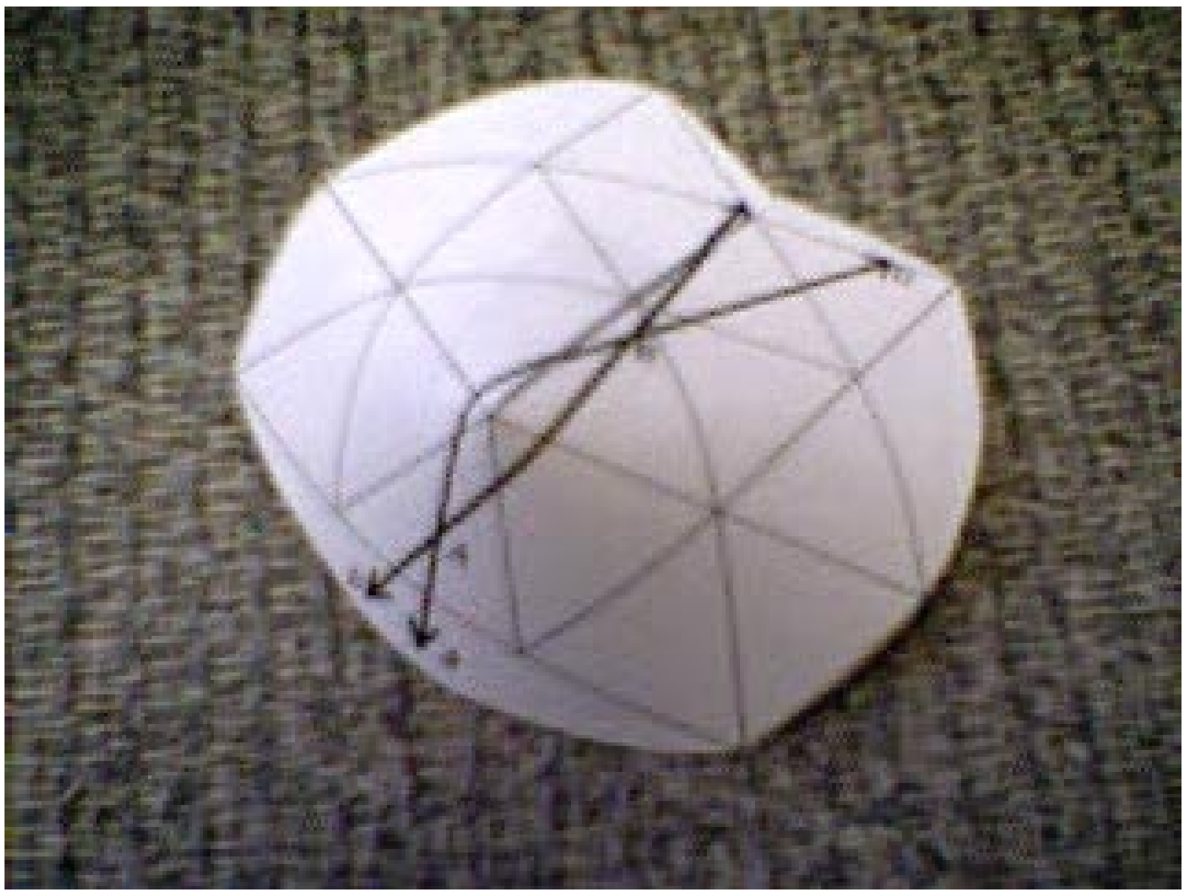


Figure 2. Paper model corresponding to Figure 1.

This result can be extended easily to other polygons. For example, a triangle with the elliptic vertex O in its interior will have an angle sum of 240º. If one of the sides of the triangles passes through O, then the angle sum is 210º. Otherwise, the angle sum is the Euclidean 180º.

Alignment regions

With respect to the point A in Figure 3, all of the points in the interior of the shaded region are 2-aligned, except for the points on the ray OC (which are in the interior), which are 3-aligned (not including O) (see the posts of Mike Antholy **[2]**). The s-lines just missing the elliptic vertex O, like the line *a* in Figure 3, will approach the lower boundary of the shaded region, but the singular line through O will continue through C. The region including the boundaries and below consists entirely of points that are uniquely aligned with A.

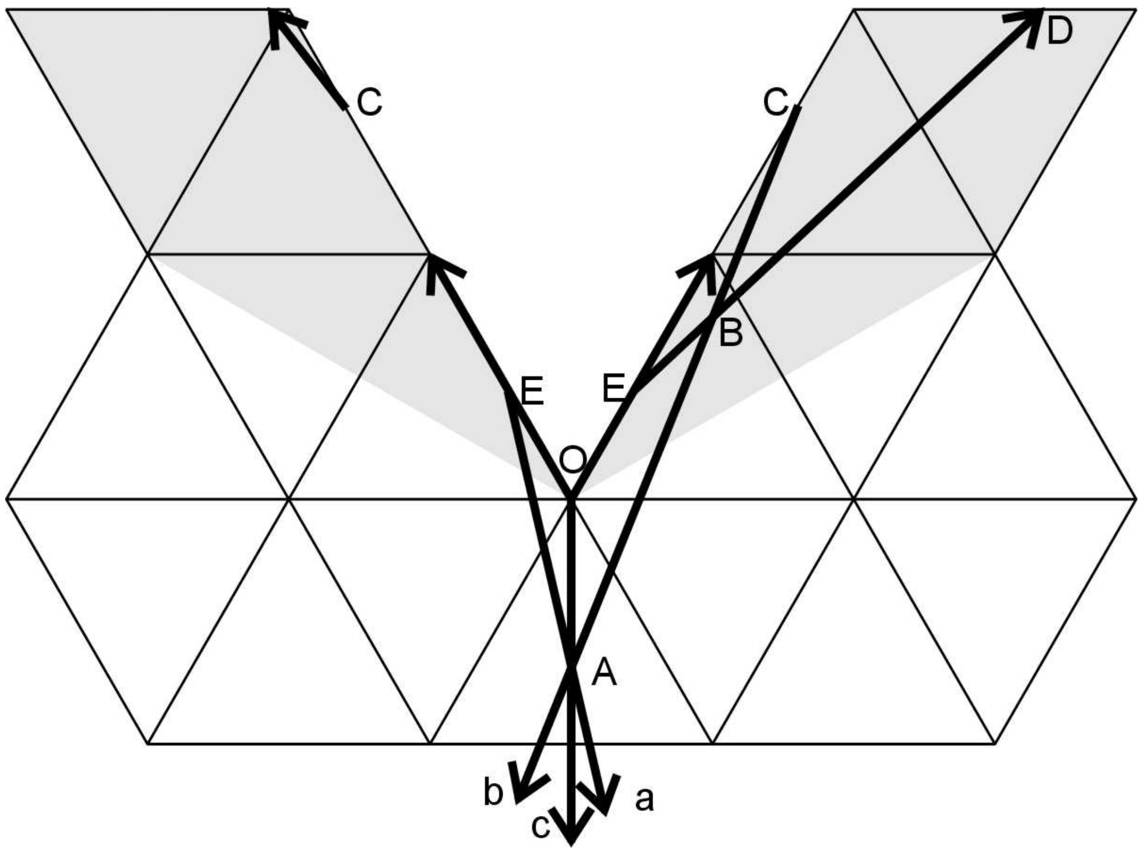


Figure 3. Regions that are 1-, 2-, and 3-aligned with A.

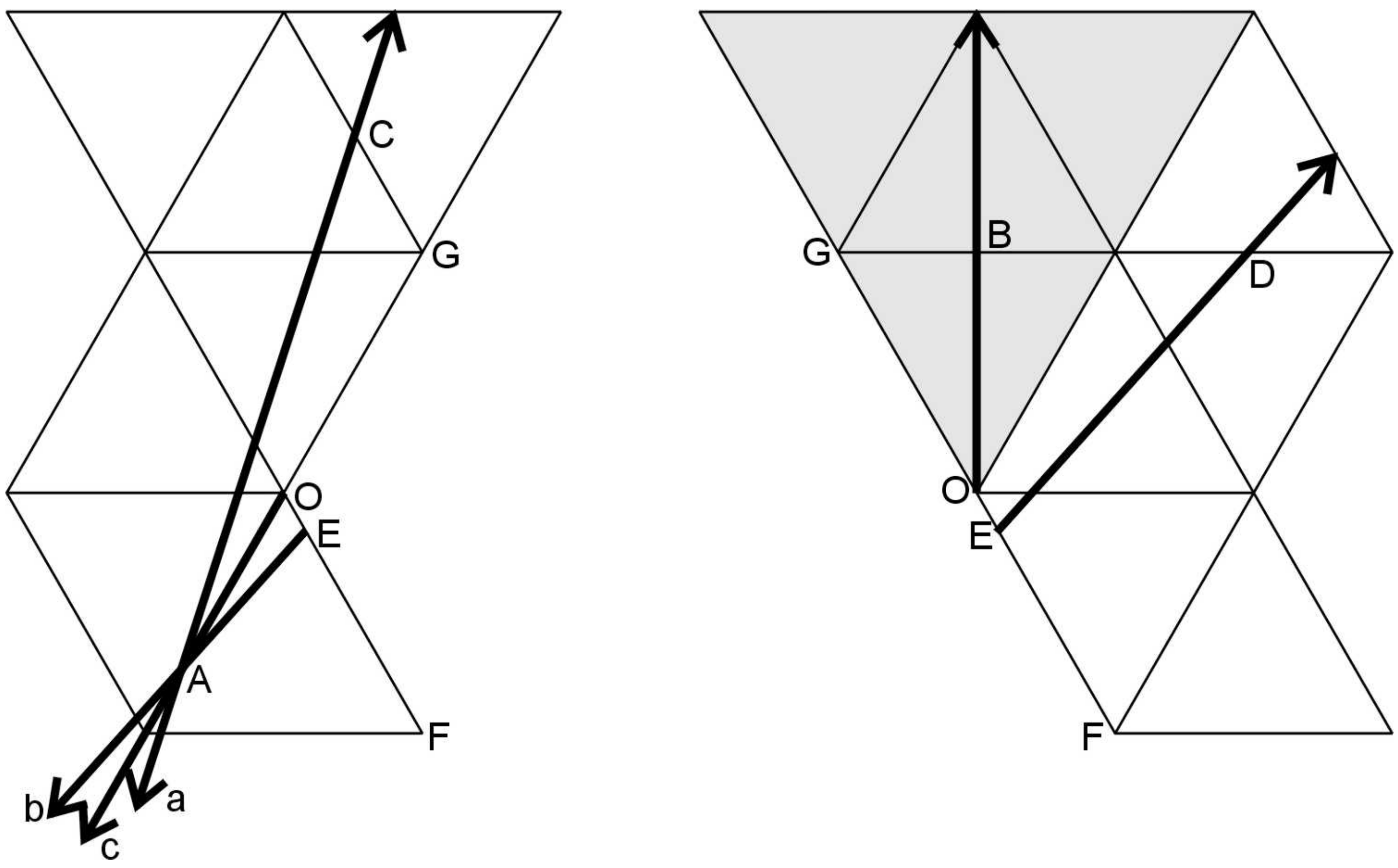


Figure 4. The hyperbolic cone-space.

Alignment around a hyperbolic vertex

The **hyperbolic cone-space** is an s-manifold that has a single hyperbolic vertex, as indicated in Figure 4. A paper model of this cone-space would not be cone shaped in the usual sense, but it is common to call the object formed by joining all of the points on a curve to a single point with line segments a cone. In Figure 4, it appears that the left and right regions will overlap if the picture is extended upwards, but these regions are meant to be disjoint except for the boundaries, which are identified. In particular, the s-lines *a* and *c* only intersect at A.

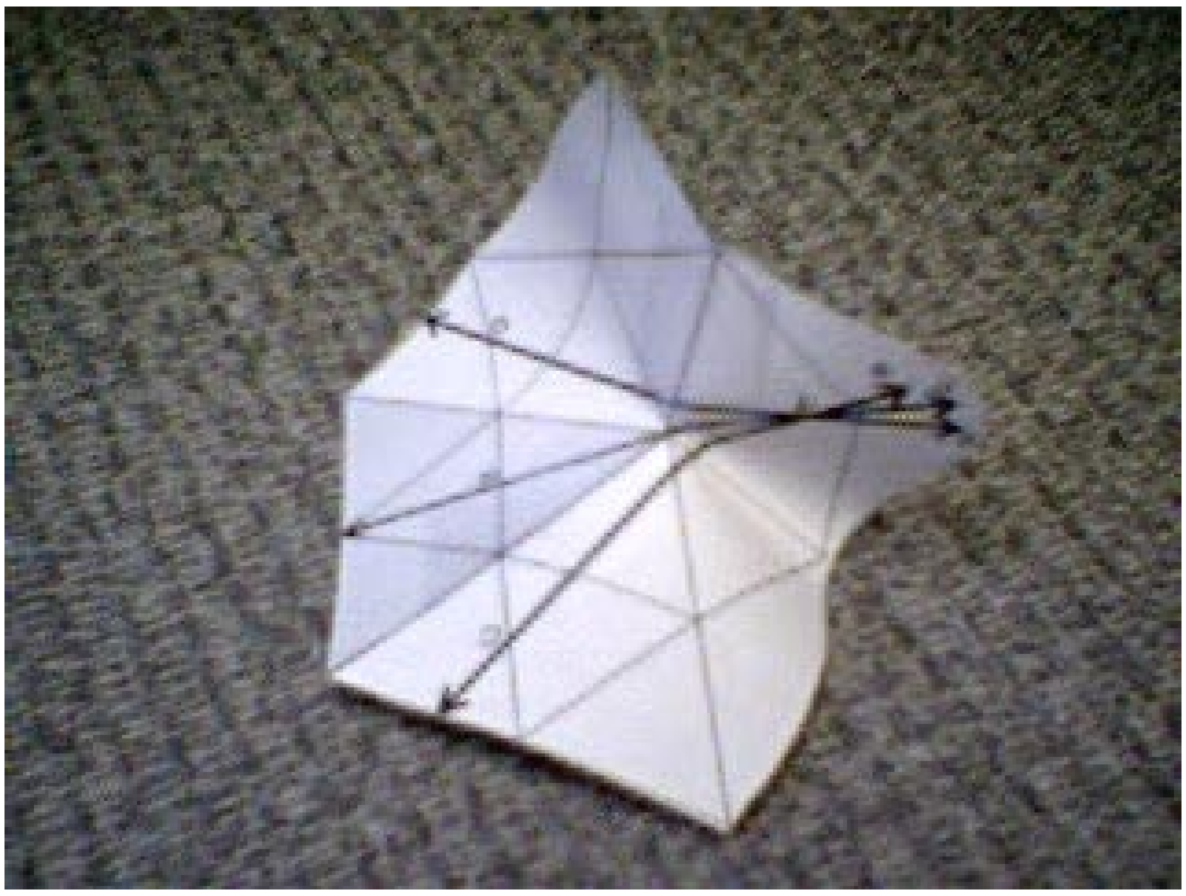


Figure 5. Paper model corresponding to Figure 4.

Alignment regions

The only s-line through A that enters the shaded region of Figure 4 is the singular s-line *c*. Therefore, all of the points in this region, except for the points on the ray OB, are 0-aligned with A. Except for the vertex O, the boundaries are included in the 0-aligned region. All other points are uniquely aligned, so Hilbert's axiom I-1 requiring that any pair of points determine at most one line is Smarandachely denied, and the hyperbolic cone-space is a Smarandache geometry relative to this axiom.

Higher Order Alignment

It is certainly possible in an s-manifold to get higher orders of alignment. For example, in Figure 6, we have a pair of points that are 5-aligned. From P, there is an s-line that goes directly to Q, one that goes around the elliptic vertex B, one that passes through both elliptic vertices, one that goes around the elliptic vertex A, and one that goes around both elliptic vertices. Using more non-Euclidean vertices opens the possibility of even more s-lines through a pair of points.

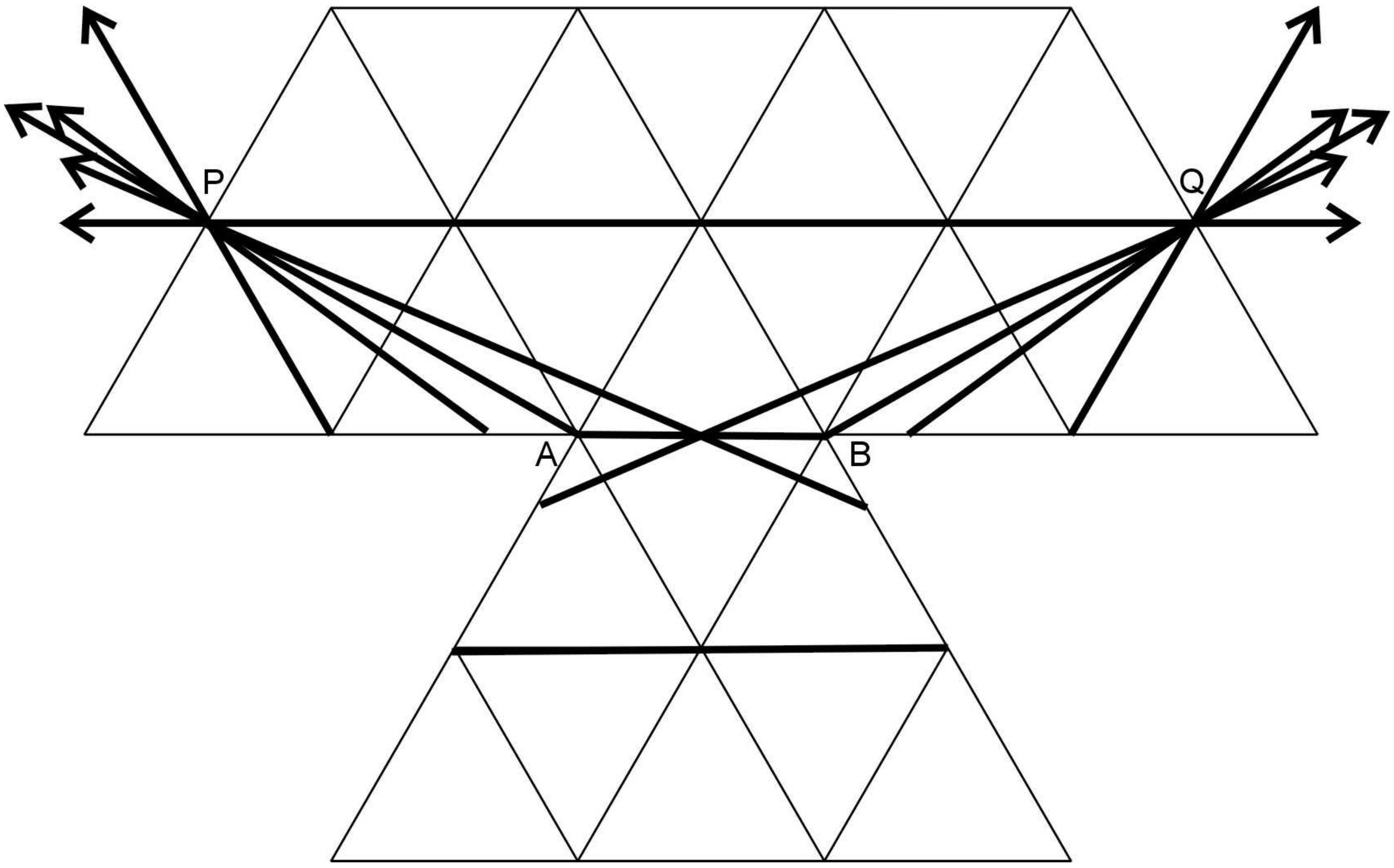


Figure 6. The points P and Q are 5-aligned.

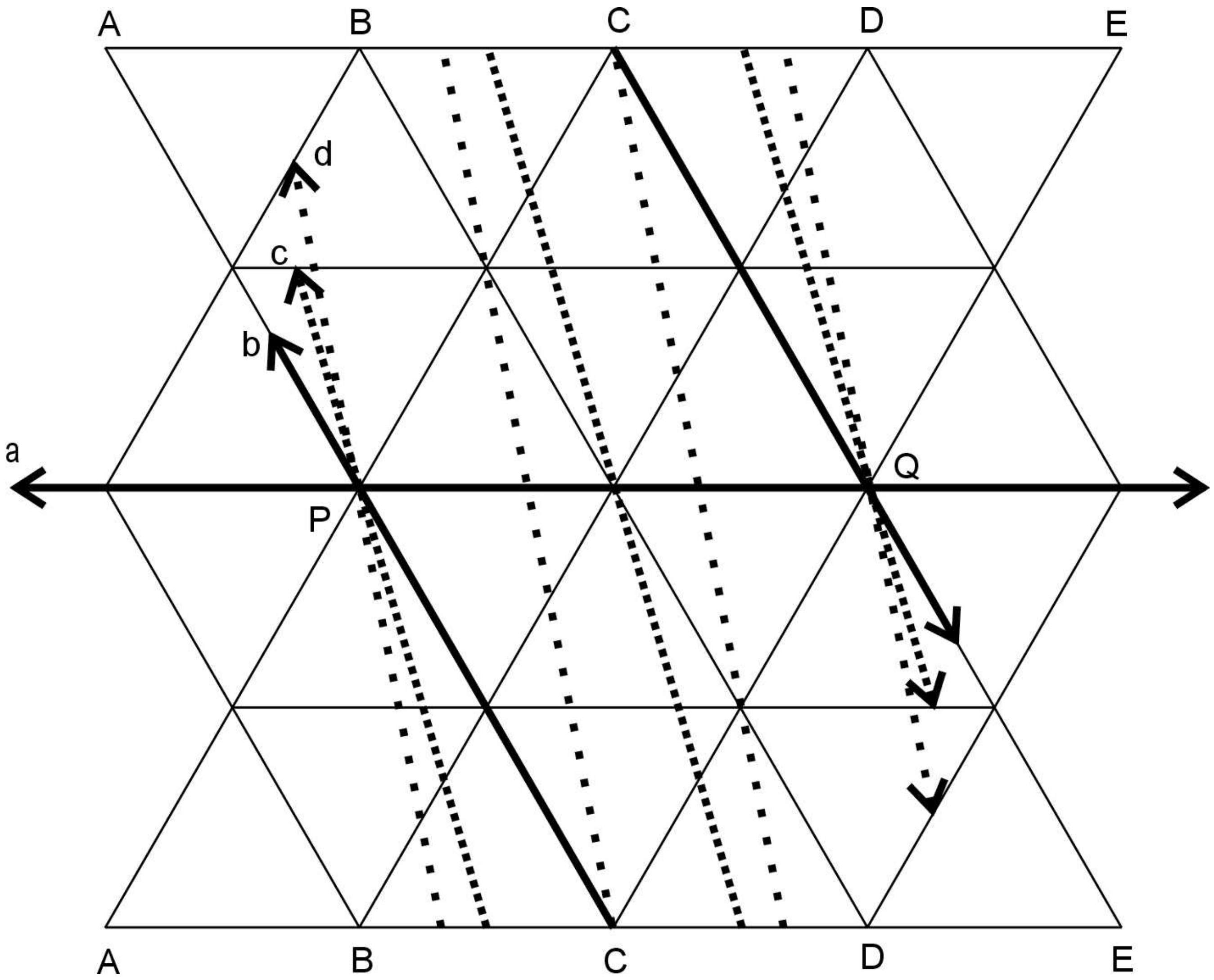


Figure 7. The points P and Q are ∞-aligned.

We can also get multiple alignments as a result of topology. In Figure 7, we have a cylindrical s-manifold. The points on the top and bottom are identified, and the figure extends indefinitely to the right and left. The s-line *a* runs along the cylinder and passes through both P and Q. The s-line *b* wraps around the cylinder once, while s-lines *c* and *d* wrap around twice and three times. Instead of passing through P in a downward direction, there are also s-lines passing through upwards that wrap around the cylinder one, two, and three times. In fact, for every positive integer *n*, there are two s-lines that pass through both P and Q and wrap around *n* times. The points P and Q are therefore ∞-aligned. All pairs of points on the cylinder are ∞-aligned, except for points that are lined up vertically. These are uniquely aligned.

Figure 8. Paper model corresponding to Figure 7.

Angle sums of triangles

There are no 2-gons in the hyperbolic cone-space, but we can determine the angle sum of a triangle in this space (see the posts beginning with those of Dacosta Teresinha and Joan Duncan **[2]**). A triangle will have an angle sum of 120º, 150º, or 180º depending on whether the hyperbolic vertex O is inside the triangle, on the interior of one of its sides, or otherwise. In Figure 9, triangle ABC has angle sum 120º, and triangle DEF, which contains no non-Euclidean vertices, has angle sum 180º

For triangle ABC in Figure 9, we can compute its angle sum as follows. The non-convex pentagon ABGOH is a Euclidean figure, so 540º = ∠HAB + ∠ABG + ∠BGO + ∠GOH + ∠OHA, and ∠GOH = 240º. Therefore, ∠HAB + ∠ABG + ∠BGO + ∠BGO + ∠OHA = 300º. The triangle CGH is also a Euclidean figure, so 180º = ∠CGH + ∠GHC + ∠HCG. The vertical angles across the cut at G and H are congruent, so ∠BGO + ∠CGH = 180º, and ∠OHA + ∠GHC = 180º. Adding the angles of the pentagon ABGOH to the angles of the triangle CGH, and subtracting the straight angles at G and H, we have that ∠HAB + ∠ABG + ∠HCG = 120º.

This angle sum analysis can, of course, be extended to polygons with four or more sides.

Questions about alignment

Is it possible to have pairs of points of all possible varieties of alignment in a single s-manifold?

Given a point A in an s-manifold, is it possible that there are points of all possible varieties of alignment with A?

The 2-aligned region in the elliptic cone-space is open (does not contain boundary points), and the 0-aligned region in the hyperbolic cone-space is closed (contains all boundary points). Is there some underlying principle being manifested here?

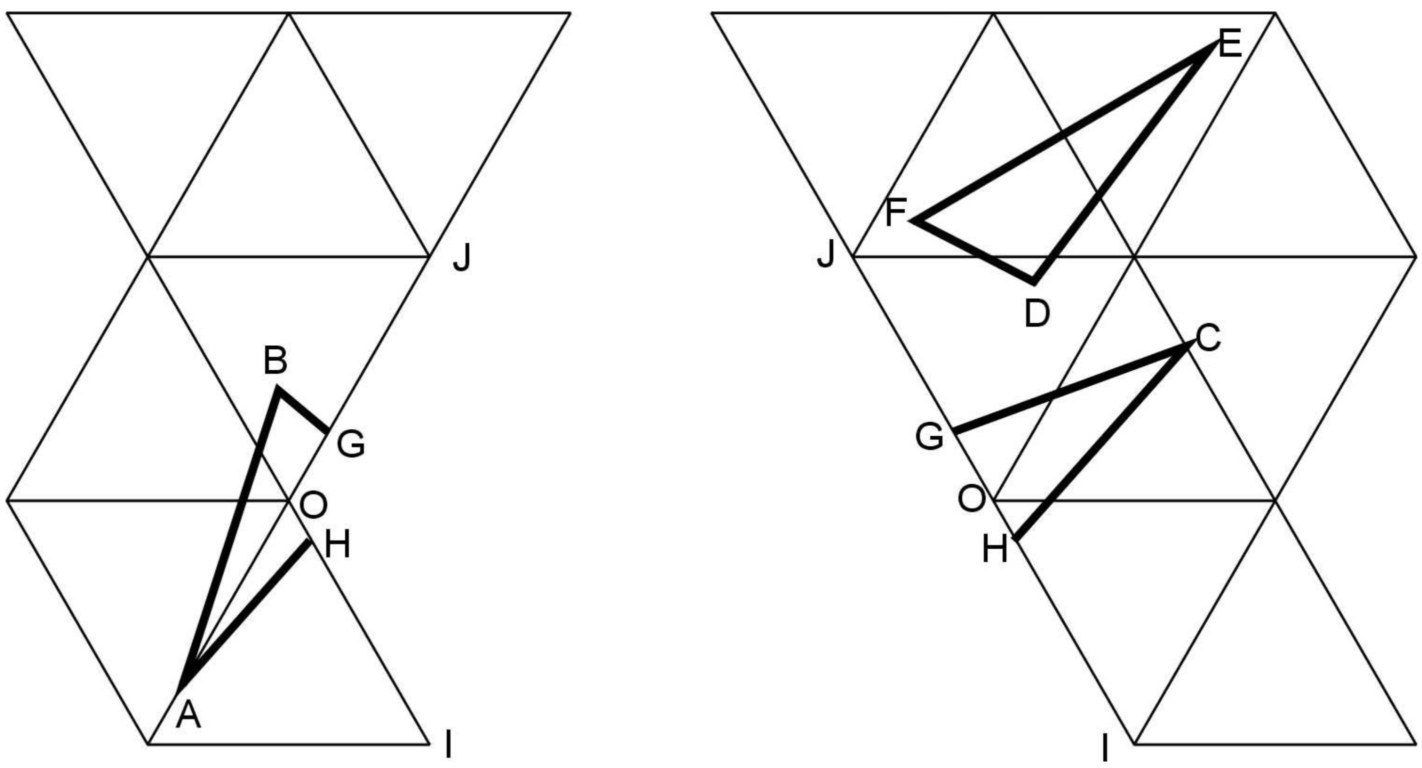


Figure 9. These are triangles around and near a hyperbolic vertex.

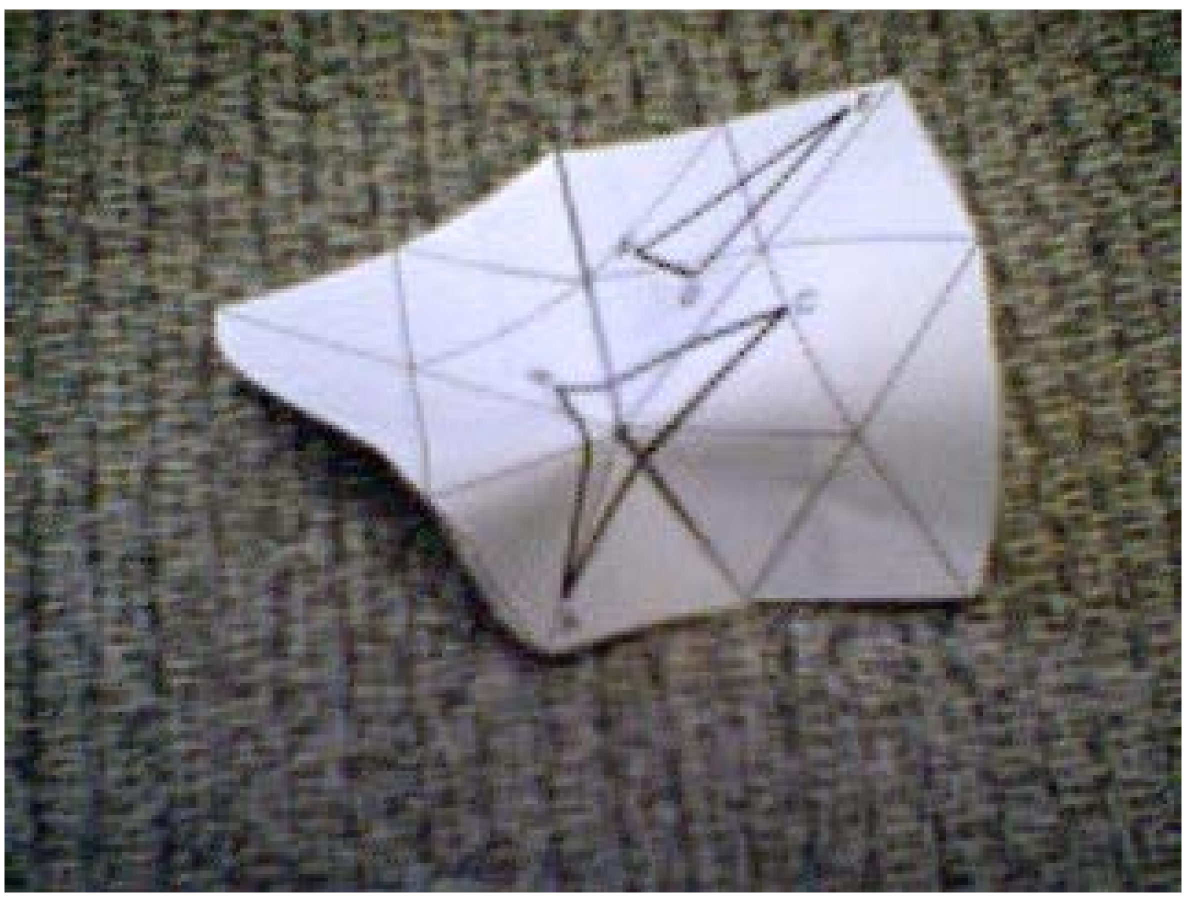


Figure 10. Paper model corresponding to Figure 9.

Betweenness

Hilbert's axioms of betweenness are as follows **[14]**.

II-1. If a point B lies between points A and C, then the points A, B, and C are distinct points of a line, and B also lies between C and A.

II-2. For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.

II-3. Of any three points on a line, there exists no more than one that lies between the other two.

II-4. Let A, B, and C be three points that do not lie on a line, and let *a* be a line which does not meet any of the points A, B, and C. If the line *a* passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.

Probably the most intuitive notion of a point C lying between two other points A and B corresponds to C lying in the interior of the line segment AB. This works perfectly well in the Euclidean plane, but it introduces some ambiguity when pairs of points are not 1-aligned, and we have seen that we have points that are not 1-aligned in s-manifolds. We would expect a variety of structures, therefore, regarding betweenness in an s-manifold.

We will say that C **lies completely between**, or simply **between**, A and B, if it lies in the interior of every possible s-segment AB. We will also say that C **lies partially between** A and B, if it lies in the interior of at least one of the s-segments AB, but not all of them. Otherwise, C is **not between** A and B.

If a point C is partially between A and B, we may describe the situation more explicitly by saying that C is ***x%*-between** A and B, where *x* is the percentage of s-segments that C lies on out of all that are possible. For example, on the sphere, if A and B are two non-antipodal points, they are joined by two segments, the short and long arcs of the unique great circle through A and B. If C lies on one of these segments, it will not lie on the other, so C would be 50%-between A and B. We will only use this description for points that are partially between, so 100%-between is distinct from completely between, and 0%-between is distinct from not between. For example, every point on the sphere, other than the north and south poles, is partially between the two poles. Each of these points lies on exactly one segment joining the poles out of infinitely many that are possible. Each of these points, therefore, would lie 0%-between the north and south poles. It may be difficult or impossible to determine a well-defined percentage in the case that there are infinitely many possible segments when 0%- and 100%-between do not apply.

We have defined (completely) between with the idea that there may be multiply aligned pairs of points, A and B, with a point C that lies on all of the s-segments AB. Figure 11 shows that this can occur in an s-manifold.

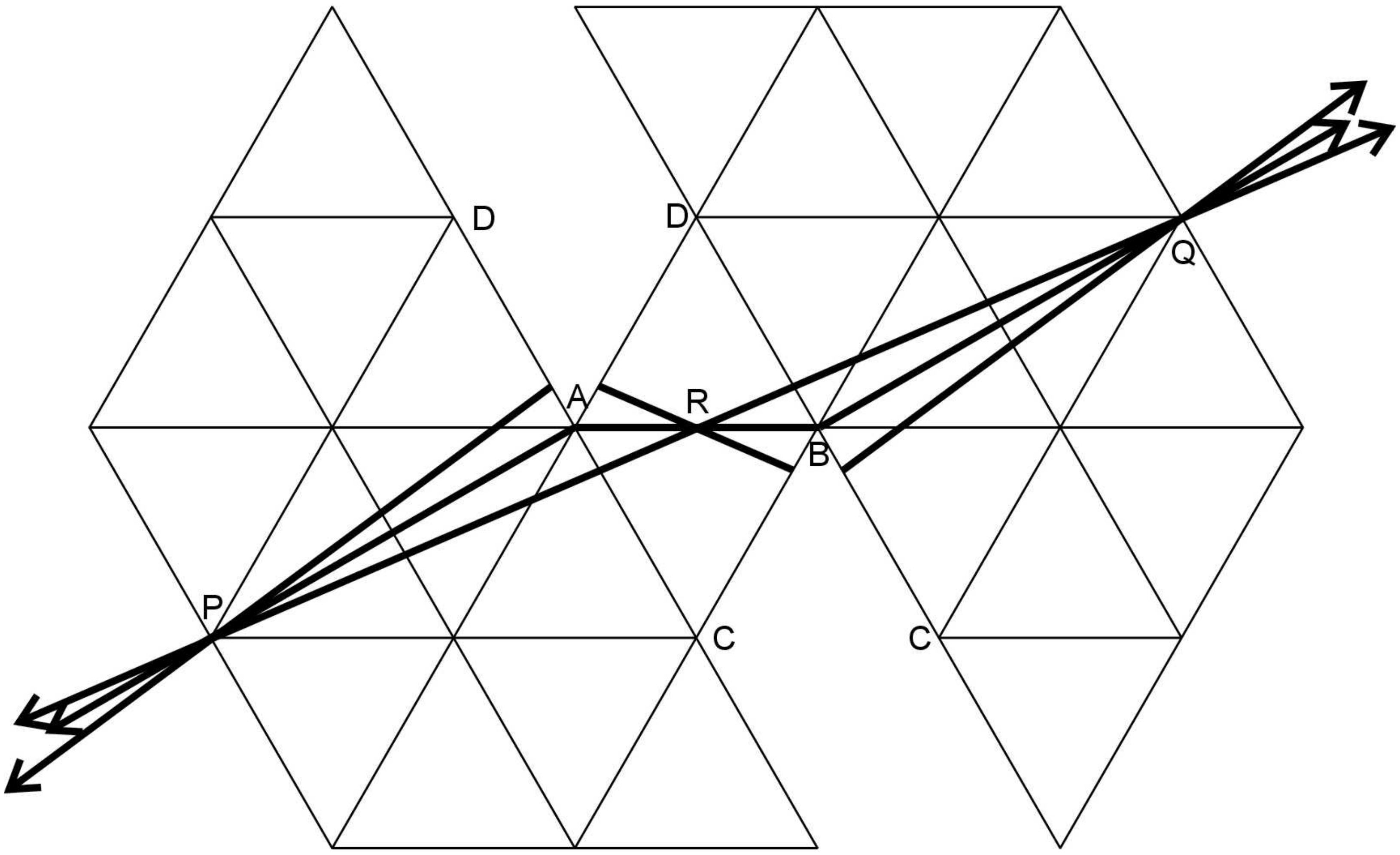


Figure 11. The point R lies completely between P and Q.

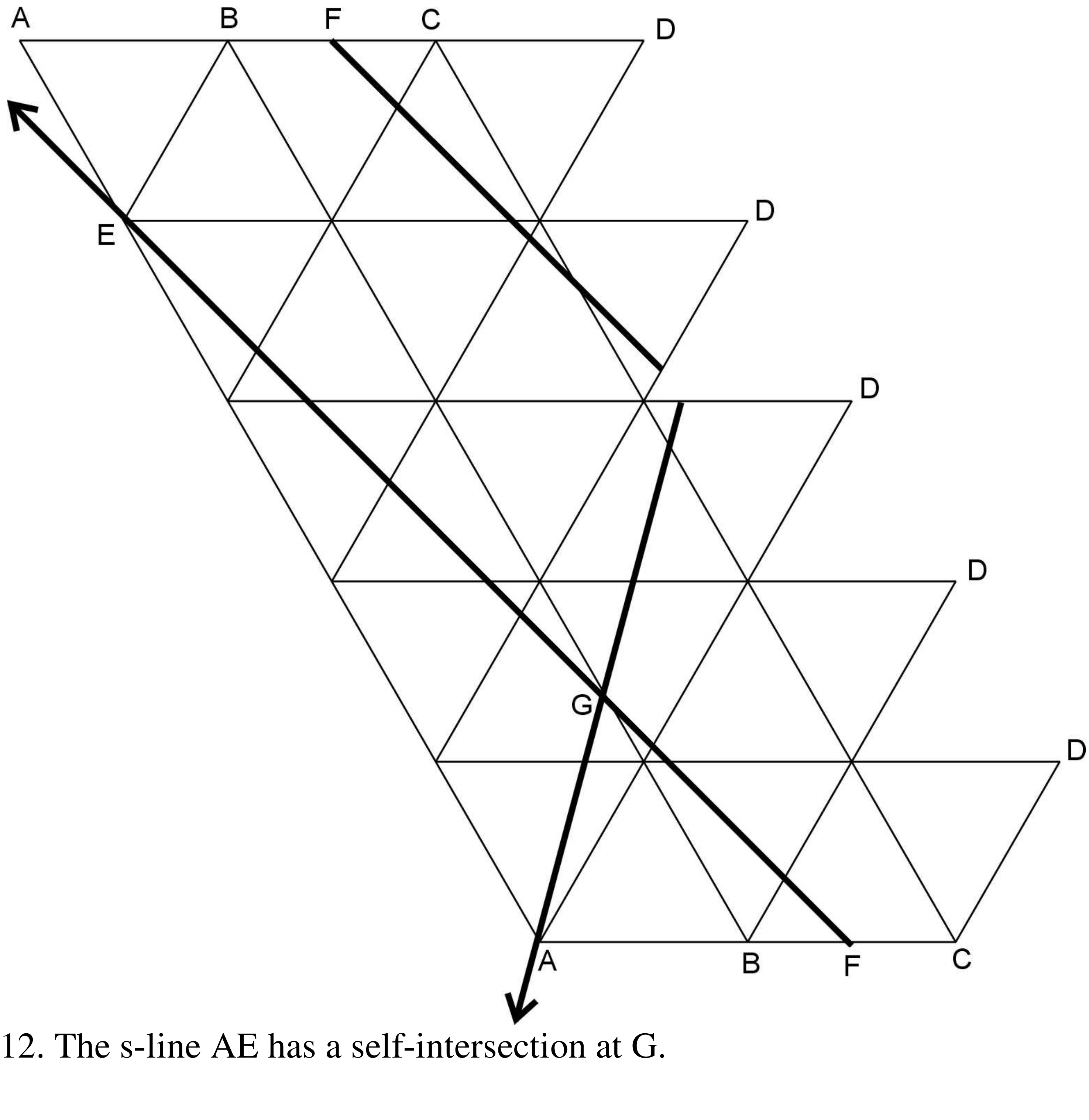


Figure 12. The s-line AE has a self-intersection at G.

In Figure 11, the two elliptic vertices A and B allow three paths from P to Q. The two hyperbolic vertices C and D prevent any s-lines through P and Q from passing on the same side of the two elliptic vertices, so these three s-lines are the only ones that pass through P and Q. The point R, therefore, lies completely between P and Q.

In the following, reference is made to s-lines with self-intersections. Figure 12 indicates one way that this can happen. Here, we have a cylinder capped off with six elliptic vertices. The s-line shown actually has infinitely many self-intersections, since each end is essentially a helix. We could prevent further self-intersections by inserting five hyperbolic vertices at the left edge, which will open the cylinder out into a sort of cone.

Figure 13. Paper model corresponding to Figure 13.

Axiom II-1. If a point B lies between points A and C, then the points A, B, and C are distinct points of a line, and B also lies between C and A.

This axiom holds, for the most part, in any s-manifold, since our definition of between states that the point must lie on an s-segment. The one exception is that the points A, B, and C need not be distinct. Since an s-line can have a self-intersection, a point A could lie in the interior of an s-segment AC according to our definition. Figure 14 illustrates how this could happen. The s-segment GFGA in Figure 12 is an example of such an s-segment. Therefore, S-denials of axiom II-1 occur in s-manifolds, but only in regards to the distinctness of the three points.

Axiom II-2. For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.

If the s-line AC exists, then there clearly must be a continuum of points B such that C lies at least partially between A and B. This axiom may fail in several ways. The s-line AC may not exist. We have seen examples of pairs of points that are remote. We have also seen pairs of points A and B that are 2-aligned forming two segments AC with disjoint interiors. In this case, as in Figure 15, C lies partially between A and B, but not completely between A and B. S-denials of this axiom are, therefore, quite common around non-Euclidean vertices.

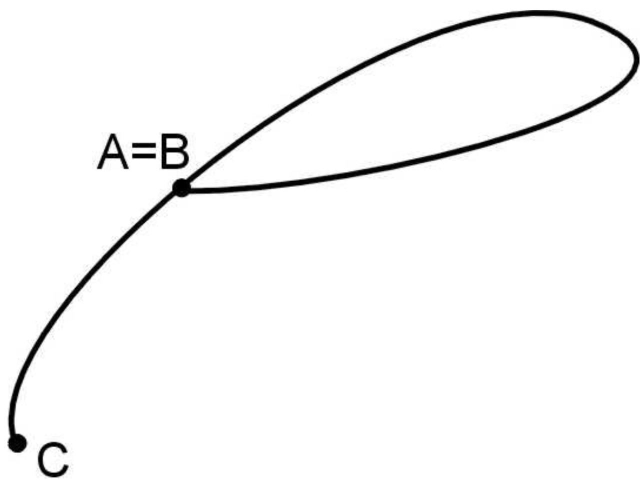


Figure 14. The point A lies in the interior of the segment AC.

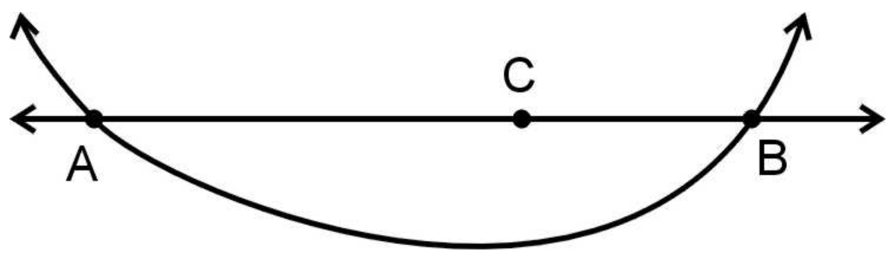


Figure 15. The point C lies only partially between A and B.

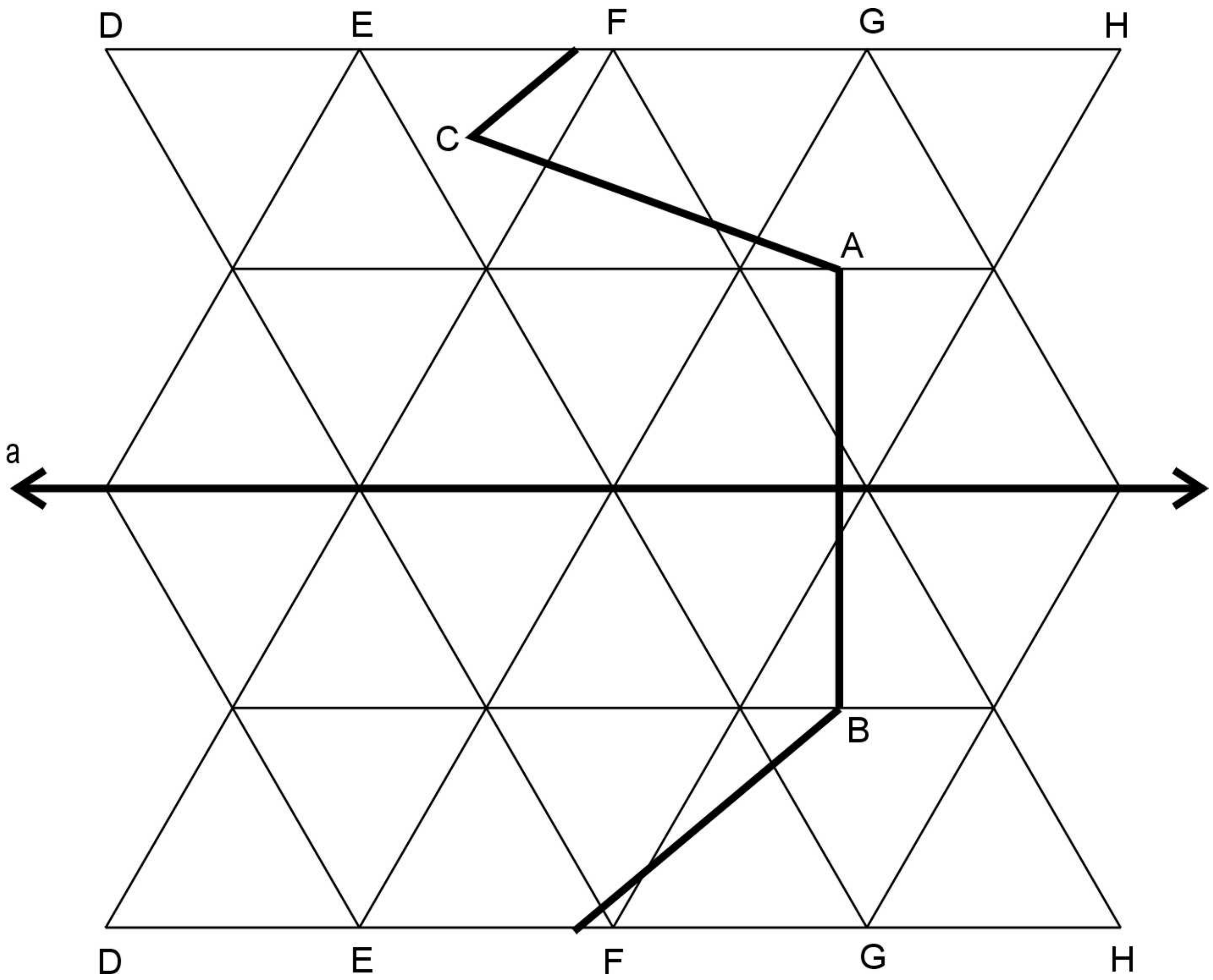


Figure 16. The s-line *a* enters triangle ABC through side AB, but never leaves.

Axiom II-3. Of any three points on a line, there is no more than one that lies between the other two.

There are closed s-lines (closed like a circle), so each of three points on a closed s-line is partially between the other two.

Conjecture. This axiom holds in every s-manifold when *completely between* is used for between.

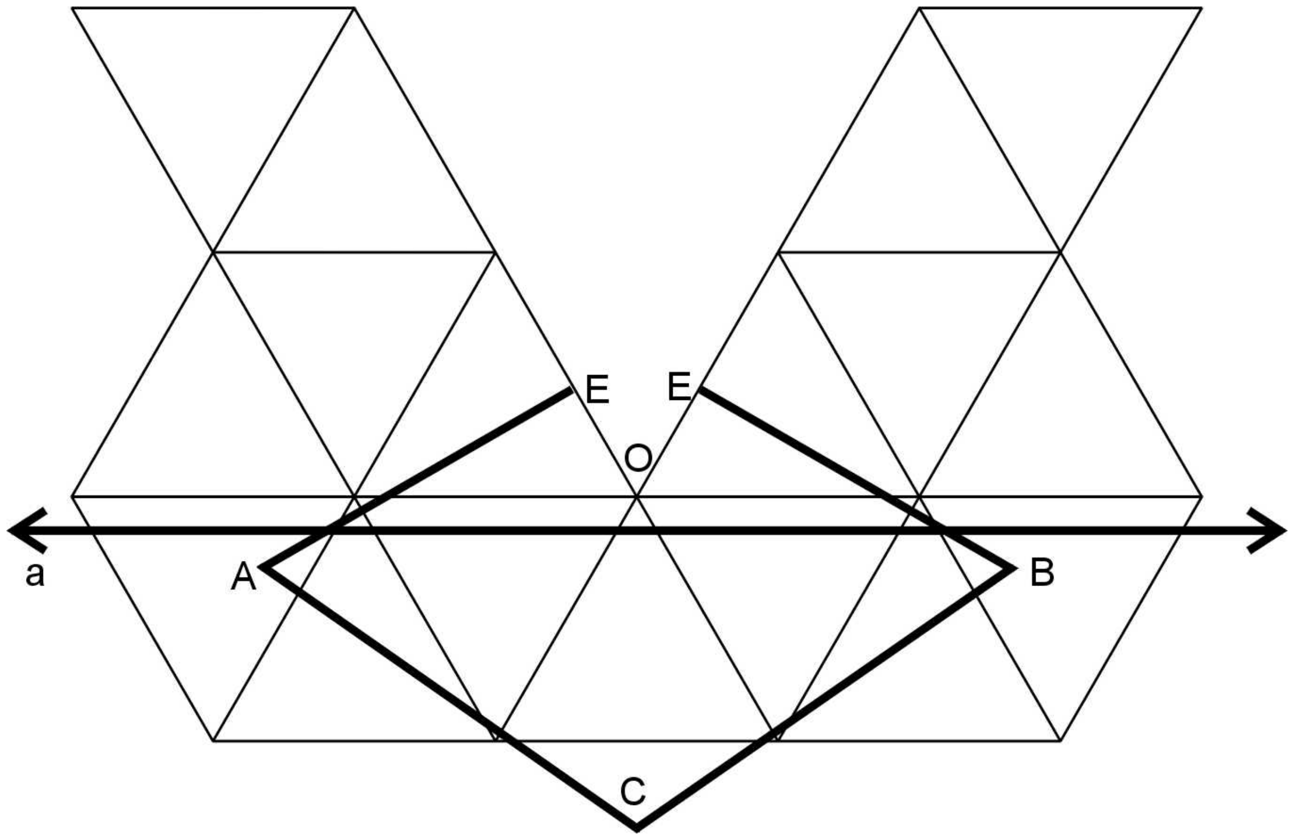


Figure 17. The s-line *a* enters and leaves the triangle ABC through the side AB.

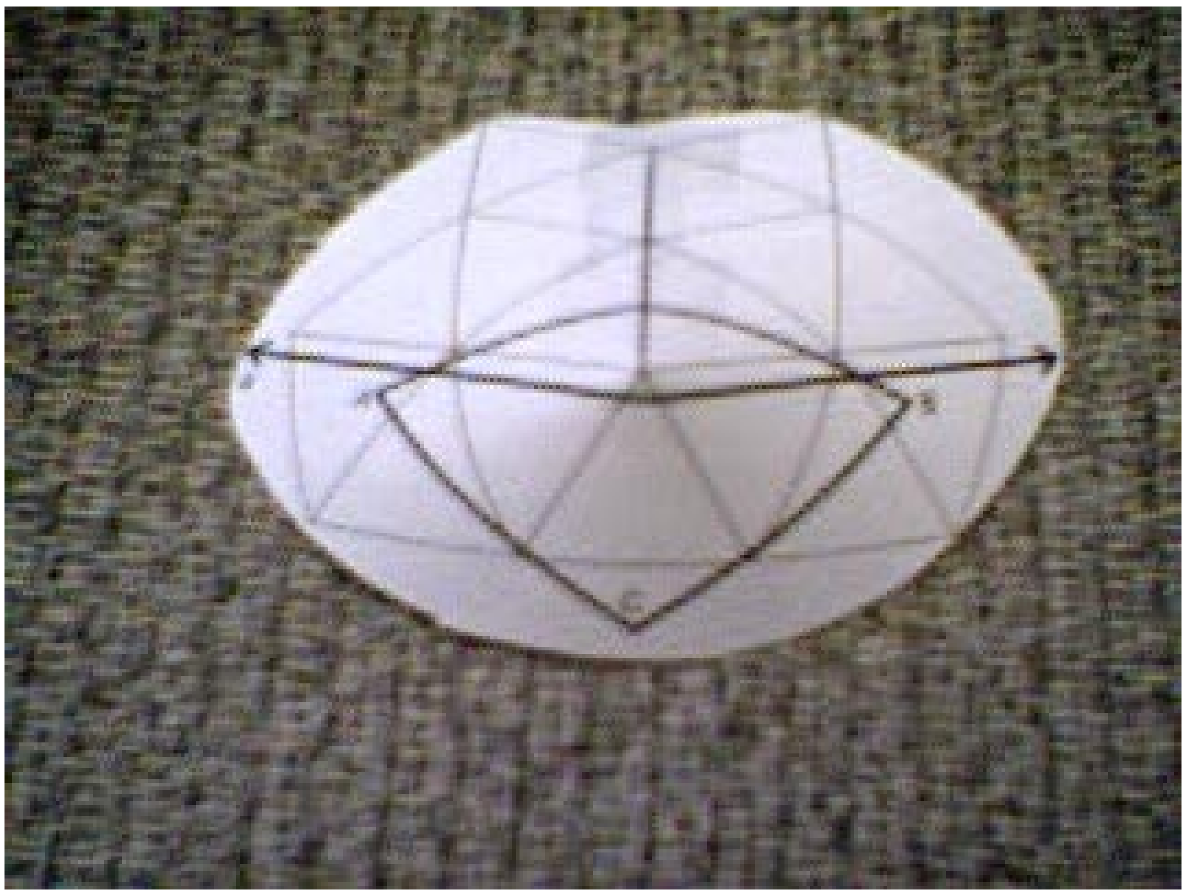


Figure 18. Paper model corresponding to Figure 17.

Axiom II-4. Let A, B, and C be three points that do not lie on a line, and let *a* be a line which does not meet any of the points A, B, and C. If the line *a* passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.

This is Pasch's axiom, and it states roughly that an s-line that enters a triangle through one of its sides must exit through one of the others. This axiom can fail in an s-manifold in several ways.

One would be that the s-line enters, but never leaves. This can happen if the inside of the triangle is connected to the outside or is infinite in extent, and would be an indication that the topology is non-trivial. In Figure 16, triangle ABC wraps around a cylindrical s-manifold. The s-line *a* intersects side AB, but it does not intersect either of the other two sides.

This axiom also fails if an s-line were to exit through the same side. This occurs in the case that two points on one side are multiply aligned, as shown in Figure 17. Here the s-lines *a* and AEB pass on either side of an elliptic vertex.

Congruence

Hilbert's axioms of congruence are as follows **[14]**.

III-1. If A and B are two points on a line *a*, and A′ is a point on a line *a*′ then it is always possible to find a point B′ on a given side of the line *a*′ through A′ such that the segment AB is congruent to the segment A′B′.

III-2. If a segment A′B′ and a segment A″B″ are congruent to the segment AB, then the segment A′B′ is also congruent to the segment A″B″.

III-3. On the line *a*, let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or on another line *a*′, let A′B′ and B′C′ be two segments, which except for B′ also have no point in common. In that case, if AB is congruent to A′B′ and BC is congruent to B′C′, then AC is congruent to A′C′.

III-4. Every angle can be copied on a given side of a given ray in a uniquely determined way.

III-5. If for two triangles ABC and A′B′C′, AB is congruent to A′B′, AC is congruent to A′C′, and ∠BAC is congruent to ∠B′A′C′, then ∠ABC is congruent to ∠A′B′C′.

Given two points A and B, the s-segment AB is not necessarily well-defined, if it is exists at all, since A and B may not be uniquely aligned. We can talk about *some* s-segment AB, however, with the understanding that we are talking about a particular s-segment AB and that we will make it clear if we wish to consider a different s-segment with the same endpoints. We will assume that an s-segment AB has a direction associated with it and that A is the starting point and B is the ending point. Given two s-segments AB and CD, we will define the **distance map** from AB to CD as follows. The point A maps to the point C, and each point P on AB is mapped to the point Q on CD such that the distance from A to P along the s-segment AB is the same as the distance from C to Q along the s-segment CD. We will say that the s-segments AB and CD are **s-congruent**, if the distance map exists and is a one-to-one correspondence. If two s-segments have no self-intersections, they are s-congruent if they have the same length, so this definition agrees with the notion of congruence of segments in Euclidean geometry.

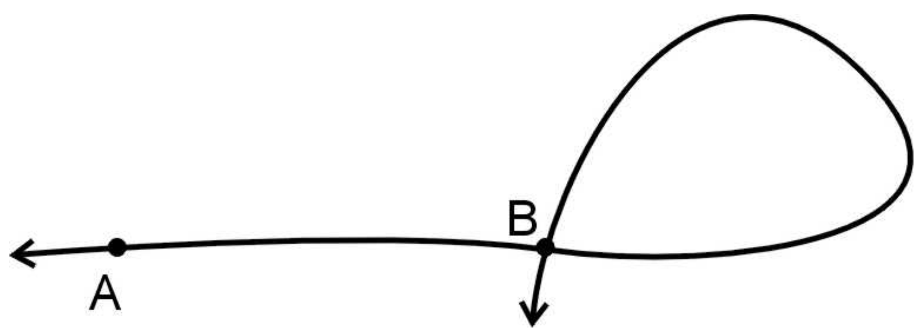


Figure 19. The s-segment AB (the longer one) is not s-congruent to the s-segment BA.

With s-congruence defined this way, an s-segment AB is always s-congruent to itself, but not necessarily to the inverse s-segment BA. If B lies in the interior of AB, as in Figure

19, the distance map from AB to BA would have A and some interior point mapping to B, and so it could not be one-to-one, and since B would have to map to two points, the distance map would not even be well-defined.

Axiom III-1. If A and B are two points on a line *a*, and A′ is a point on a line *a*′ then it is always possible to find a point B′ on a given side of the line *a*′ through A′ such that the segment AB is congruent to the segment A′B′.

In an s-manifold, it is always possible to find a point on an s-line any distance from a point in either direction. There may not be an s-segment with this length between the two points, however, since we require that an s-segment cannot completely cover an s-line. For example, if the s-line is closed like a circle and has length 2, then starting from some point A, we may travel a distance 3 in one direction and there will be a point on the s-line at this position, but since s-segments cannot cover an s-line completely, there is no associated s-segment. If the s-segment exists, then s-congruence is not guaranteed if there are self-intersections. For example, in Figure 20, even if AB and CD are the same length, they could not be s-congruent. We have seen examples of closed s-lines and s-lines with self-intersections, so this axiom is S-deniable in an s-manifold.

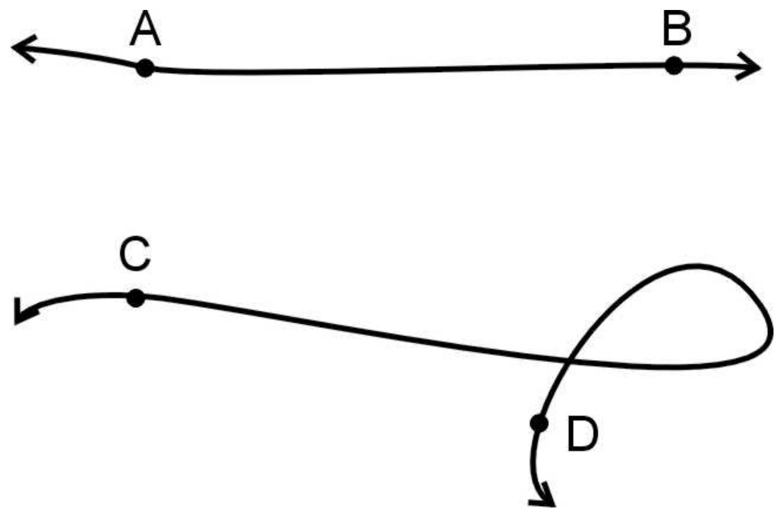


Figure 20. The s-segments AB and CD are not s-congruent.

Axiom III-2. If a segment A′B′ and a segment A″B″ are congruent to the segment AB, then the segment A′B′ is also congruent to the segment A″B″.

We have chosen a relatively conservative definition for s-congruence, so we cannot S-deny this axiom in an s-manifold. Clearly, since one-to-one correspondence is preserved under composition, s-congruence of s-segments is transitive. In fact, our definition of s-congruence satisfies the properties of an equivalence relation, i.e., it is reflexive, symmetric, and transitive. Using a notion of congruence that was not an equivalence relation would complicate the study of all related issues immensely. As it is, this axiom always holds in an s-manifolds.

One alternative to s-congruence that has greater "S-deniability" is the following. We only mention this, and we will not pursue this further in this book. We could define the **q-segment** AB to be the collection of all s-segments with endpoints A and B. The "q" refers to quantum physics, which inspires this definition, although fuzzy logic would be a more appropriate reference. In any case, each particular mention of the q-segment AB refers to a particular s-segment with probability determined by the total number of s-segments. Two q-segments AB and CD are ***x%*-q-congruent**, if x% is the probability that the particular s-segments AB and CD have the same length. For example, let A and B be two non-antipodal points on the sphere. Then the q-segment AB consists of the two arcs of the great circle through A and B. Since there are two segments AB, each one occurs with probability 50%. Comparing AB with itself, there are four possibilities all occurring with equal probability, short-short, short-long, long-short, and long-long. The short-short and long-long possibilities compare segments of the same length, so the q-segment AB is 50%-q-congruent with itself.

Axiom III-3. On the line *a*, let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or on another line *a*′, let A′B′ and B′C′ be two segments, which except for B′ also have no point in common. In that case, if AB is congruent to A′B′ and BC is congruent to B′C′, then AC is congruent to A′C′.

We have additivity of length for s-segments in an s-manifold, but this does not necessarily extend to s-congruence. In Figure 21, the s-segments AB and DE could be s-congruent, as could s-segments BC and EF, but the s-segments AC and DF, while being the same length, would not be s-congruent. Of course, the s-segments DE and EF have more than the point E in common, but it seems that they satisfy the basic intent of the axiom. Therefore, this axiom is S-deniable in the sense that s-congruence is not additive, but as stated, it holds in all s-manifolds, as long as the s-segments AC and A′C′ exist.

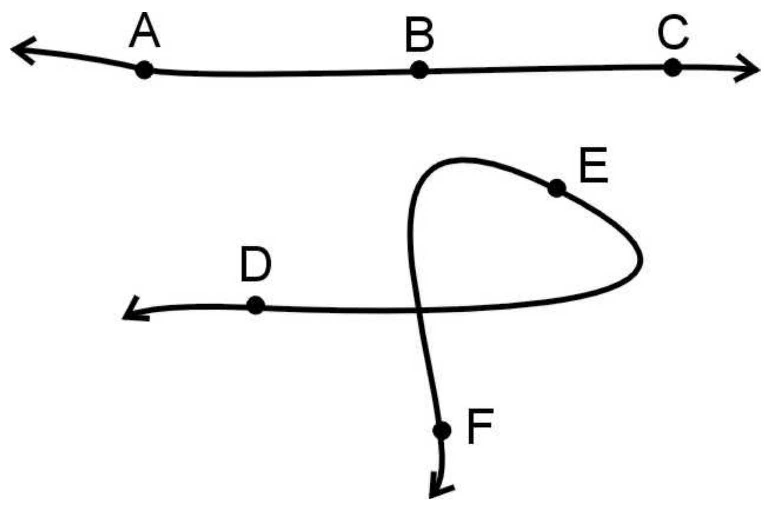


Figure 21. The s-segments AC and DF could not be congruent.

The measure of an angle carries over reasonably from Euclidean geometry. By the measure of a given angle, we will mean the smallest non-negative one. For example, in the plane, we can say that an angle that measures 90° also measures 270°, but we will think of these as being equivalent. We will say that two **angles are congruent** if their measures are the same.

Axiom III-4. Every angle can be copied on a given side of a given ray in a uniquely determined way.

An angle that emanates from an elliptic vertex can measure at most 150°, and one that emanates from a hyperbolic vertex can measure up to 210°. There are limitations, therefore, in copying an angle emanating from a hyperbolic vertex and in copying an angle to an elliptic vertex. For any s-manifold that has a hyperbolic vertex and an elliptic vertex, any angle measuring more than 150° cannot be copied to a ray emanating from an elliptic vertex, so this axiom would be S-denied. A similar statement can be made for any s-manifold with a non-Euclidean vertex.

Axiom III-5. If for two triangles ABC and A′B′C′, AB is congruent to A′B′, AC is congruent to A′C′, and ∠BAC is congruent to ∠B′A′C′, then ∠ABC is congruent to ∠A′B′C′.

In Euclidean geometry, this axiom extends to the SAS theorem for congruence of triangles. In an s-manifold, however, the angle sums can vary, so this axiom will generally be false if the angle sums of the two triangles are different. In Figure 22, we have three points A, B, and C near an elliptic vertex, and the points B and C are 2-aligned. Therefore, there are two possible sides BC, even though the angle ∠BAC and sides AB and AC are the same for both triangles. This axiom is S-denied in any s-manifold with an elliptic vertex.

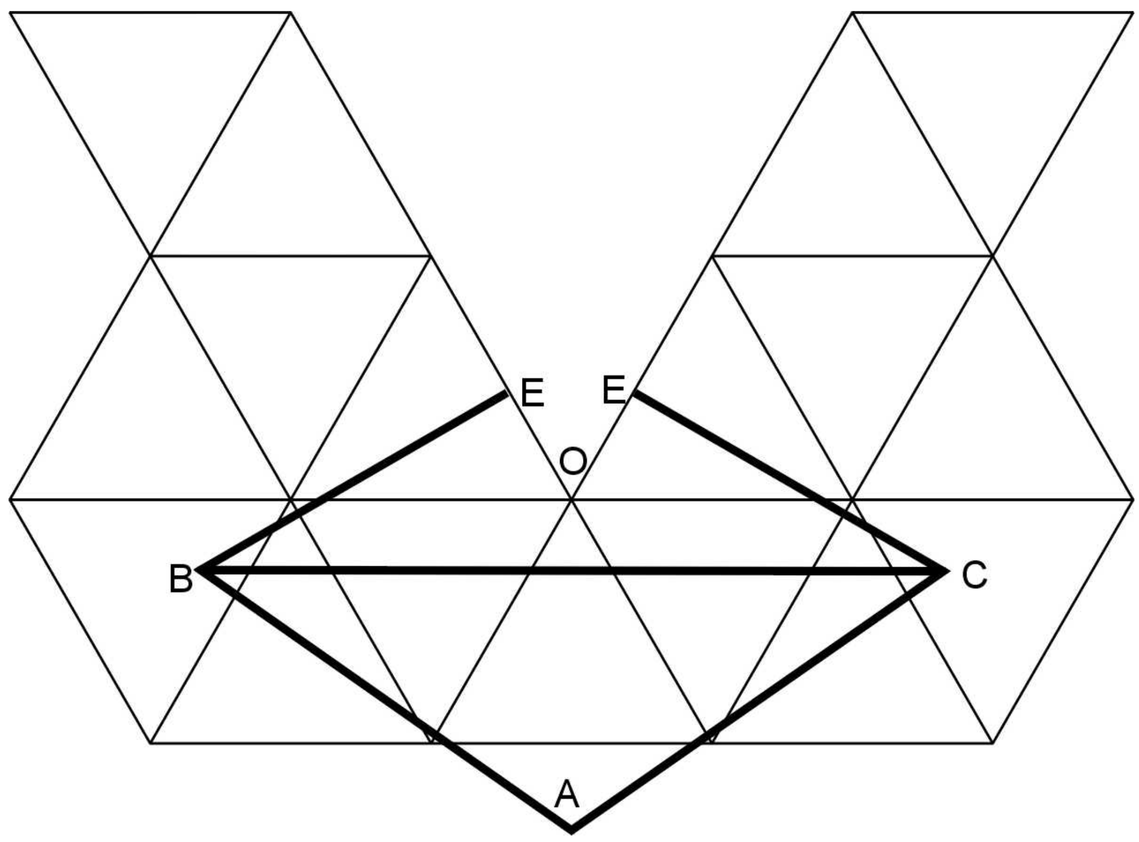


Figure 22. The SAS criteria are satisfied for two different triangles near this elliptic vertex.

Parallels

Euclid's fifth postulate states, "If two lines are cut by a transversal so that the interior angles on one side are less than two right angles, then the two lines will intersect on that side," **[9]**. Implicit here is that if the angles are greater than or equal to two right angles, then the lines will not intersect on that side. Clearly then, the two lines will be parallel, if, and only if, the angles equal two right angles. In other words, given a line *a* and a point P not on *a*, there is exactly one line through P that is parallel to *a*. This is essentially Playfair's postulate, although Hilbert was able to weaken this to, "there is *at most* one line through P that is parallel to *a*," since the existence of parallels can be established from other axioms or postulates.

Non-Euclidean geometry started, for the most part, with Bolyai, Lobachevski, and Gauss, and their hyperbolic geometry can be obtained from Hilbert's axioms by replacing the parallel axiom with a statement like, "Given a line *a* and a point P not on *a*, there are at least two lines through P that are parallel to *a*" (see **[3]**). Hilbert's other axioms establish that all of the lines "in between" these two parallels must also be parallel. In hyperbolic geometry, therefore, there are always infinitely many parallels.

The elliptic geometry of Riemann requires that there be no parallel lines (and clearly Hilbert's other axioms are not consistent with this requirement, so other axioms will differ).

In a Smarandache geometry in which the parallel postulate is S-denied, the number of parallels will change throughout the space, and this will depend on the point P and the line *a*. We will define our notions of Euclidean, elliptic, and hyperbolic, therefore, as a relationship between a point and an s-line. We will say that a point P is **Euclidean relative to an s-line *a***, if there is exactly one s-line through P that is parallel to *a*. We will define the other terms similarly. Let P be a point not on an s-line *a*. The point P is **elliptic relative to *a***, if there are no parallels through P, and P is **hyperbolic relative to *a***, if there are at least two parallels through P.

If a point P is hyperbolic relative to an s-line *a*, Smarandache recognized a variety of ways in which this could happen **[19]**. There could be a finite number of parallels, and there could be infinitely many parallels. It could also be that all or almost all of the s-lines through P are parallel. We will expand the definition of hyperbolic as follows. If there are only finitely many parallels through P (but at least two), P is **finitely hyperbolic relative to *a***. In the case that there are infinitely many parallels through P, we will say that P is **regularly hyperbolic** if there are infinitely many non-parallels through P, **extremely hyperbolic** if there are only finitely many non-parallels (but at least one), and **completely hyperbolic** if there are no non-parallels. We can make the finitely hyperbolic definition more explicit by saying that P is ***n*-hyperbolic** when there are exactly *n* parallels through P.

In the Euclidean plane, all points are Euclidean relative to every line. We will shorten this to all points are Euclidean. Similarly, all points are hyperbolic in the hyperbolic geometry of Bolyai, Lobachevski, and Gauss, and all points are elliptic in the elliptic geometry of Riemann. All points are elliptic in spherical geometry, as well.

Elliptic points

There are elliptic points in the elliptic cone-space. Roughly, a point will be elliptic relative to an s-line, if the elliptic vertex lies between them. More precisely, given an s-line *a* that does not pass through the elliptic vertex, there is a continuum of s-lines that are parallel to *a* in the Euclidean sense that approach the elliptic vertex. All points beyond these s-lines are elliptic relative to *a*. This is illustrated in Figure 23.

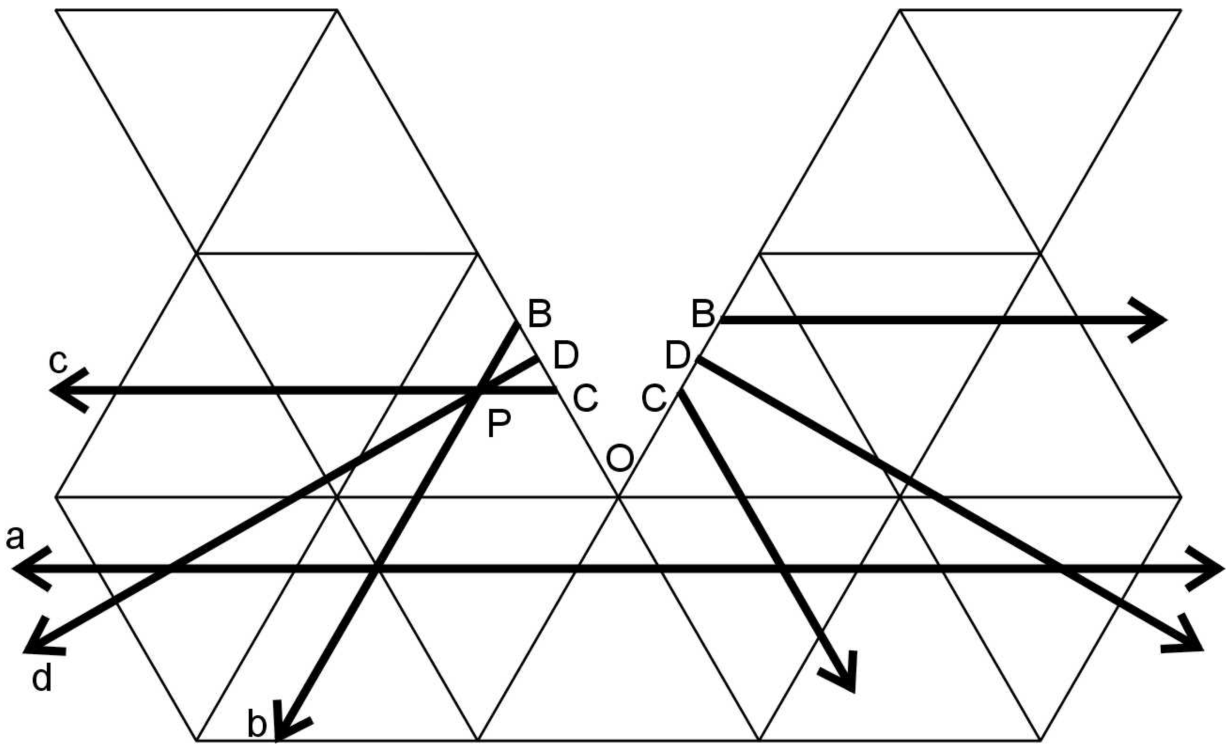


Figure 23. The point *P* is elliptic relative to the s-line *a*.

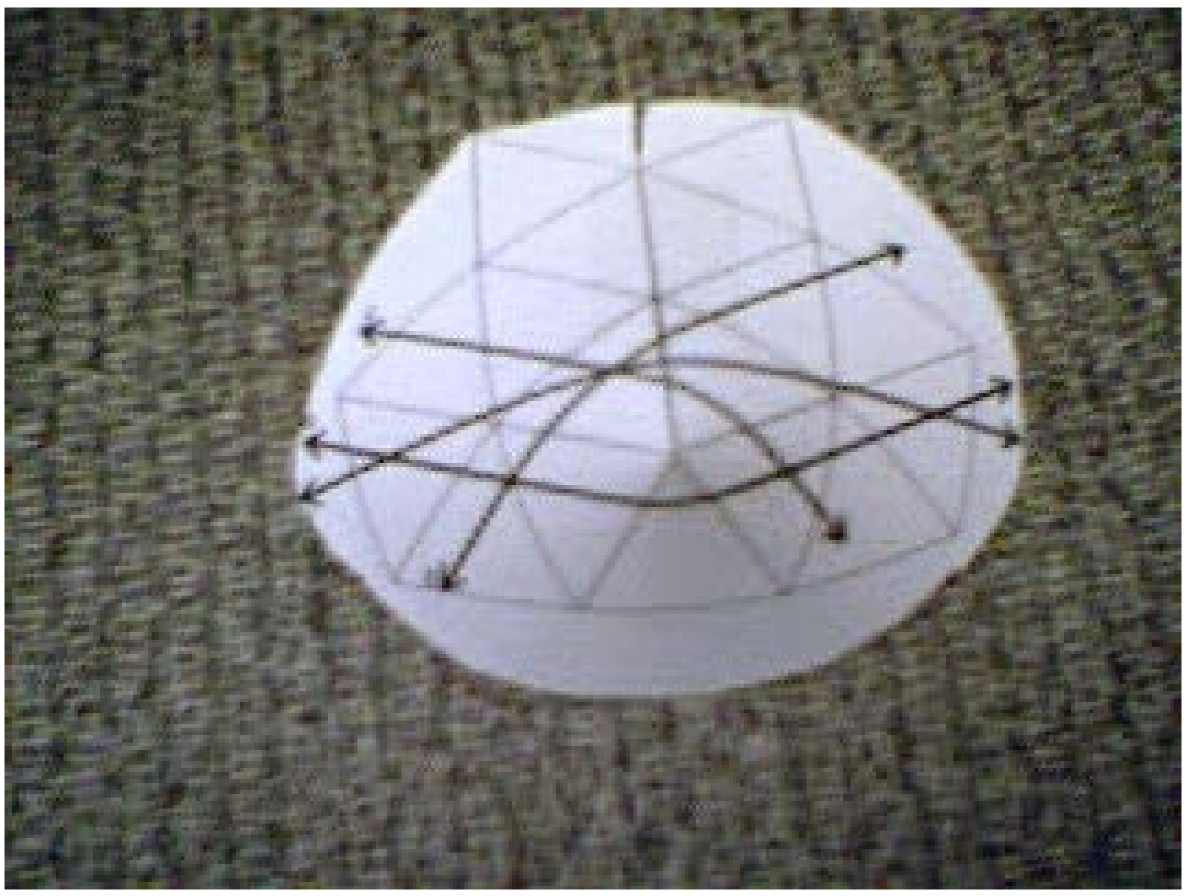


Figure 24. Paper model corresponding to Figure 23.

Towards the right, the angle for *c* at P relative to *a* is 90º (recall the definition of a relative angle), so *c* is parallel at P. Since the rest of the space is Euclidean, *c* is parallel to *a* everywhere to the left. Clearly, any s-line through P with an angle greater than 90º (towards the right) will intersect *a* on the left. Since the angle decreases by 60º as we move to the right, any s-line through P with an angle less than 150º will intersect *a* on the right. It follows that all s-lines through P will intersect *a* at least once, and those lines with angles strictly between 90º and 150º will intersect *a* twice. P is, therefore, elliptic relative to the s-line *a*.

Regularly hyperbolic points

We can find hyperbolic points in the hyperbolic cone-space. In general, if there is a hyperbolic vertex between a point P and an s-line *a*, then the point P will be regularly hyperbolic relative to *a*.

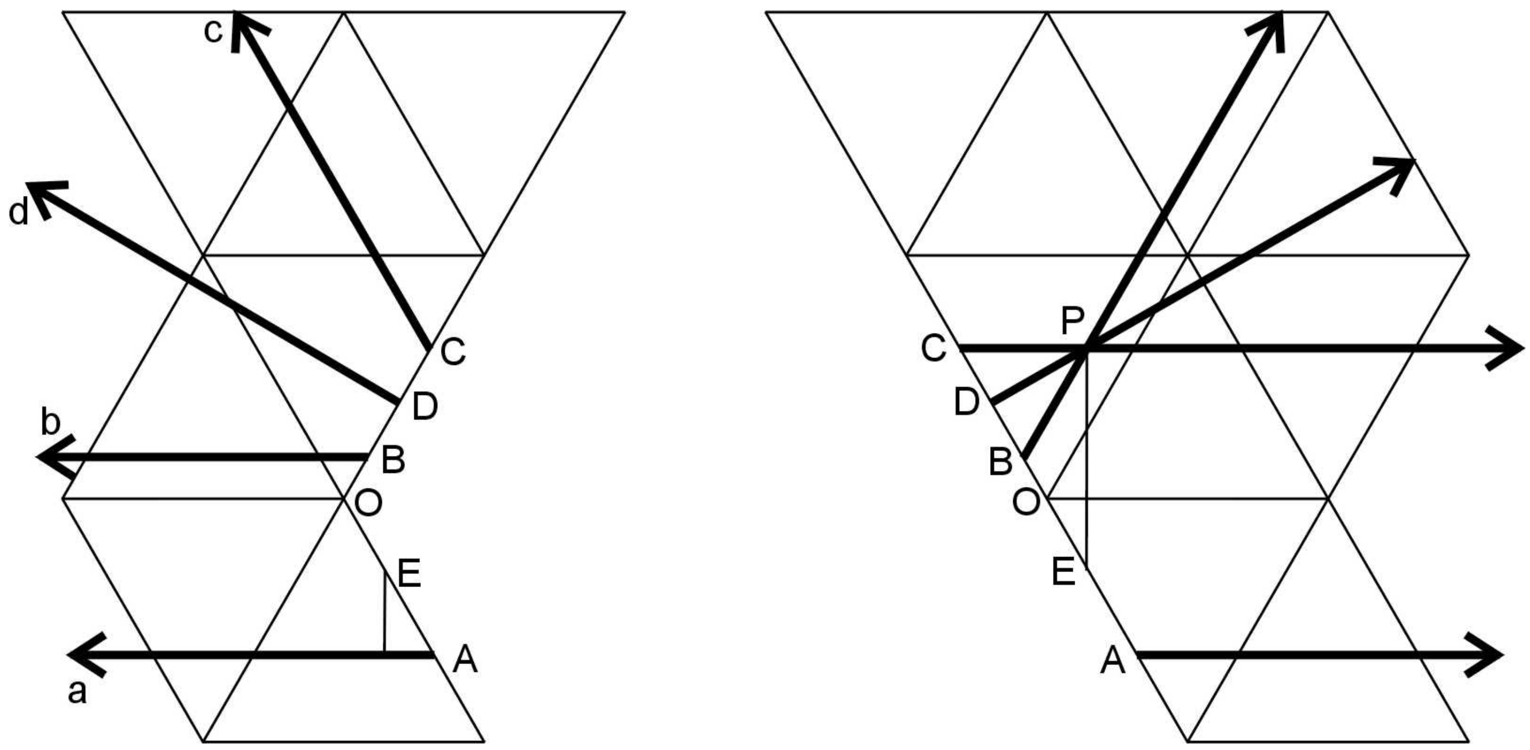


Figure 25. The point P is regularly hyperbolic relative to the s-line *a*.

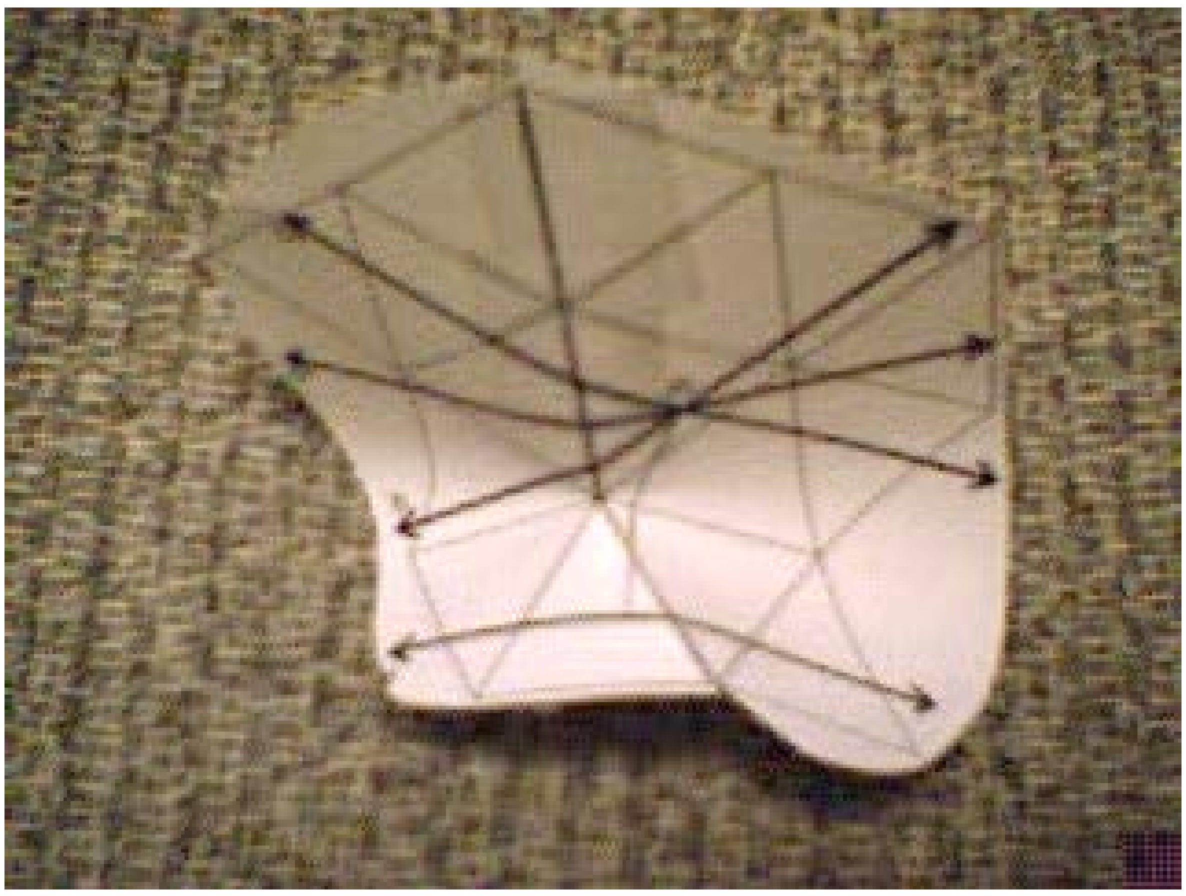


Figure 26. Paper model corresponding to Figure 25.

In Figure 25, towards the left, the line *c* is parallel to *a* at P. The angle for *c* increases by 60º to the left of O, so *c* is parallel. Since the angles increase by 60º as we move to the left of O, all of the s-lines with angles between 30º and 90º (inclusive) are parallel to *a*. All other s-lines will intersect *a*. There are a continuum of s-lines through P that are parallel and a continuum of s-lines that are not parallel. P is, therefore, regularly hyperbolic relative to *a*.

Finitely hyperbolic points

In the hyperbolic geometry of Bolyai, Lobachevski, and Gauss, where all of the points are regularly hyperbolic, it is sufficient to require that there exist at least two lines through a given point and parallel to a given line, since it can be proven that the in between lines are also parallel. This is generally the case in an s-manifold, as can be seen in the previous section, but not always.

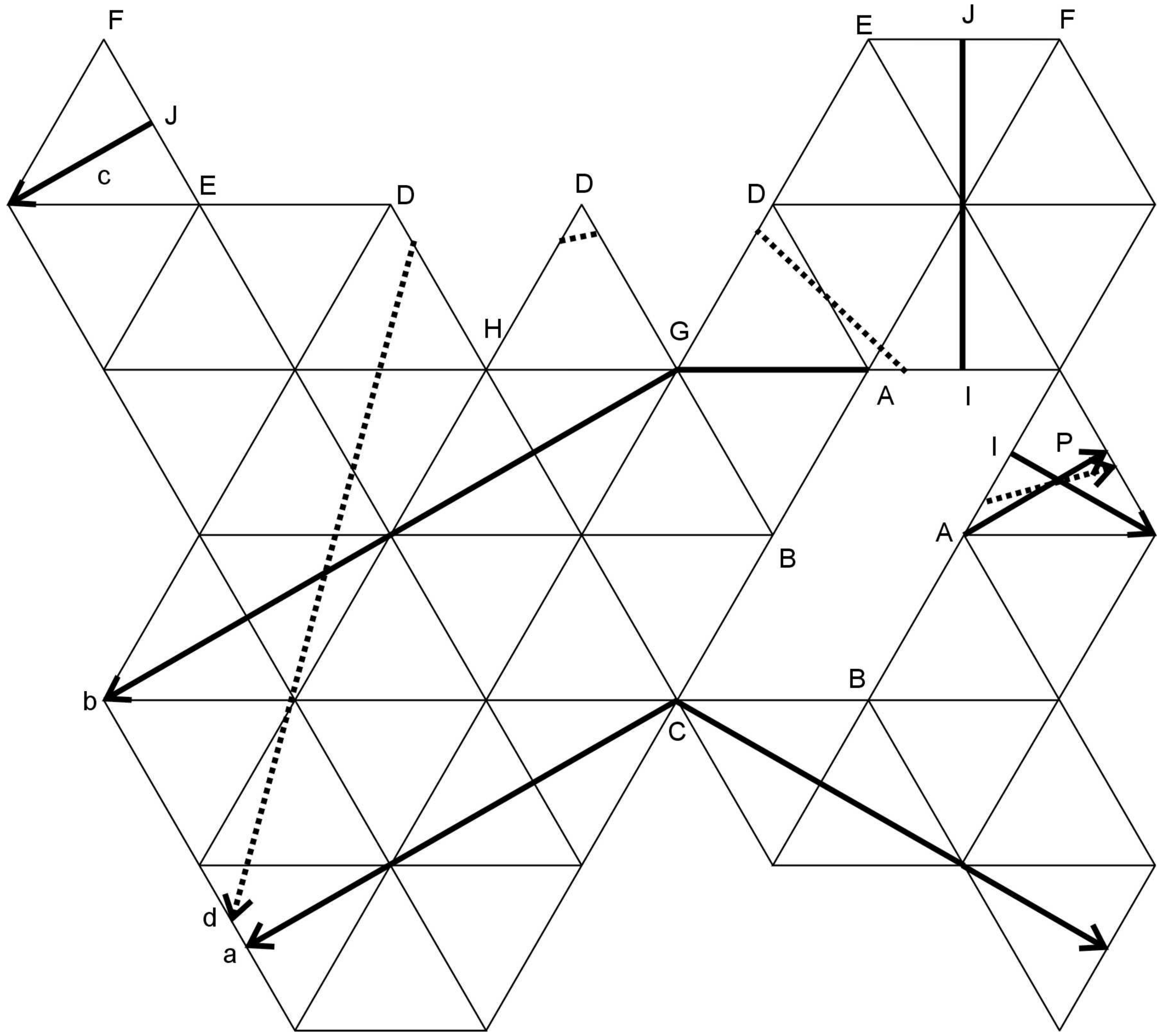


Figure 27. The point *P* is finitely hyperbolic relative to the s-line through C.

In Figure 27, the vertex B is a hyperbolic vertex, and the point P is hyperbolic relative to the line *a*. Towards the left, the s-lines *b* and *c* have angles 30º and 90º at P relative to *a*.

The space is Euclidean to the right of P, so the s-lines *b* and *c* will not intersect *a* to the right of P.

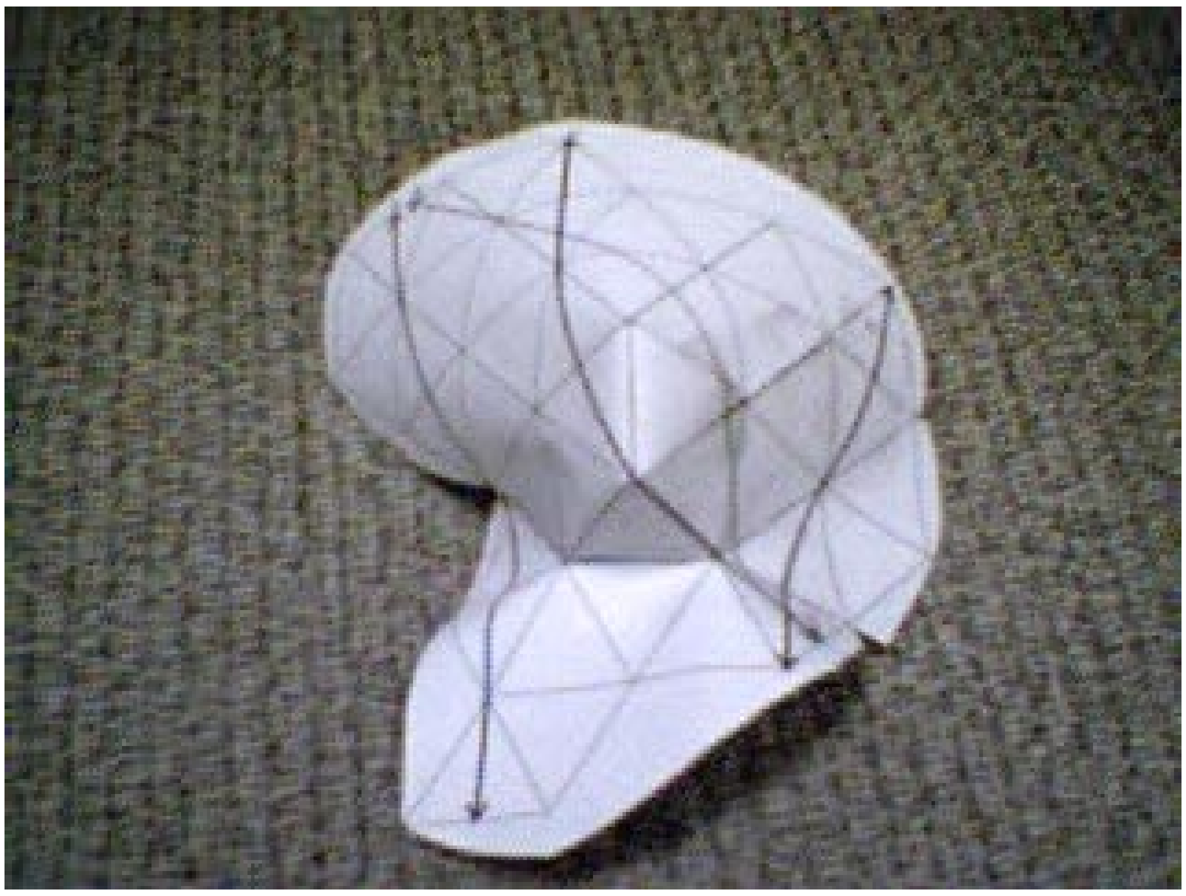


Figure 28. Paper model corresponding to Figure 27.

There are three other non-Euclidean vertices in this space, one additional hyperbolic vertex, A, and two elliptic vertices, G and H. One effect of these vertices is that they cause all of the s-lines between *b* and *c* to turn towards *a* while leaving *b* and *c* parallel.

The s-line *b* has an angle 30º relative to *a* at P. The hyperbolic vertex B increases this to 90º. The s-line *a* passes through the hyperbolic vertex A and the elliptic vertex G. This increases the angle by 30º, and then decreases it by 30º back to 90º. Therefore, *b* remains parallel to *a* to the left.

The s-line *c* has both hyperbolic and both elliptic vertices between it and *a*. The angle for *c* is 90º relative to *a* at A, and the changes towards the left are +60º, +60º, –60º, and –60º. Therefore, *c* is also parallel to *a* on the left.

The four non-Euclidean vertices also lie between *a* and those s-lines with angles strictly between 30º and 90º. Therefore, all of these s-lines will have an angle strictly between 30º and 90º to the left of B, A, G, and H. They must, therefore, intersect *a*. The s-line *d* shown in Figure 27 is typical of these intermediate s-lines.

All of the other s-lines through P are clearly not parallel, so there are exactly two s-lines through P that are parallel to *a*, and therefore, P is finitely hyperbolic relative to *a*.

Question. It seems reasonable to expect that more extensive configurations would yield points with exactly there or exactly four parallels to a given s-line. Do these exist?

Extremely hyperbolic points

A basic assumption in geometry is that there is a line through any two points. In this case, for a point P not on a line *a*, there are infinitely many lines through P that intersect *a*, one passing through each point of *a*.

In an s-manifold, we have seen that there can be pairs of points that are remote, or 0-aligned. This means that there are pairs of points that do not lie on the same s-line. This opens up the possibility of extremely hyperbolic and completely hyperbolic points, which do not have an infinite number of non-parallels relative to some s-line.

We know that each hyperbolic vertex that lies between two s-lines increases the relative angle by 60º. Three hyperbolic vertices, therefore, could take two s-lines that are ± εº relative to a third s-line to ± (90 + ε)º. This could force all, or almost all, of the s-lines through some point to be parallel to some s-line.

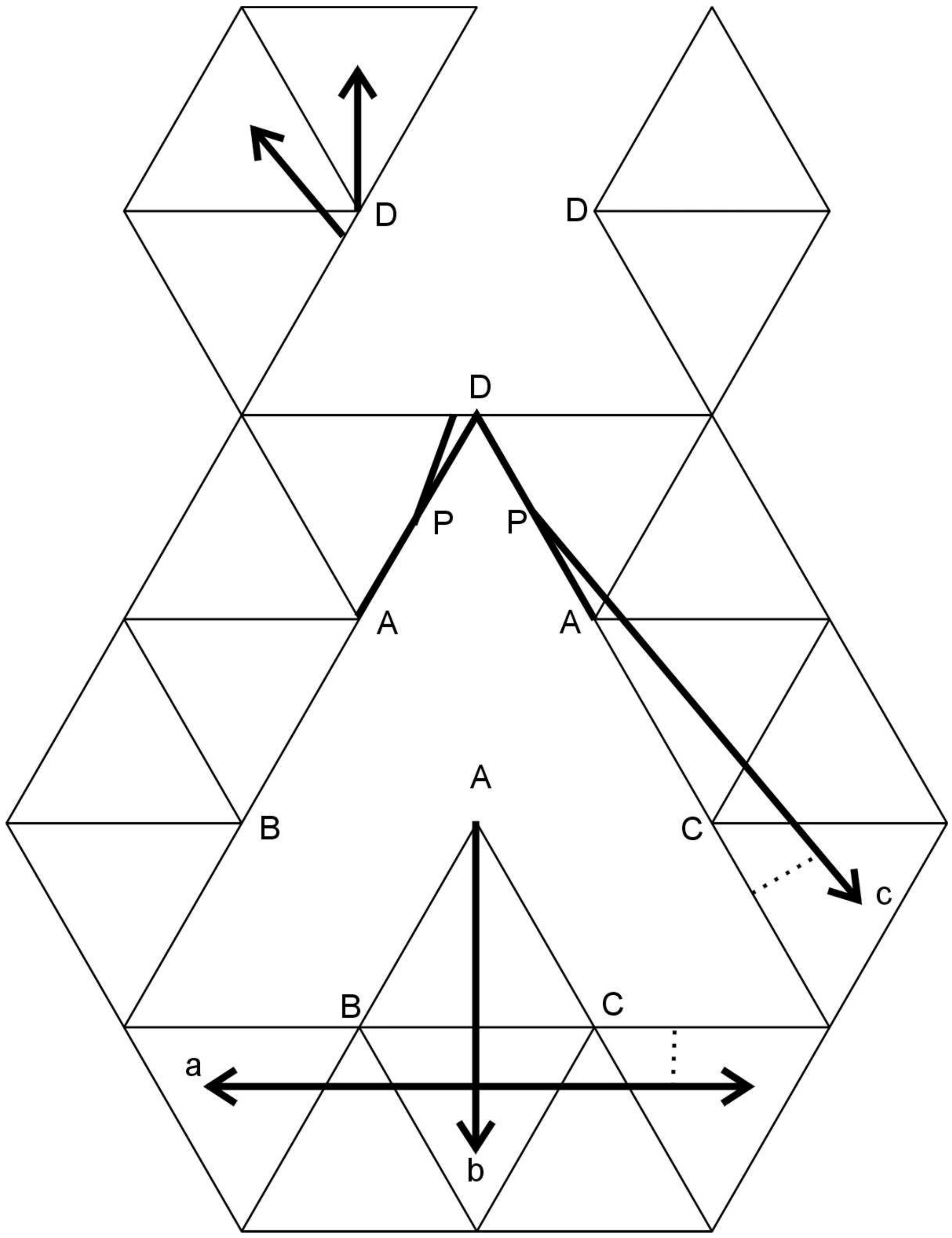


Figure 29. The point P is extremely hyperbolic relative to the s-line *a*.

This is illustrated in Figure 29. The vertices A, B, and C are hyperbolic (D is also hyperbolic, but will be discussed later), and these lie in between the point P and the s-line *a*. The s-line *b* runs through the points D, P, and A, and it intersects the s-line *a*. In the downward direction, the s-line *c* has an angle larger than 90º near P, relative to *b*. Since the s-line *b* passes through the hyperbolic vertex A, this angle is increased by 30º (when the hyperbolic vertex lies on one of the s-lines, the effect is half of what it would be if the vertex were strictly between the s-lines). The angle is increased further by 60º, since the hyperbolic vertex B lies between the s-lines. Therefore, the angle of *c* is eventually more than 180º relative to *b*. It follows that the angle of *c* is greater than 90º relative to *a*, and *a* and *c* are parallel. This would be true of any s-line through P that passed to the right of A. It would be equally true of any s-line through P that passed to the left of A. Therefore, only one s-line through P intersects the s-line *a*, and P is extremely hyperbolic relative to *a*.

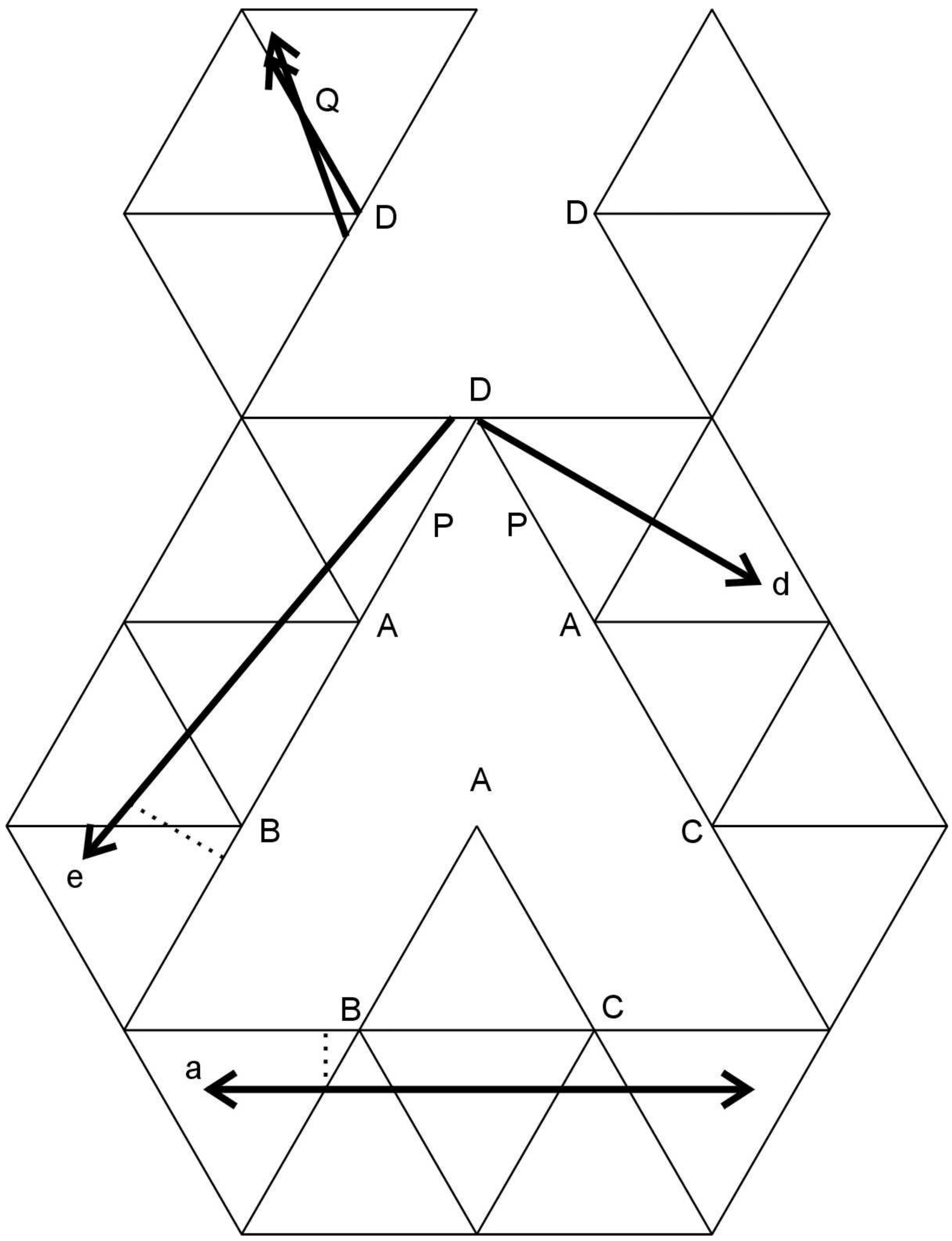


Figure 30. The point Q is completely hyperbolic relative to the s-line *a*.

Completely hyperbolic points

A completely hyperbolic point would have no non-parallels relative to some s-line *a*. The space shown in Figure 29 contains completely hyperbolic points relative to the s-line *a*. One of these is shown in Figure 30.

In Figure 30, the point Q has four hyperbolic vertices between it and the s-line *a*. It is also off the s-line *b* from Figure 29. The s-line *d* passes through the hyperbolic vertex D, and then passes to the right of A. The s-line *d* is clearly parallel to *a*, as is any line through Q to the right of *d*. Any s-line through Q that passes to the left of the hyperbolic vertex D and also the vertex A is also parallel to the s-line *a*. Therefore, every s-line through P is parallel to the s-line *a*, and P is completely hyperbolic relative to *a*.

Question. We have an example here of an extremely hyperbolic point with exactly one non-parallel. It seems reasonable to expect that some configuration like that used in the finitely hyperbolic example could yield exactly two or exactly three non-parallels. Does such a configuration exist?

Chapter 3. Smarandache Geometries

The reasons for investigating s-manifolds grew out of the search for examples of several particular types of Smarandache geometries proposed by Smarandache **[19]**. This chapter presents examples and partial examples of s-manifold Smarandache geometries in these categories.

Paradoxist Geometries

Smarandache called a geometry **paradoxist** if there are points that are elliptic, Euclidean, finitely hyperbolic, regularly hyperbolic, and completely hyperbolic **[19]**. We will add extremely hyperbolic to Smarandache's definition of a paradoxist geometry. We will also say that a geometry is **semi-paradoxist**, if it has Euclidean, elliptic, and regularly hyperbolic points.

Smarandache asked (see question 21 of **[19]**), "Let's have the case of Euclid + Lobachevsky + Riemann geometric spaces (with corresponding structures) into one single space. What is the angles sum of a triangle with a vertex in each of these spaces equal to? And is it the same anytimes?" He perhaps envisioned a space that has elliptic regions and hyperbolic regions. We will see, and we have seen, that in our s-manifolds, there really are not regions where certain properties hold, but properties are determined by the relationships between objects. For example, a point is elliptic relative to an s-line, if there is an elliptic vertex in between. Smarandache might view this as a pleasant surprise.

Paradoxist s-manifolds

A paradoxist geometry will have points that are Euclidean, elliptic, and finitely, regularly, extremely, and completely hyperbolic. We have seen that all of these occur in s-manifolds. It is also clear that these phenomena result from non-Euclidean vertices lying between the points and s-lines in question. There generally will be Euclidean points in an s-manifold, and it will be typical for elliptic and regularly hyperbolic points to exist around elliptic and hyperbolic vertices. The idea used to construct an s-manifold with a finitely hyperbolic point used two elliptic and two hyperbolic vertices. One hyperbolic vertex was used to create a 60° continuum of parallels, and the other was used to split one of the boundary parallels away from the other. One elliptic vertex essentially cancelled out the effect of the splitting hyperbolic vertex, and the other turned all but one of the parallels 60° towards the base s-line. This left both boundary parallels at the same angle, and all the in-between parallels at an angle headed towards the base s-line.

The idea used to construct an s-manifold with an extremely and completely hyperbolic point used four hyperbolic vertices. One hyperbolic vertex was used to split the s-lines through the hyperbolic point so that there was one isolated s-line in the middle of a 60° gap. Two more hyperbolic vertices in the gap increased the angle to 180°. A base s-line, therefore, could be found that intersected only the isolated s-line. To increase one of the 30° gaps on either side of the isolated s-line to 180° or more, three hyperbolic vertices were needed. This idea of spreading the gap to 180° or more, therefore, seems to need four hyperbolic vertices to get a completely hyperbolic point.

Question. Is it possible to have finitely hyperbolic points in an s-manifold with fewer than two elliptic and two hyperbolic vertices? Is it possible to have a completely hyperbolic point in an s-manifold with fewer than four hyperbolic vertices?

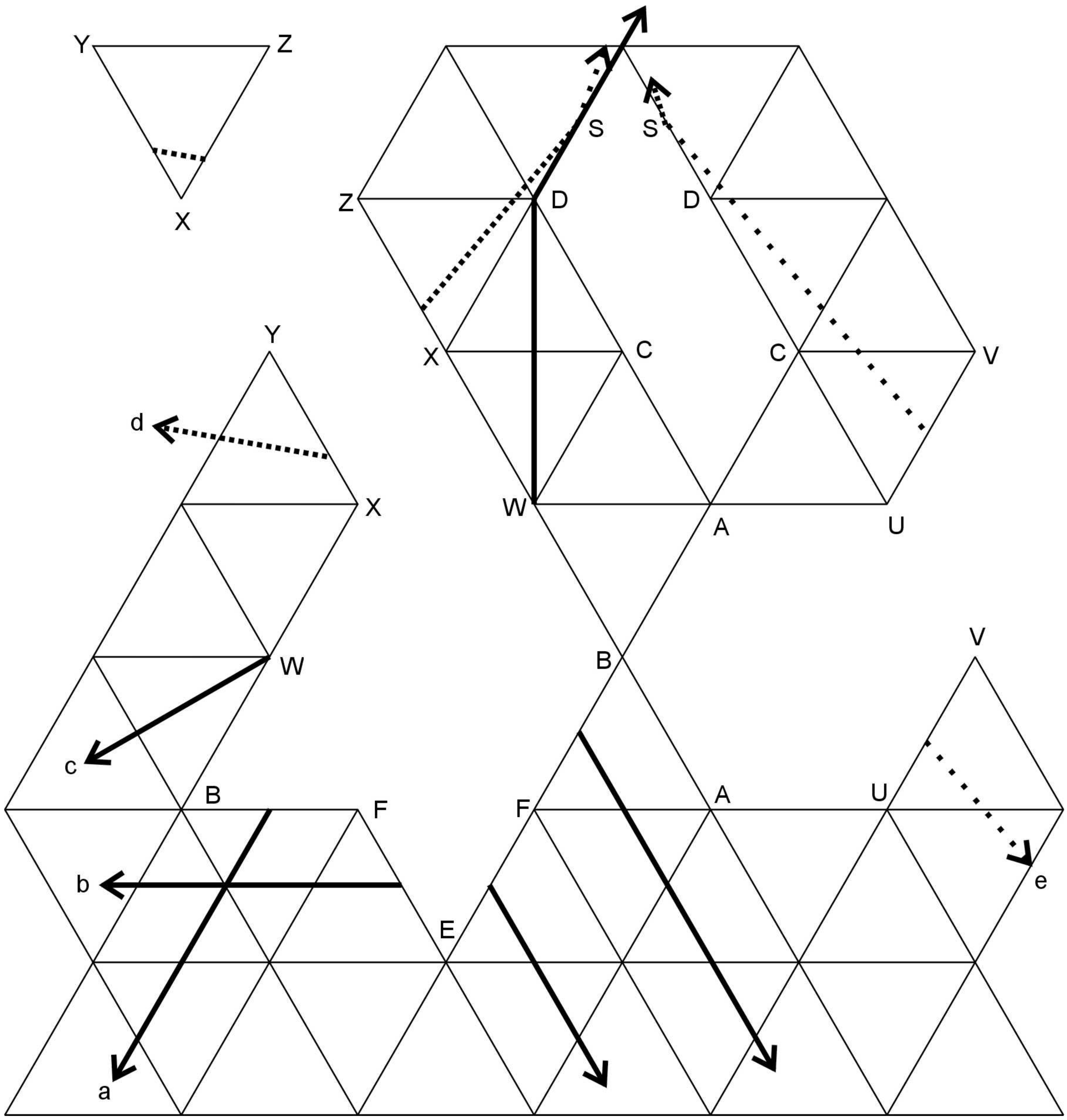


Figure 1. The point S is extremely hyperbolic relative to *b* and completely hyperbolic relative to *a*.

From what is known, we can construct an s-manifold with finitely hyperbolic points using two elliptic and two hyperbolic vertices, and one with completely hyperbolic points using four hyperbolic vertices. It may, therefore, be the best we can do to construct a paradoxist s-manifold with two elliptic and four hyperbolic vertices. It could be the case, of course, that the answers to both of the questions just posed is no, but that there is still a paradoxist s-manifold with fewer than two elliptic and four hyperbolic vertices.

In any case, it is possible to construct a paradoxist s-manifold with two elliptic and four hyperbolic vertices. One is shown in Figures 1, 2 and 3.

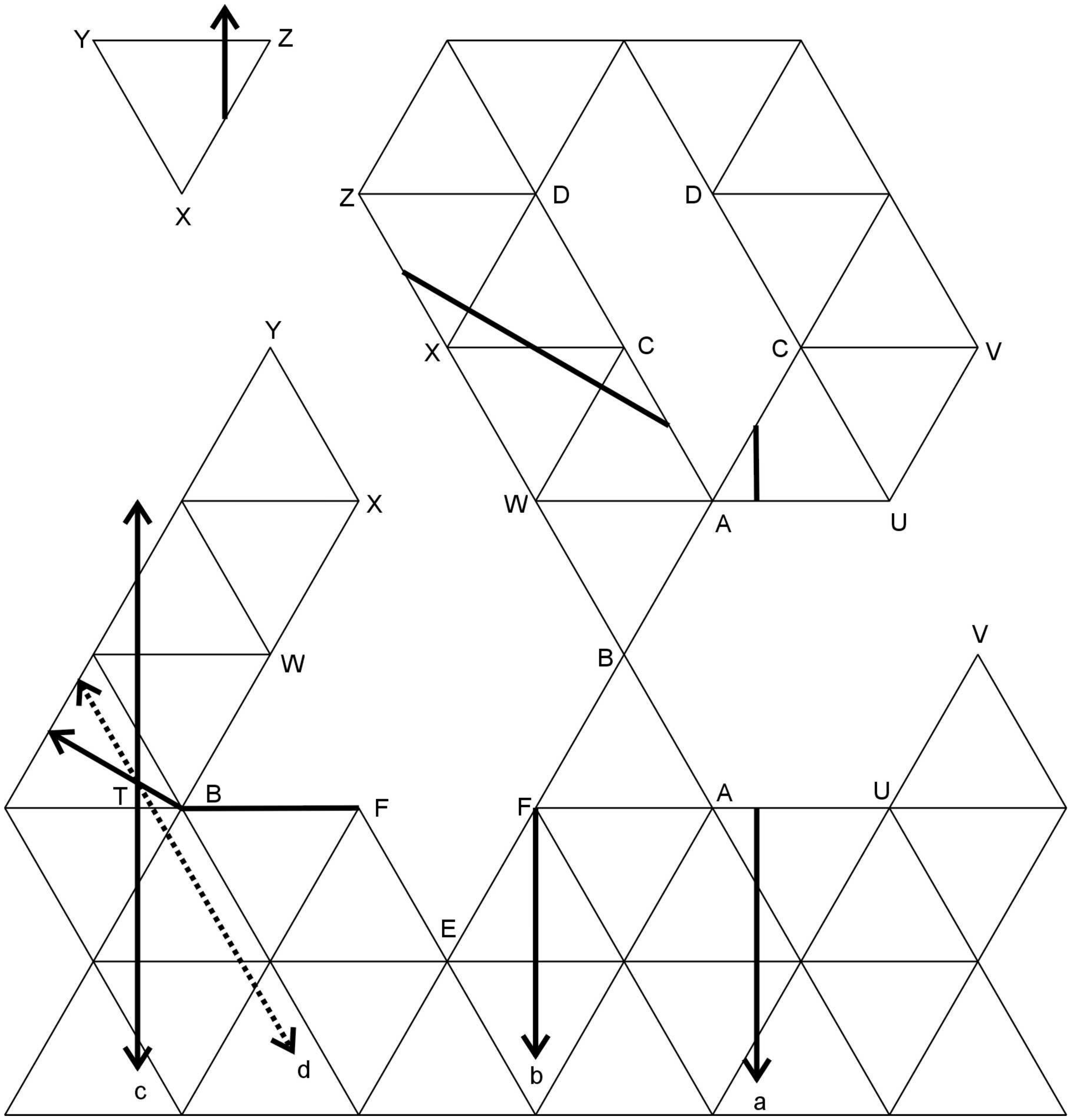


Figure 2. The point T is finitely hyperbolic relative to the s-line *a*.

In Figure 1, the point S is completely hyperbolic relative to the s-line *a*. It is necessary here that the s-line *a* lies between the four hyperbolic and the two elliptic vertices. Otherwise, the elliptic vertices could cancel the effect of the hyperbolic vertices. It is also necessary that the hyperbolic vertices A, B, and C lie inside the gap formed at vertex D between the isolated s-line *c* and the s-lines like *e* that pass to the right of D. Towards the left, the s-line *c* has an angle of 120° relative to the s-line *a*, and since the region between them is Euclidean outside of the diagram, this angle will be preserved and the s-lines will not intersect. All of the s-lines through S that pass to the left of D, like the s-line *d*, will have even greater relative angles, so these will not intersect *a* either. Towards the right, all of the s-lines through S that pass to the right of D will have an angle that is greater than 90° relative to *a*, and these will not intersect *a*.

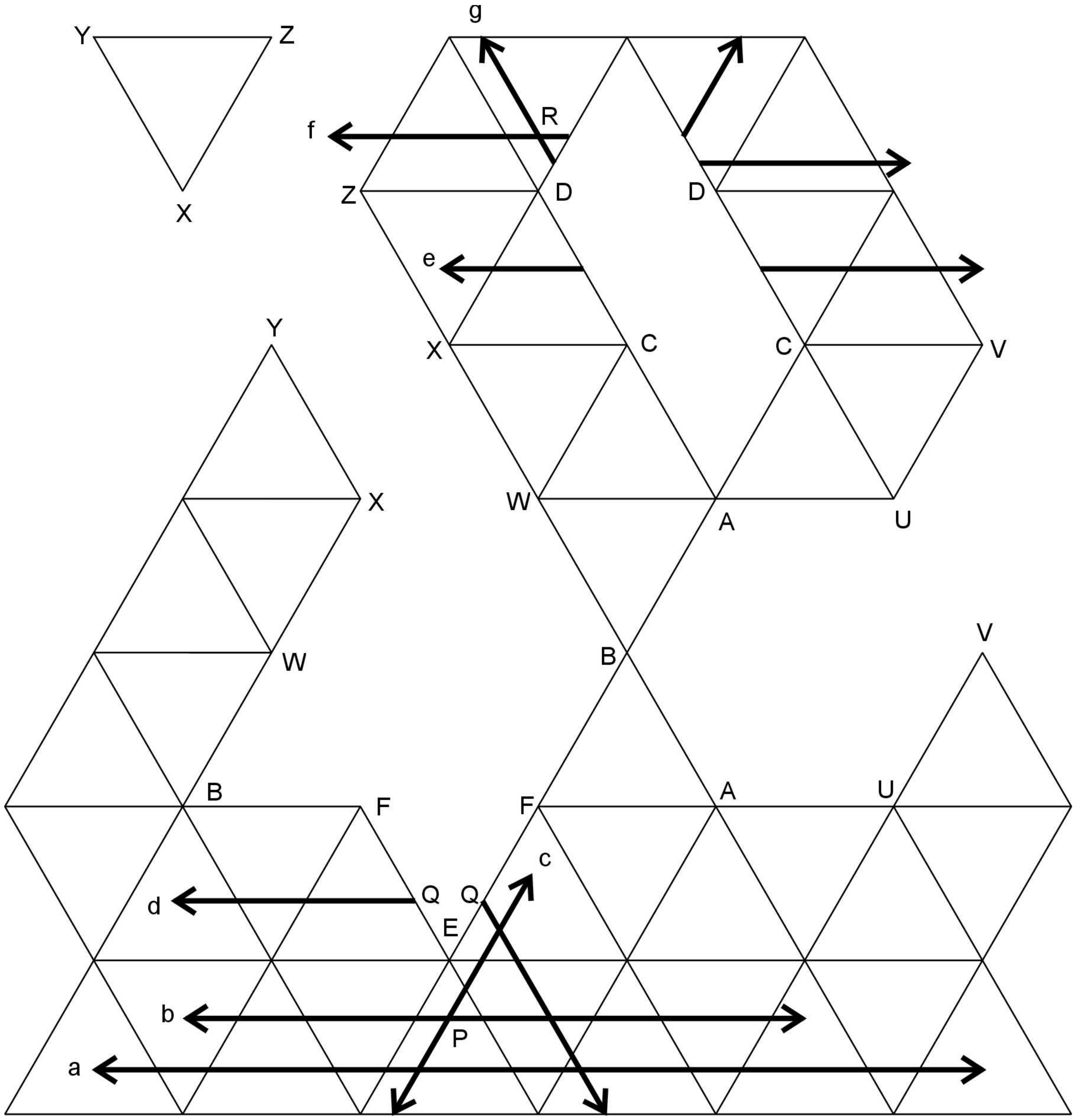


Figure 3. The point P is Euclidean and Q is elliptic relative to *a*. R is regularly hyperbolic relative to *e*.

Also in Figure 1, the s-line *b* lies below the elliptic vertex F. This cancels the effect of one of the hyperbolic vertices, and the isolated s-line *c* intersects *b*, so S is only extremely hyperbolic relative to *b*.

In Figure 2, the s-line *a* lies between the two hyperbolic vertices C and D and the other non-Euclidean vertices. This allows the two elliptic and two hyperbolic vertices to act as they would alone, and here the point T is finitely hyperbolic relative to *a*. If only the hyperbolic vertex A were present, the s-lines *b* and *c* would be the “last” parallels through T. Towards the top of the diagram, both *a* and *c* are vertical in the Euclidean region, and these will not intersect. Clearly any s-line through T and between *c* and the vertex B will intersect *a*. Due to the effects of the two elliptic vertices E and F, the angles of both *b* and *c* are 90° relative to *a* towards the bottom of the diagram. Any of the s-lines between *b* and *c*, like the s-line *d*, will have relative angles less than 90° and will intersect *a*. Therefore, only *b* and *c* are parallel to *a*, and T is finitely hyperbolic relative to *a*.

Figure 3 shows how Euclidean points, like P, arise naturally within bands of adjacent triangular disks. If the angle of an s-line through P is less than 90° to the right relative to *a*, it will intersect *a* on the right within this band that extends infinitely in either direction. If this relative angle is greater than 90°, the s-line will intersect on the left.

On the other hand, elliptic points, like Q, and regularly hyperbolic points, like R, are common around elliptic and hyperbolic vertices. The band of triangles just above the one mentioned is Euclidean everywhere, except for the triangular disk that is missing at the vertex E. The effect here is that any s-line through Q will have its relative angle reduced by 60° as it passes E. Therefore, any s-line through Q will have an angle less than 90° relative to *a* on at least one side of E. Having all of the other non-Euclidean vertices outside of these two bands makes them irrelevant to whether Q is elliptic or not. A similar case can be made for the hyperbolic point R relative to the s-line *e*.

Question. The paradoxist s-manifold just described has two elliptic vertices and four hyperbolic vertices. What is the minimum possible number of non-Euclidean vertices in a paradoxist s-manifold?

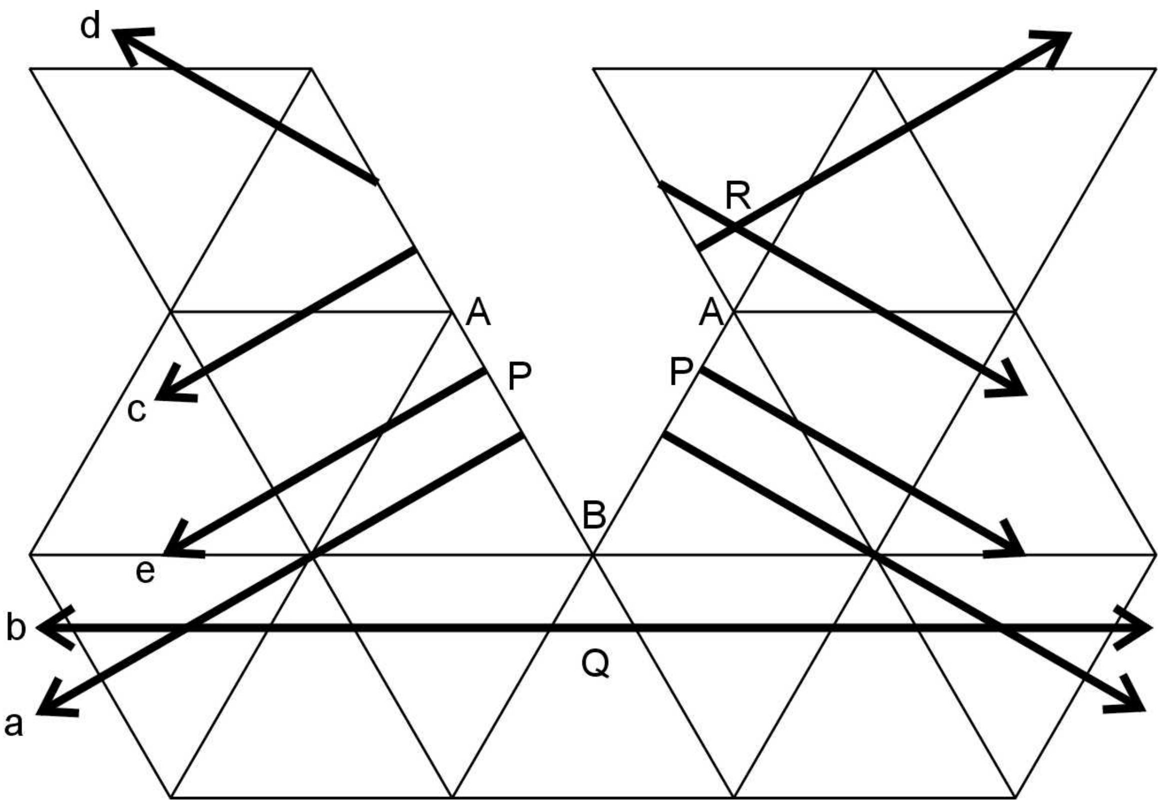


Figure 4. A semi-paradoxist s-manifold.

Semi-paradoxist s-manifolds

A semi-paradoxist geometry will have points that are Euclidean, elliptic, and regularly hyperbolic. The paradoxist s-manifold just presented is, of course, also semi-paradoxist, as can be seen in Figure 3. The ideas in obtaining the finitely, extremely, and completely hyperbolic points, however, seem to be peculiar to s-manifolds, since they consider lines that pass through non-Euclidean vertices.

We can illustrate a simpler semi-paradoxist s-manifold whose properties can be reproduced in a smooth manifold. As seen in Figure 3, the presence of Euclidean points is almost automatic, and hyperbolic and elliptic points come along with elliptic and hyperbolic vertices. It seems that the simplest semi-paradoxist s-manifold would have a single elliptic and a single hyperbolic vertex. In particular, an s-manifold that is topologically equivalent to the plane will be semi-paradoxist if all vertices are Euclidean except for exactly one elliptic and one hyperbolic vertex. An example of an s-manifold of this type is shown in Figure 4.

In Figure 4, the point P is Euclidean relative to the s-line *a*. The one parallel through P is the line *e*. The point Q is elliptic relative to *a*. The point R is regularly hyperbolic relative to *a*. The s-lines *c* and *d* are the “last” parallels, and all the s-lines between *c* and *d* are also parallel.

Question. Is the presence of both an elliptic and a hyperbolic vertex sufficient to guarantee that an s-manifold is at least semi-paradoxist? If not, are there additional conditions that would? Are there semi-paradoxist s-manifolds with fewer than two non-Euclidean vertices?

A planar s-manifold with one elliptic and one hyperbolic vertex is semi-paradoxist. This is achieved by referring only to s-lines and points away from the non-Euclidean vertices. These spaces, therefore, remain semi-paradoxist even after smoothing the two non-Euclidean vertices. It follows from this that there are semi-paradoxist geometries among the class of smooth surfaces. This, of course, is not surprising, since the curvature, and therefore the geometry, can vary on a smooth surface. This is not a completely trivial observation, however, since defining Euclidean and non-Euclidean geometry in terms of parallel lines does not correspond exactly to definitions in terms of curvature.

Euclidean theorems

It would be most interesting to find general theorems for s-manifolds that are peculiar to s-manifolds and that somehow capture the essence of an s-manifold. This should always be a goal, but if this is possible, we should expect it to come as a result of having a deeper understanding. One obvious possibility for exploration lies in comparing the geometry of s-manifolds to Euclidean geometry, and this should offer opportunities to understand both kinds of geometry better. Along these lines of thought, each theorem of Euclidean geometry is an object ready for an s-manifold analysis. Here, we will consider one example.

The alternate interior angles theorem is an important one, so we will look at it. Note that there are actually two alternate interior angle theorems. One has congruent angles implying parallel lines, and the other has parallel lines implying congruent angles. We consider the first, which is independent of Hilbert's parallel axiom and can be used to establish the existence of parallels.

(Euclidean) alternate interior angles theorem. If two lines *a* and *b* are cut by a transversal *c* such that alternate interior angles are congruent, then *a* and *b* are parallel.

Even in the statement of this theorem, there is an s-manifold configuration that does not exist in Euclidean geometry. It is possible that, of the two possible, one pair of alternate interior angles is congruent and the other is not. For example, if *a* meets *c* at a hyperbolic vertex, the two interior angles there will sum to 210°. Since the other pair of interior angles may sum to 180°, we can have one pair of congruent alternate interior angles measuring 80° each and one pair of non-congruent alternate interior angles measuring 130° and 100°.

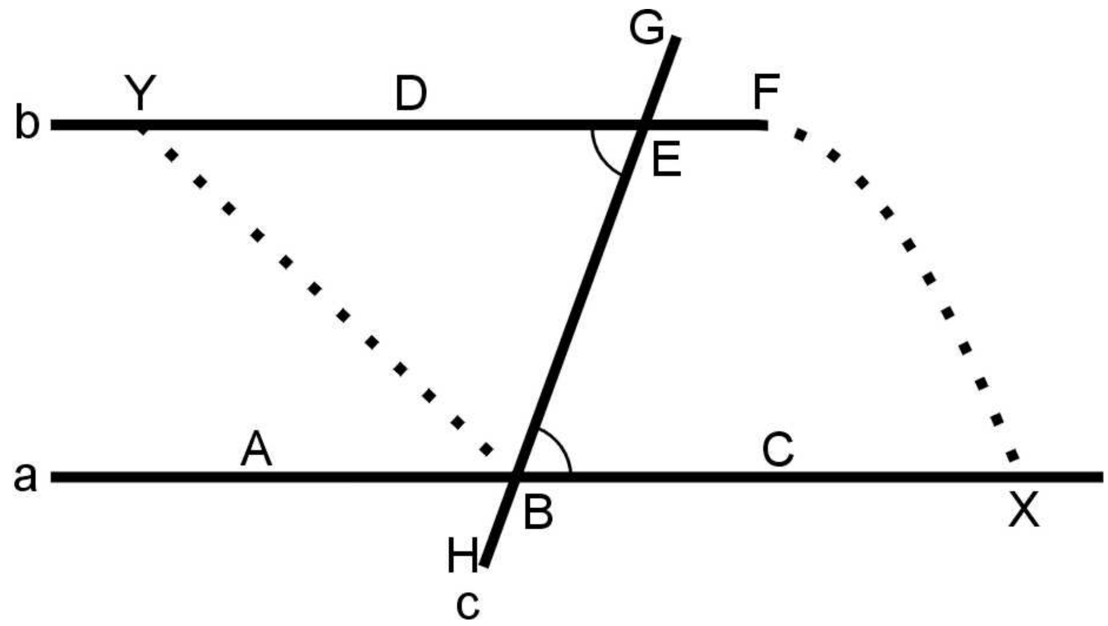


Figure 5. Alternate interior angles are congruent.

A typical proof of this theorem might go as follows (see **[12]**). We have lines *a* and *b* cut by a transversal *c*, as in Figure 5. Suppose angles ∠CBE and ∠DEB (alternate interior angles) are congruent, and suppose that the lines *a* and *b* are not parallel. The lines *a* and *b* must therefore intersect in some point X. There must also be a point Y on *b* such that Y is on the opposite side of *c* from X and the segment EY is congruent to the segment BX. By the SAS axiom, angles ∠EBY and ∠BEX must be congruent. Since the alternate interior angles ∠BEX and ∠EBA must be congruent, we have that ∠EBA and ∠EBY are congruent. It follows that the rays BY and BA must be the same, and therefore, the lines *a* and *b* also intersect at the point Y. Since the lines *a* and *b* cannot intersect in two distinct points, we must have that they are parallel (or coincident).

There are several parts of this proof that may not be valid for s-lines in an s-manifold. First of all, since the other pair of alternate interior angles need not be congruent, the point X needs to lie on the ray BC. Next, having X and Y on opposite sides of an s-line *c* might not make sense in a closed or non-orientable s-manifold (see the next chapter). If X lies on the ray BC, however, we can require that Y lie on the ray ED. We have also seen that the SAS axiom need not hold in an s-manifold. The existence of multiply aligned points allowed a counter-example, but there might be other ways that this axiom fails. That angles ∠EBA and ∠EBY are congruent assumed that both pairs of alternate interior angles are congruent, and this might not be true in an s-manifold. Finally, two distinct s-lines can share two or more points, so we would not necessarily have a contradiction.

Let us look at the semi-paradoxist s-manifold presented in the previous section to see how this proof holds up there, and to see how this theorem might be rephrased to make it true in this particular s-manifold. From a mathematician's point of view, it would seem natural to try to make a statement that is sometimes true into a theorem by imposing additional conditions.

A fairly obvious sufficient condition (i.e., a condition strong enough to make the statement true, but perhaps stronger than necessary) is a requirement that there be no non-Euclidean vertices between the s-lines *a* and *b*. This would make the region around the pair of lines essentially Euclidean. It is not clear how best to define what it means to be between two s-lines, but since all s-lines separate this s-manifold, we can use the following.

Theorem. (In the s-manifold of Figure 4) Suppose two s-lines *a* and *b* are cut by a transversal *c* such that alternate interior angles are congruent and the point B of Figure 5 is on the opposite side of *b* from both non-Euclidean vertices, then *a* and *b* are parallel.

The added condition guarantees that neither non-Euclidean vertex lies on either *a* or *b*. Therefore, both pairs of alternate interior angles must be congruent. If we suppose that the s-lines *a* and *b* are not parallel, then there is at least one point X common to both. Choose X so that its distance from B along *a* or its distance from E along *b* is a minimum, and without loss of generality, suppose that it lies as in Figure 5. We can then choose Y as in the proof given above. Our added condition and choice of the closest X guarantees that the triangles EBX and BEY have Euclidean interiors, and the SAS theorem must hold for these triangles. It follows that ∠EBY and ∠EBA are congruent, and that Y must lie on *a*. We now have that X and Y are vertices of a non-degenerate 2-gon with Euclidean interior, which is a contradiction.

The theorem can be improved by weakening the added condition. Some insight into this might come with the following observations. If two s-lines *a* and *b* are parallel in the s-manifold of Figure 4, then it is intuitively obvious that the two s-lines divide this s-manifold into three regions and exactly one of these lies between the two s-lines. One of the following must be true. The elliptic vertex does not lie on or between *a* and *b*, the elliptic vertex lies on *a* or *b* and the hyperbolic vertex lies on *a* or *b* or between, or both non-Euclidean vertices lie between *a* and *b*.

Problem. Find a necessary and sufficient added condition for the alternate interior angles theorem in the s-manifold of Figure 4. Is there one that works for any s-manifold?

Smarandache Non-Geometries

Smarandache defined a non-geometry to be one that Smarandachely denies each of the five postulates of Euclid **[19]**. These are **[9]**:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

We would like to find an s-manifold that is a Smarandache non-geometry, and we will start the search by looking at the postulates one by one.

Postulate I

It is quite normal to expect that given two points in an s-manifold, there is an s-line through them. We have seen, however, that there are pairs of points that do not lie on a single s-line. We called these points 0-aligned or remote, and we have seen these around the hyperbolic vertex of the hyperbolic cone-space. It is also generally assumed in this context that the straight line postulated is unique. This would be false around an elliptic vertex, where we have 2- and 3-aligned pairs of points. We can be quite sure, therefore, that any s-manifold with non-Euclidean vertices will S-deny postulate I.

Postulate II

In regards to this postulate, Smarandache asks that, “It is not always possible to extend by continuity a finite line to an infinite line.” This postulate has been interpreted to mean that any line segment can be extended indefinitely, and in the context of manifolds in particular, this does not imply that the line must be infinite. For example, an arc of a circle can be extended indefinitely, while tracing the circle an infinite number of times. The circle, however, is not itself infinite.

Depending on our interpretation, two ways of S-denying this postulate come to mind. One is to find an s-manifold that has s-lines that are closed like a circle. Another is to introduce the concept of a boundary to an s-manifold. Here, an s-line may extend up to this boundary, but since the space does not continue, the s-line cannot either. We will consider both.

Postulate III

In order it violate this postulate, we would need a center and radius that does not correspond to a circle, or in our s-manifold terminology, that the s-circle corresponding to this center and radius does not exist. The definitions of an s-proto-circle and s-circle were formulated with this postulate in mind. Let us look at Euclid's definition. In **[9]** we see, "A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying with the figure are equal to one another; and the point is called the centre of the circle." Here, Euclid uses the word *line* in the sense that we would use *curve*. We defined the set of points that have an s-segment between it and the center of a fixed length to be an s-proto-circle, and this corresponds roughly with the "straight lines falling upon it from one point." We then want that "A circle is a plane figure contained by one line," so we defined that an s-proto-circle is an s-circle, if it also is a simple closed curve. It seems natural, therefore, to say that if an s-proto-circle is in fact an s-circle, then the s-circle exists.

Two ways that an s-circle can fail to exist are as follows. If the center is near a boundary of an s-manifold with boundary and a radius is larger than the distance from the point to the boundary, then a section of the s-proto-circle will be missing, and it will not be closed.

Another way arises for a center near a non-Euclidean vertex. Around an elliptic vertex, an s-proto-circle can have a self-intersection, endpoints, and an isolated point as shown in Figure 24 of Chapter 1. Around a hyperbolic vertex, as shown in Figure 6, an s-proto-circle can have gaps and an isolated point, since there is a region of 0-aligned points on the other side of the hyperbolic vertex from the center, and only one radial s-segment can enter this region. Again in this case, the s-proto-circle will not be closed, and the s-circle does not exist.

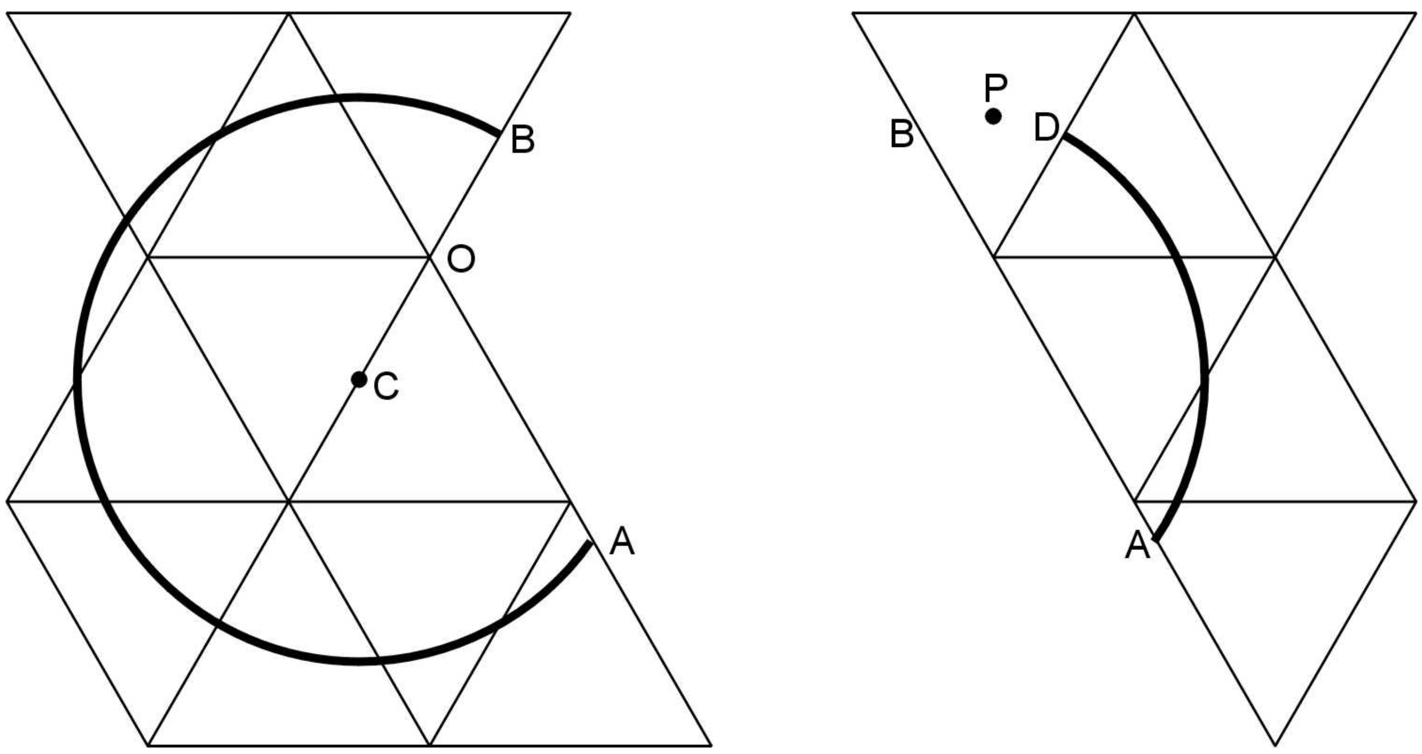


Figure 6. An s-proto-circle can have gaps near a hyperbolic vertex.

Postulate IV

Smarandache mentions a fairly standard definition for right angles, “a right angle is an angle congruent to its supplementary angle” **[19]**. We will use this definition as well. Therefore, an **s-right angle** is congruent (has the same angle measure) to its supplementary angles (two supplementary angles make a straight angle). Since there are 300° around an elliptic vertex and 420° around a hyperbolic vertex, congruent supplementary angles at an elliptic vertex will measure 75°, and at a hyperbolic vertex they will measure 105°. Since congruence has been defined to coincide with angle measure, not all s-right angles are congruent, since some are 75°, some are 90°, and some are 105°.

Postulate V

Here, we need in some instances pairs of s-lines cut by a transversal so that interior angles on one side of the transversal add up to less than two s-right angles and also do not intersect on that side. Of course, since not all right angles are congruent, it is not completely clear what “less than two s-right angles” means. Whether a single angle is less than an s-right angle or not is well-defined, so two angles less than an s-right angle or one s-right angle and one angle less than an s-right angle will clearly satisfy the conditions of the postulate. Also, if both angles emanate from points other than non-Euclidean vertices, it is easy to determine if the conditions are met.

A non-geometry s-manifold with boundary

Most of the required properties of a non-geometry can come as a result of a boundary. The example here is similar to that given by Smarandache **[19]**. Our definition of an s-manifold does not allow for a boundary. We can define an **s-manifold with boundary** by allowing in the definition of an s-manifold that some edges of triangular disks need not be identified. We will require that vertices of non-identified edges share at most four triangular disks and exactly two non-identified edges. We will also require that two non-identified edges share at most one vertex. The space will end at these boundary edges and vertices.

In Figure 7, we have an example of an s-manifold with boundary that is a non-geometry. The space stops at the boundary marked with heavy dots, but continues with Euclidean vertices around the rest of the diagram.

Postulate I is S-denied, since the points C and D cannot be joined by an s-line, but other pairs of points like I and J can be.

Postulate II is S-denied, since the segment EF cannot be continued further, while there are s-lines that can be extended to infinity. Any s-line parallel to the segment JK and between segment JK and the boundary can be extended to infinity.

Postulate III is S-denied, since the s-proto-circle shown with center G cannot be completed. A smaller s-circle centered at G or larger s-circles centered at points elsewhere in this space clearly do exist.

Postulate IV is S-denied, since angles JBI and angles HBI are supplementary and congruent. They are therefore s-right angles even though they measure 75º. The s-right angles at points other than B measure 90º.

Finally, Postulate V is S-denied, since the s-line LI and EF are cut by a transversal JK so that the sum of the angles LJK and JKE is less than 180º, but the s-lines LI and EF do not intersect. Clearly, there are s-lines meeting these conditions that do intersect on the side of the transversal with the angles summing to less than two s-right angles.

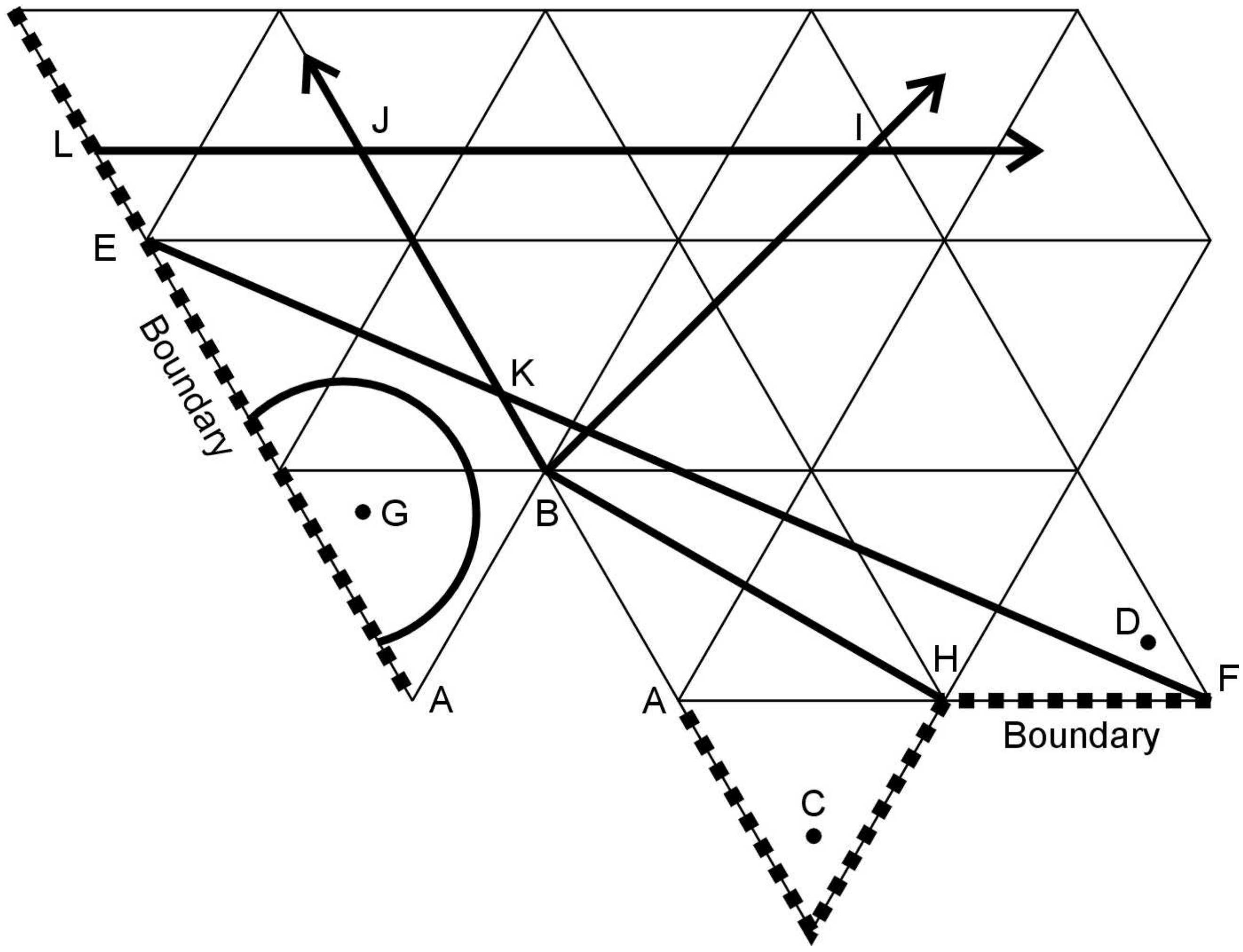


Figure 7. A non-geometry s-manifold with boundary.

A non-geometry s-manifold without boundary

We can construct an s-manifold without using a boundary by using the interpretation that an s-line that is closed like a circle is not continuously extendable. It is certainly not extendable to infinity, as required by Smarandache, since a circle has a finite length.

An example of a non-geometry s-manifold without boundary is shown in Figures 8 and 9. Figure 8 shows what is essentially the Euclidean plane with a triangle cut out. Attached to this triangular hole is the ring shown in the bottom of Figure 9. The ring shown in the top of Figure 9 sits on top of this. Similar rings are stacked indefinitely on top of this. All vertices are Euclidean except for the six hyperbolic vertices A, B, C, D, E, and F joining the planer **base** to the vertical cylinder, which we will call the **tube**. The band of triangular disks between the hyperbolic vertices will be called the **collar**.

Postulate I is S-denied, since there are 0-aligned points around the hyperbolic vertices, like the points H and I. Most other pairs of points are at least 1-aligned. There are also 2-aligned points, like H and J, and ∞-aligned points on the tube.

Postulate II is S-denied, since s-lines like LT around the cylindrical part of the space cannot be extended "to infinity." These are essentially circles, which have no endpoints, but have finite length. Except for those s-lines on the tube that are horizontal, all other s-lines can be extended to infinity.

Postulate III is S-denied, since some s-proto-circles, like the one shown centered at M, have gaps in them, and do not exist as s-circles. Recall that an s-circle is an s-proto-circle that is a simple closed curve.

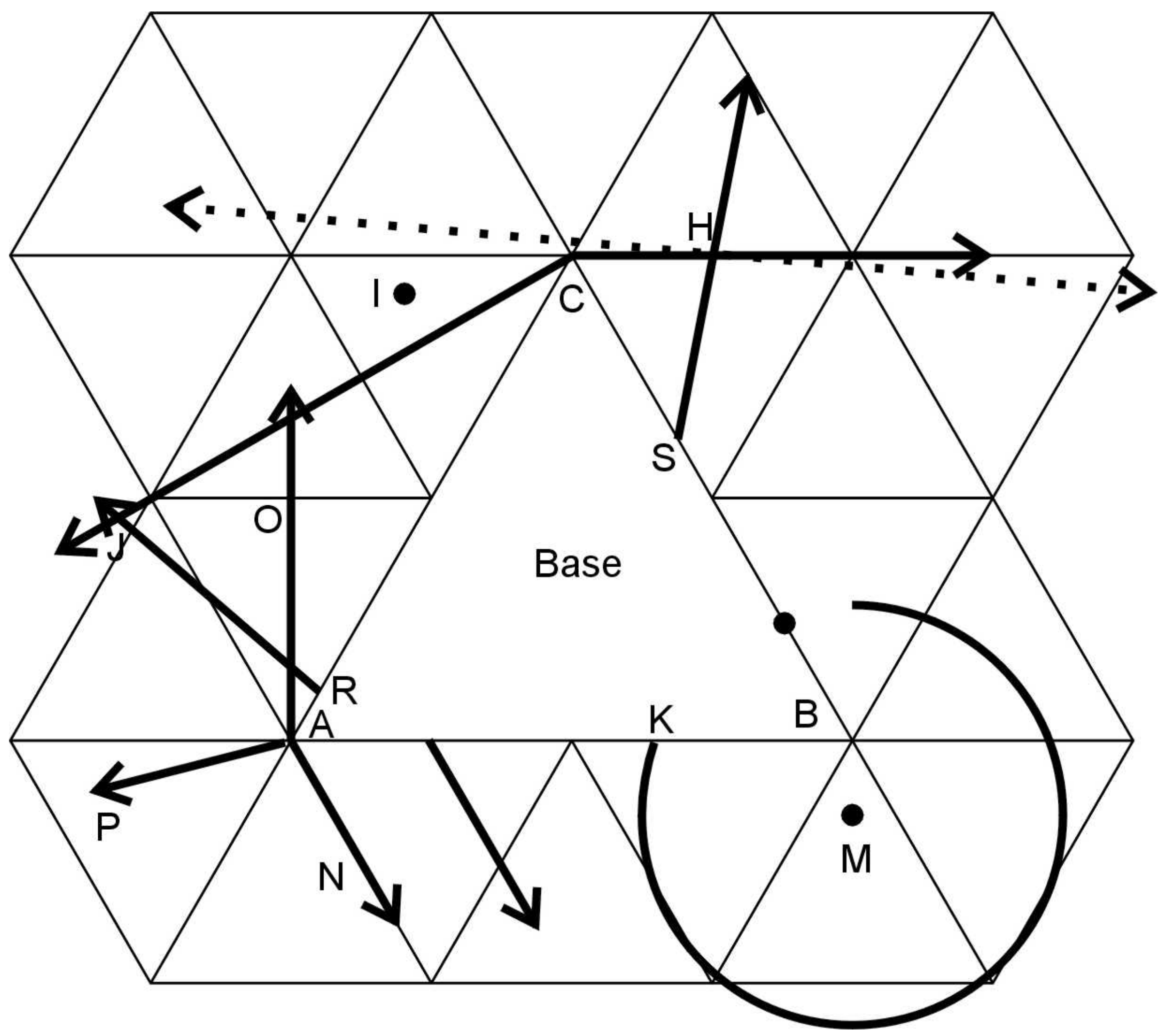


Figure 8. This is the base of the non-geometry s-manifold without boundary.

Postulate IV is S-denied, since some s-right angles do not measure 90°. For example, at the hyperbolic vertex A, the congruent supplementary angles (the definition of a right angle) ∠OAP and ∠NAP, measure 105°. The s-right angles at all points that are not non-Euclidean vertices measure 90°.

Postulate V is S-denied, since there are pairs of s-lines that satisfy the conditions of postulate V, but do not intersect. For example, the s-lines LT and US are cut by the transversal UT such that ∠LTU on the right and ∠TUS sum up to less than two right angles, but do not intersect to the right (or to the left). This is clear, since the line US never enters the tube. Except for the triangular cutout, the base is essentially the Euclidean plane, so there are many examples of lines that satisfy this postulate.

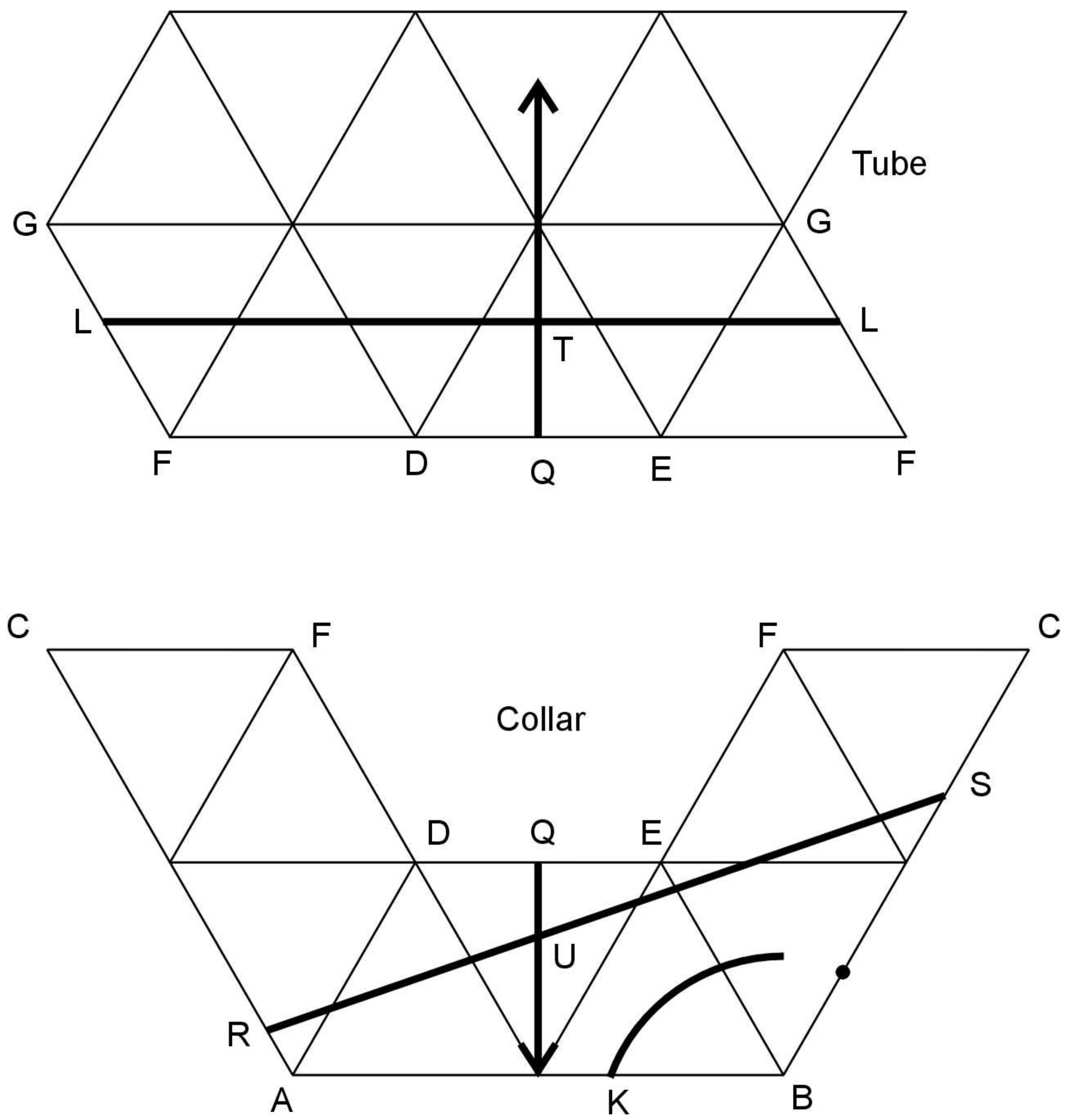


Figure 9. This is the collar and tube for the non-geometry s-manifold without boundary.

Other Smarandache Geometries

Anti-geometries

Smarandache defines an anti-geometry to be one that S-denies all of Hilbert's axioms for Euclidean geometry **[19]**. These axioms apply to three-dimensional spaces, so we would be interested here in only the two-dimensional axioms. These were discussed in the previous chapter. There we saw that axioms I-3, II-3, and III-2 hold in every s-manifold, so there can be no s-manifold anti-geometry. Combining the examples given, it should be easy to construct an s-manifold that S-denies the remaining axioms. This would be a repetition of what has already been discussed. We can, of course, go a bit further by using q-congruence. Also by using s-manifolds with boundary and extending the definition of an s-line, it is quite plausible that an anti-geometry could be constructed. See the counter-projective s-manifold below, and the post of Mike Antholy at **[2]**.

Counter-projective geometries

Smarandache's definition for a counter-projective geometry requires that the following axioms be S-denied **[19]**.

I. Given two distinct points, there is a unique line through them.
II. Given three non-collinear points P, Q, and R, and two distinct points S and T such that S lies between P and Q, and T lies between P and R, then the line QR intersects the line ST.
III. Every line contains at least three distinct points.

Axiom III is always true in an s-manifold, since all lines have infinitely many points. Axiom I is S-denied in virtually every s-manifold, but there are examples of s-manifolds, like the s-sphere to be defined later, where every pair of points is multiply aligned. Axiom II is generally false in an s-manifold. It can hold in certain circumstances, however.

In order to S-deny Axiom III, we will use a boundary in the example given here. It may seem appropriate that a counter-projective geometry have a boundary, since projective geometries are closed (in the sense described in the next chapter). We will also introduce enough structure so that each of the axioms is true in some cases and false in different ways. We also define s-lines differently from before. Here, s-lines are constructed as if the space were extended beyond the boundaries with Euclidean vertices. This will allow s-lines with only finitely many points.

The presence of both elliptic and hyperbolic vertices will force the existence of remote, uniquely aligned, and multiply aligned pairs of points. Therefore, Axiom I will be satisfied in some cases and denied in other cases, both by the non-existence and the non-uniqueness of the line.

In Euclidean geometry, the lines QR and ST of Axiom II will sometimes intersect and sometimes not intersect. This will occur for most triples P, Q, and R in this model, but there will also be triples where all possible lines ST intersect QR and triples where all of the lines ST do not intersect QR.

As for Axiom III, this model will have lines that contain exactly one, two, and three points, as well as lines that have infinitely many points.

In Figure 10, as we have seen before, there are pairs of points around the elliptic vertices W and X that are multiply aligned. For example, D and Y are 3-aligned, since there is an s-line to the left of W, to the right of W, and through W that pass through both D and Y. This pair of points, therefore, violates the uniqueness condition of Axiom I. We have also seen before that there are remote points around hyperbolic vertices. For example, the points U and L have no s-line through them. Here the existence condition of Axiom I is violated.

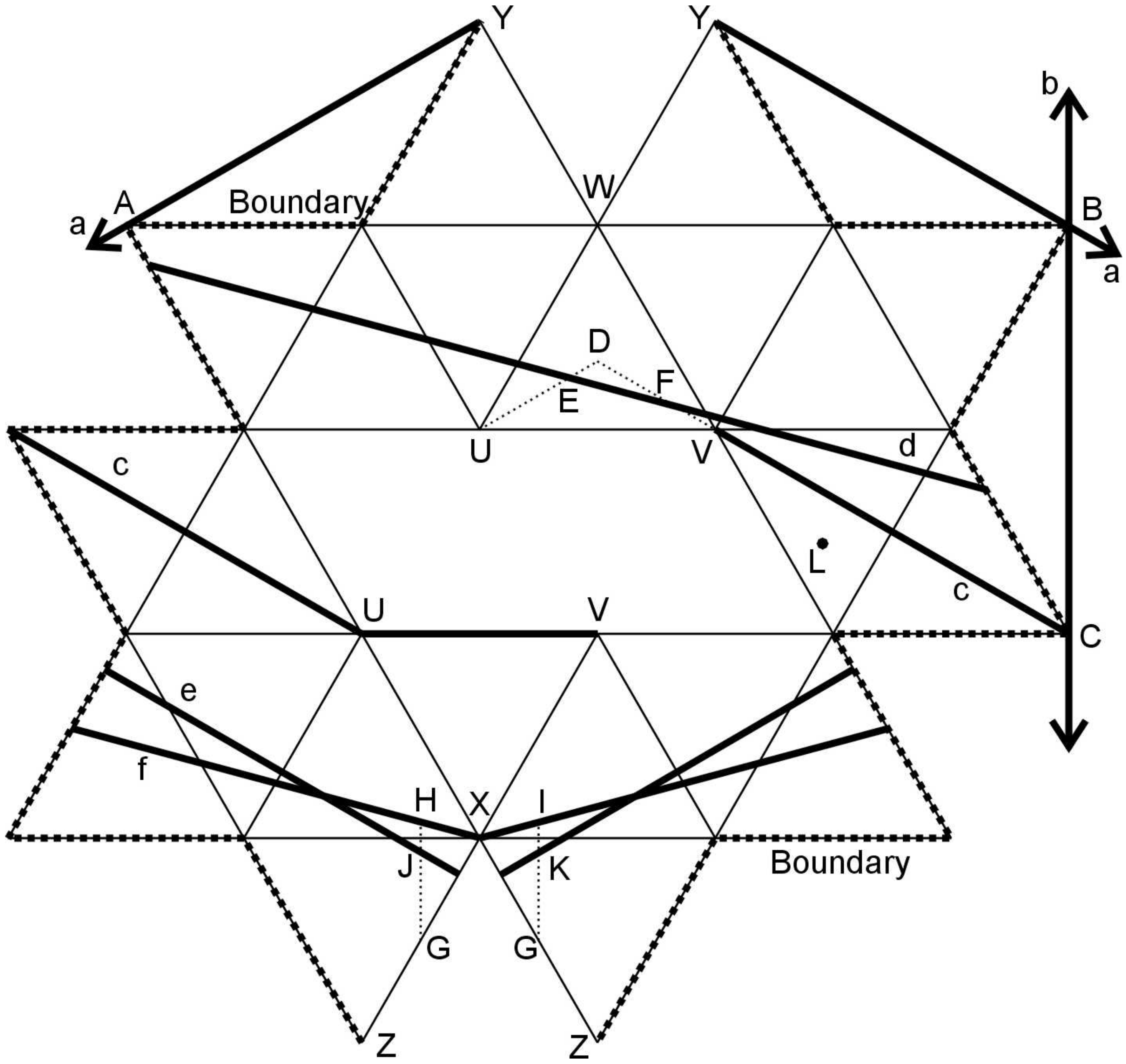


Figure 10. A counter-projective s-manifold.

Consider the non-collinear triple of points D, U, and V. The vertices U and V are hyperbolic, and the angles ∠DUV and ∠DVU both measure 30°. For any point E on segment DU and point F on segment DV, therefore, the angle of the s-line *d* through E and F relative to the s-line *c* (between U and V) must be greater than 60°. Since the s-line *d* must pass above the hyperbolic vertices U and V, the relative angles must increase by 30° to more than 90°. It follows that the s-lines *d* and *c* will never intersect.

For the non-collinear triple G, H, and I, a similar analysis shows that the s-lines *e* and *f* must always intersect. It is easy to find triples where the s-lines under consideration will sometimes intersect and sometimes not.

Finally, Axiom III is violated by s-lines like *a* and *b*. Here, the s-line *a* lies mostly outside of the boundary of this s-manifold, and so *a* contains only the points A, Y, and B. The s-line *b* contains only the points B and C. Clearly, there is an s-line that contains only B. All of the other s-lines shown, *c*, *d*, *e*, and *f*, contain infinitely many points

Having considered projective geometry (in terms of a counter-projective geometry), the concept of a dual geometry, where the ideas of points and lines are switched, presents itself (See J.M. Charrier's post at **[2]**.) It is doubtful that such a thing could be an s-manifold, but the dual of an s-manifold might be an interesting topic for further study.

Chapter 4. Closed s-Manifolds

The peculiar geometry of an s-manifold results in a variety of Smarandache structures. Being a manifold, we can also generate Smarandache structure using topology. Here we will look at some of the basic topological structures obtainable by closed s-manifolds.

Closed s-Manifolds

A manifold is closed, if it is compact and has no boundary. For an s-manifold, compactness is equivalent to an s-manifold consisting of a finite number of triangular disks. No boundary means that each edge is shared by exactly two triangular disks and each vertex is completely surrounded by triangular disks. Here the term *closed* is an extension to surfaces of the notion of a closed curve. For example, a surface would have to be closed in order for it to enclose a volume. A sphere or torus would be closed, but a flat disk or a hemisphere would not, since they have boundaries. For an s-manifold, being closed and being compact are equivalent, since s-manifolds have no boundary (although an s-manifold with boundary does have a boundary).

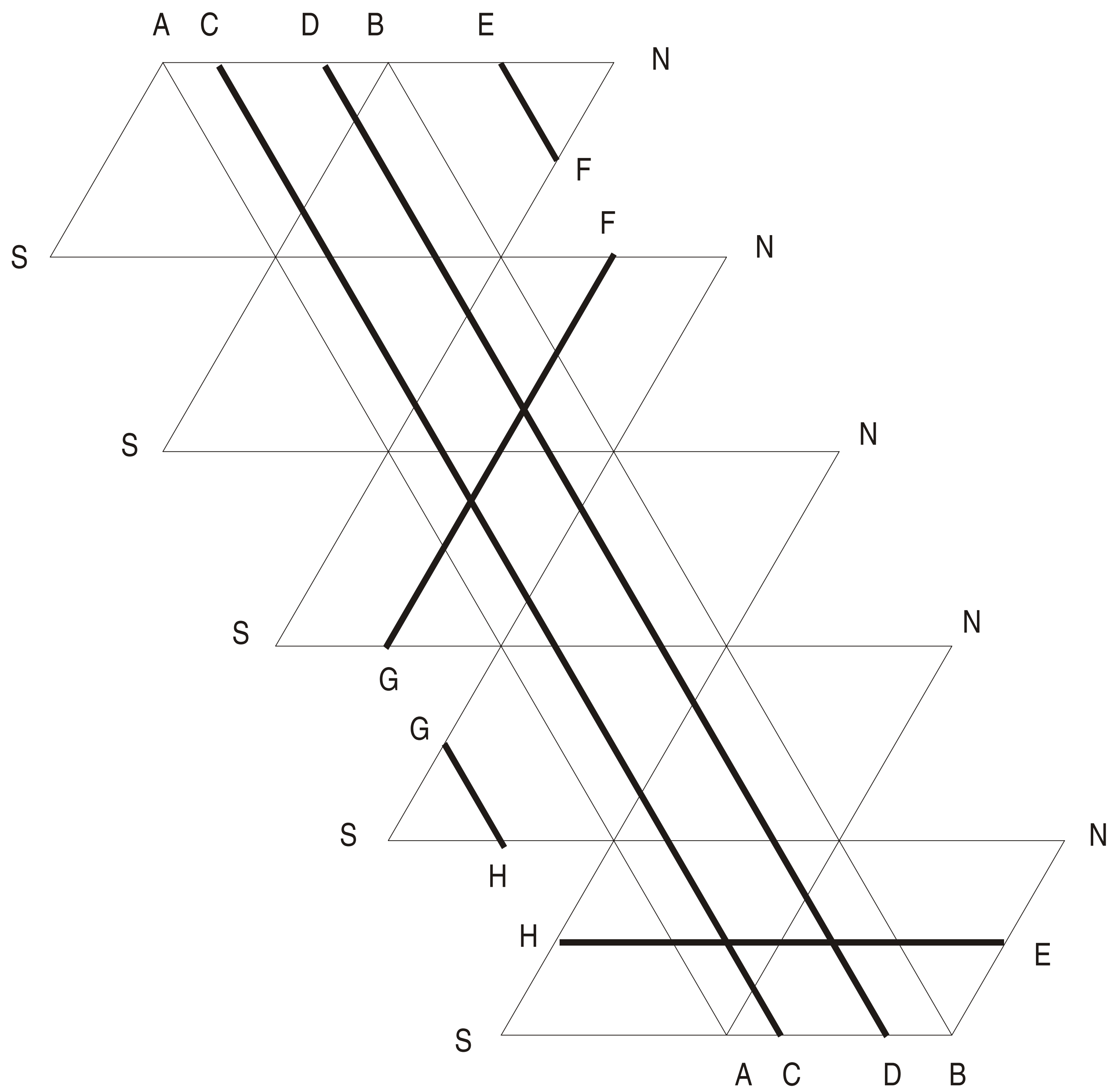


Figure 1. Non-singular s-lines parallel to edges in the s-sphere.

The s-sphere

An icosahedron has equilateral triangular faces with five triangles around every vertex. It is, therefore, an s-manifold, and we will call it the **s-sphere**. This s-manifold consists of 20 triangular faces, and has 12 vertices, all of them elliptic. It is a closed surface, and it is topologically equivalent to the unit sphere (or any sphere). Of all s-manifolds that are topological spheres, this is the most regular, and the one most closely aligned with the standard spherical geometry. Some of the s-lines in this s-manifold are very similar to the lines (great circles) in spherical geometry, but the behavior of s-lines in this space can be quite complicated.

Those s-lines running parallel to the edges of the triangular disks are simple circles, but unlike the great circles on a sphere, they may be parallel, like the s-lines through C and D shown in Figure 1. If two s-lines of this type intersect, they will intersect in two points, as do the s-lines through C and E shown in Figure 1. Note that the s-line through E can be shown in one piece, like the s-lines through C and D, by cutting the space differently.

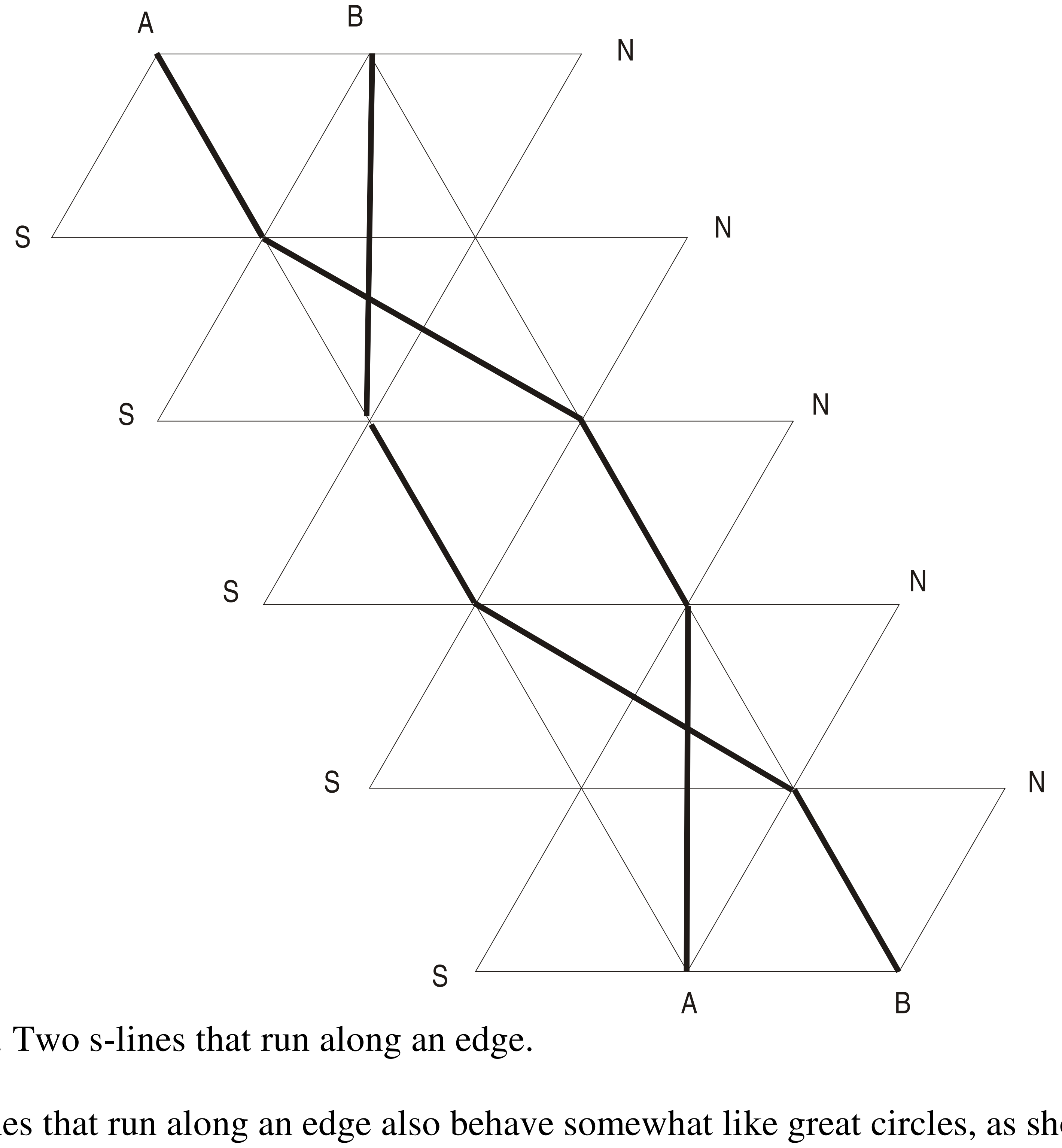


Figure 2. Two s-lines that run along an edge.

The s-lines that run along an edge also behave somewhat like great circles, as shown in Figure 2. The s-lines shown are typical of this type.

Other s-lines are more complicated and wrap around the s-sphere many times. In fact, in trying to follow an s-line different from the types shown in Figures 1 and 2, the process will typically seem to continue indefinitely. The question arises, therefore, whether some or all of these other s-lines are closed. An s-line that is not closed would necessarily have infinite length. While exploring this question, we will briefly examine a tool that may be of use in studying s-manifolds.

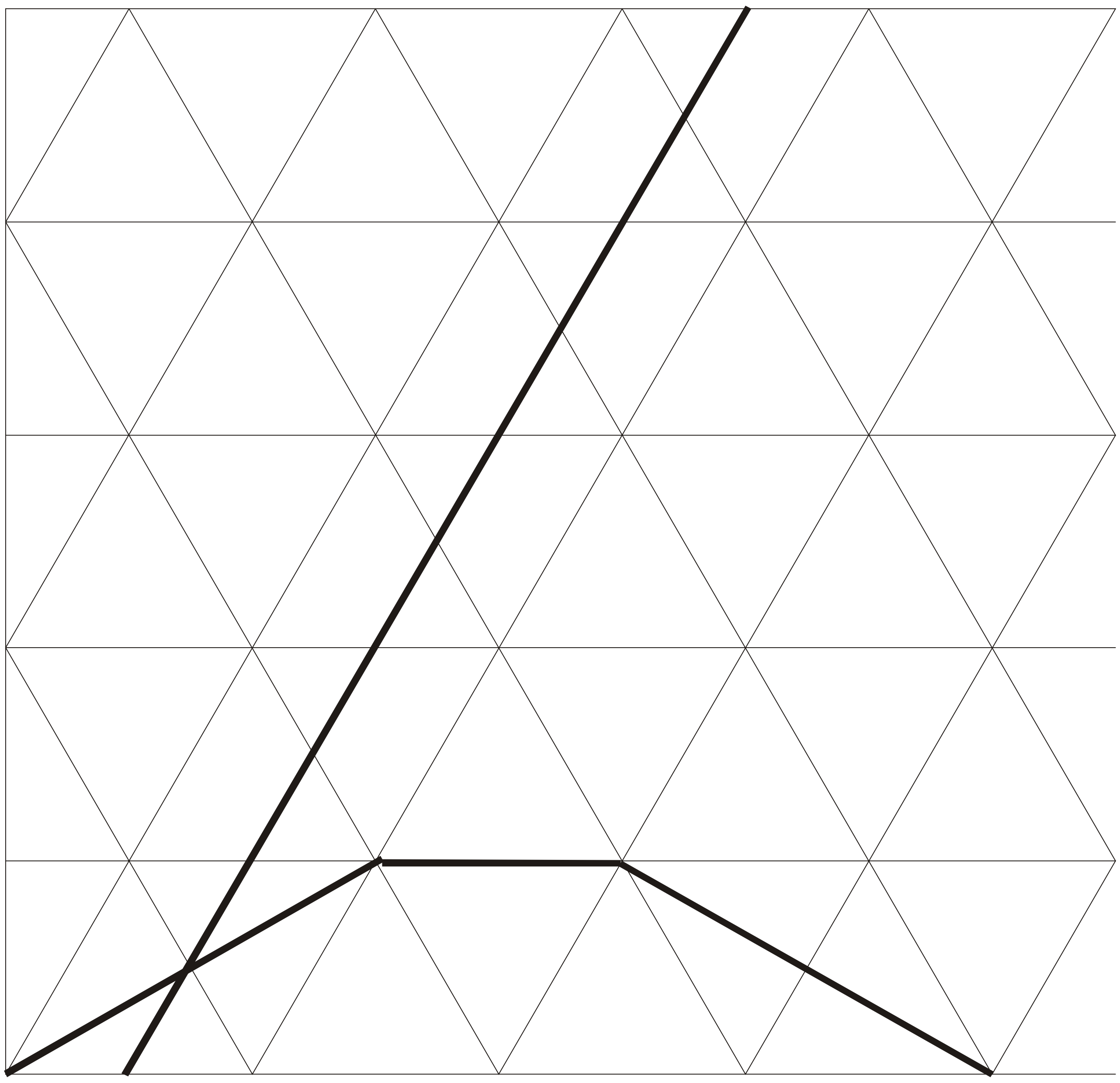


Figure 3. The projection of s-lines from Figures 1 and 2.

Locally linear projections

To help visualize situations where s-lines may or may not be closed, we will introduce a mapping of an s-line into the Euclidean plane. We will assume a tiling of the plane with equilateral triangles that have sides of length one so that one of the edges has (0,0) and (1,0) as endpoints. Given an s-line, we will start with a segment of the s-line that spans one of the triangular disks of the s-sphere. We will identify this triangular disk with the one in the plane that lies above the edge with endpoints (0,0) and (1,0), so that this segment has one endpoint on the x-axis and otherwise lies above the x-axis. If the other endpoint lies on the interior of an edge, then we will continue the projection into the corresponding adjacent triangle in the same way that the s-line extends into the adjacent triangle on the s-sphere. If the other endpoint lies on a vertex, we will continue it so that it makes a 150˚ angle measured counter-clockwise. We can continue this process indefinitely. In particular, if the original s-line contains no vertices, then its projection will be an infinite ray. In Figure 3, the s-line from Figure 2 starting at A at the bottom is projected, as is the s-line from Figure 1 that starts at G.

The s-lines in the s-sphere that do not pass through vertices will project to straight lines in the plane. If the projection of such an s-line has slope $\sqrt{3}$, then it will be parallel to the edges of the triangles, and it must be the projection of the simple circles first mentioned. In particular, the projection will intersect each horizontal edge at the same distance from the left endpoint and at the same angle. Without prior knowledge of the nature of this s-line on the s-sphere, we could see from this that the s-line is closed. This is because the projection hits an infinite number of edges in exactly the same way. Each of these corresponds to the s-line on the s-sphere intersecting an edge in exactly the same way. Since there are only a finite number of edges on the s-sphere, this s-line must hit one of these edges twice in exactly the same way. The s-line must, therefore, be closed.

If the projection of an s-line on the s-sphere does not pass through a vertex and has a slope that is a rational multiple of $\sqrt{3}$, then it must be closed. We can see this as follows. This projection passes through the segment from (0, 0) to (1, 0) at a point (x, 0) and at an angle θ. If the slope is $a\sqrt{3}/b$, then this projection will intersect the segment from $(b, a\sqrt{3})$ to $(b + 1, a\sqrt{3})$ at $(b + x, a\sqrt{3})$ and at the same angle θ. In other words, the projection intersects this segment in exactly the same way that it intersects the edge (0,0) (1,0). In fact, for every positive integer n, this projection will intersect the segment $(bn, an\sqrt{3})$ to $(bn + 1, an\sqrt{3})$ at $(bn + x, an\sqrt{3})$ and at the angle θ. Each of these corresponds to the s-line on the s-sphere intersecting an edge. Since there are only a finite number of ways that this can happen on the s-sphere, we can conclude that this is a repeating cycle, and the s-line is closed. Furthermore, any line in the plane that does not pass through a vertex must be a projection of some s-line on the s-sphere, so we see there must be many kinds of closed s-lines on the s-sphere.

On the other hand, if a line in the plane has a slope that is an irrational multiple of $\sqrt{3}$, then the corresponding s-line on the s-sphere cannot be closed. There are certainly lines in the plane of this type, so the s-sphere contains s-lines that are not closed, and these wrap around the s-sphere an infinite number of times and must have infinite length.

Alignment

The points C and D in Figure 1 are ∞-aligned. Those triangular disks that contain parts of the two s-lines shown through these two points form a cylinder. An s-line running along the edge containing C and D joins these two points, as does an s-line joining C at the top and D at the bottom that wraps around the s-sphere once between these two points. An s-line that wraps around twice, passing through the midpoint between C and D on the edge also joins these two points. There are also s-lines that start at C and wrap around the space any number of times before passing through D. There are an infinite number of s-lines, therefore, that join C and D.

Since any of the triangular disks in the s-sphere lie in three cylinders of the type just mentioned, ∞-alignment is quite common.

Euclidean bands and band spaces

As mentioned, the structure of the s-lines on the s-sphere is quite complex. We may perhaps gain some insight into this structure by considering a simpler situation. One possibility comes from the observation just made that there are cylindrical bands around the s-sphere. The s-lines through C and D in Figure 1 lie in one of these. The s-line through E lies in another. Each band is the set of triangular disks that an s-line parallel to one of the edges passes through.

We may consider a finite geometry based on the bands of the s-sphere. The b-lines in this geometry are the bands and the points are the triangular disks. Since the s-sphere consists of 20 triangles, there are 20 points in this geometry. Each band consists of 10 triangular disks, and each triangular disk is associated with three bands. Therefore, there are 20· 3/10 = 6 b-lines in this geometry.

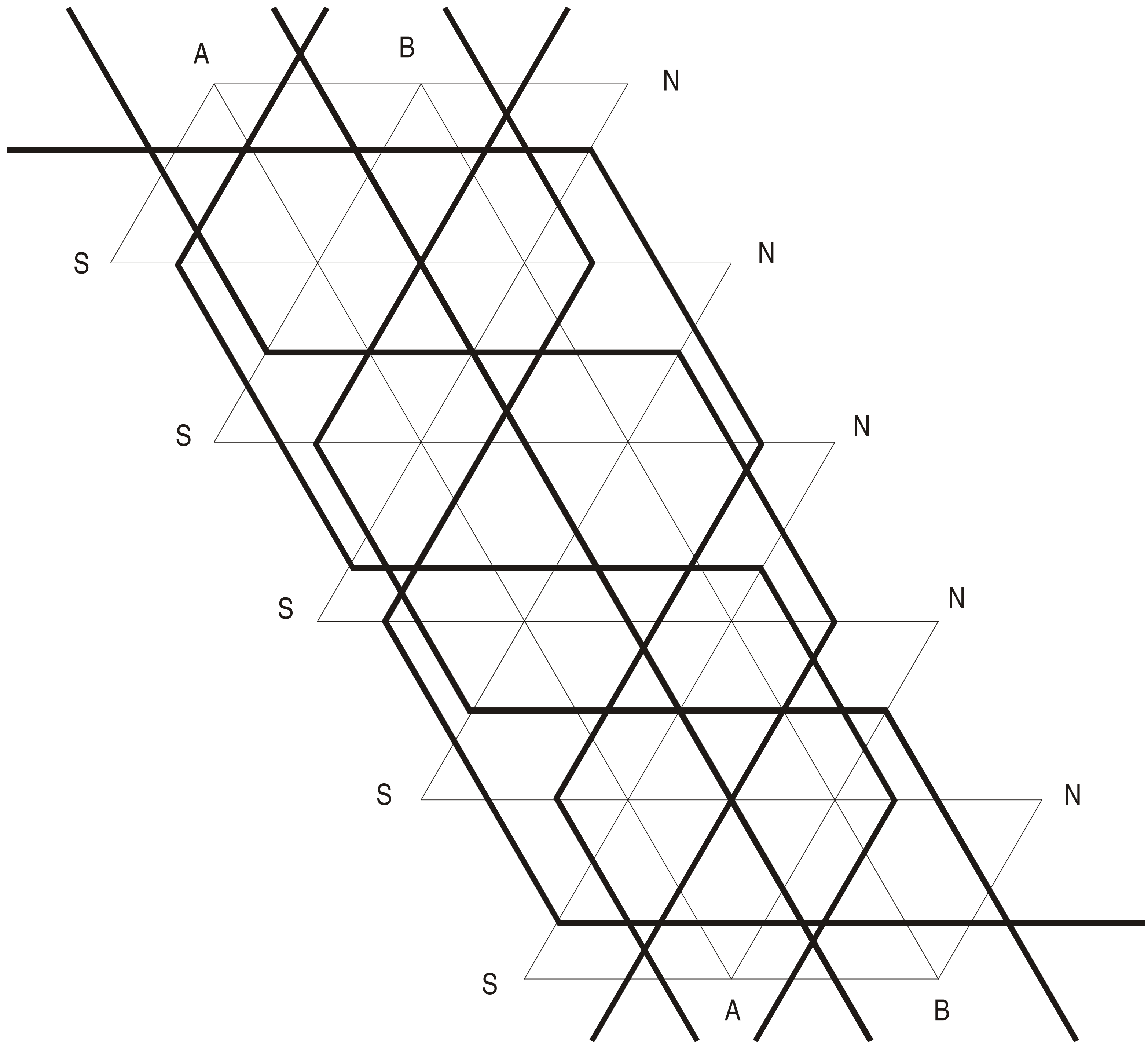


Figure 4. Each of the six bands in the s-sphere is indicated by an s-line running through it.

Interior band spaces

We can also consider a simpler space than the s-sphere by restricting attention to a special class of s-lines. The easiest s-lines are those that run along or are parallel to the edges. We will call any geometry formed by designating as i-lines, those s-lines in an s-manifold that have at least one segment that is parallel to an edge an **interior band space**. The underlying set of points is the same, and only the set of curves that are called lines is different. We will stop here saying only that this is another kind of geometry that can be studied.

The s-projective plane

A model for the standard elliptic geometry is called the projective plane, and can be obtained from the sphere by identifying antipodal points. It can also be obtained from a hemisphere or disk by identifying antipodal points on the boundary.

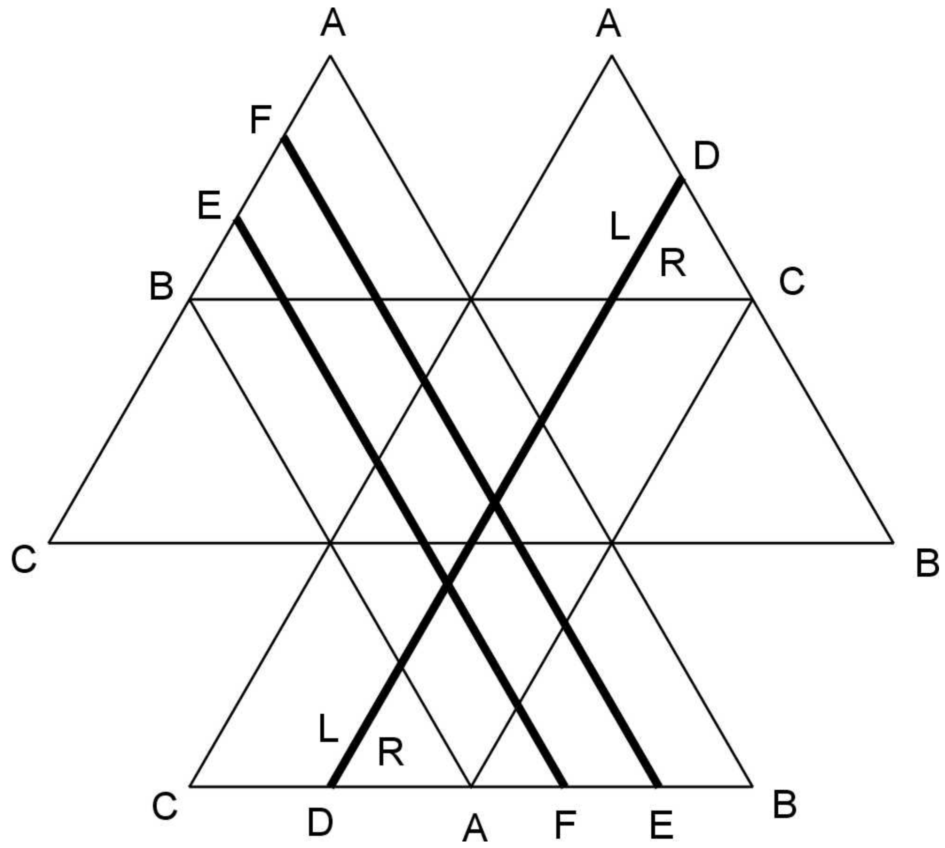


Figure 5. The s-projective plane.

The configuration of triangular disks in Figure 5 is essentially half of the configuration for the s-sphere, so this is topologically a hemisphere. The points A, B, and C are on the boundary, and identifying the segments AC, CB, and BA as marked is topologically equivalent to identifying antipodal points. This meets the requirements of an s-manifold, since each of the vertices is shared by five triangular disks. We will call this the **s-projective plane**. It clearly is not embeddable in Euclidean 3-space (i.e., it cannot be presented as a subspace without self-intersections or stretching or bending of the individual triangular disks), but it can be embedded in 4-space topologically (with stretching and bending). It is highly questionable as to whether the s-projective plane can be embedded in 4-space without bending or stretching the triangular disks. It is an important concept for s-manifolds, and in the study of manifolds in general, that we do not restrict attention to those spaces that can be visualized as a subspace of a Euclidean space. It can be an interesting problem, however, to determine whether or not a manifold can be embedded in a Euclidean space, and if so, to determine the minimum possible dimension.

The s-projective plane is non-orientable. This means, roughly, that it is impossible to impose a notion of orientation, such as clockwise/counter-clockwise or left/right, that is consistent over the entire manifold. For example, the s-line through D in Figure 5 has a designation of a left and right side marked by an L and an R. At the bottom, it seems clear that the vertex A is on the right side of this s-line, but at the top, it appears that A is on the left. The non-orientability of the s-projective plane is also manifested around the s-line through E and F. This s-line is a simple closed curve that bounds a Möbius band

The s-torus

The s-torus is a torus topologically, and has only Euclidean vertices. We will use the configuration shown in Figure 6. Here we have three rows and columns of Euclidean bands to avoid a triangular disk having a self-intersection, which would be a violation of the definition of an s-manifold. In Figure 6, two s-lines are shown, both passing through F. Both s-lines are closed, one wraps around the space once in the vertical direction, the other wraps around three times vertically and once horizontally. We can associate with these s-lines the ordered pairs (1,0) and (3,1). This extends to a notion of slope that applies to all of the s-lines in the s-torus. There is an s-line through F corresponding to any ordered pair of integers (v,h), and the relatively prime pairs are distinct. In fact, infinitely many of these s-lines also pass through G, so F and G are ∞-aligned. This extends to any pair of points, and any point in the s-torus is ∞-aligned with any other point.

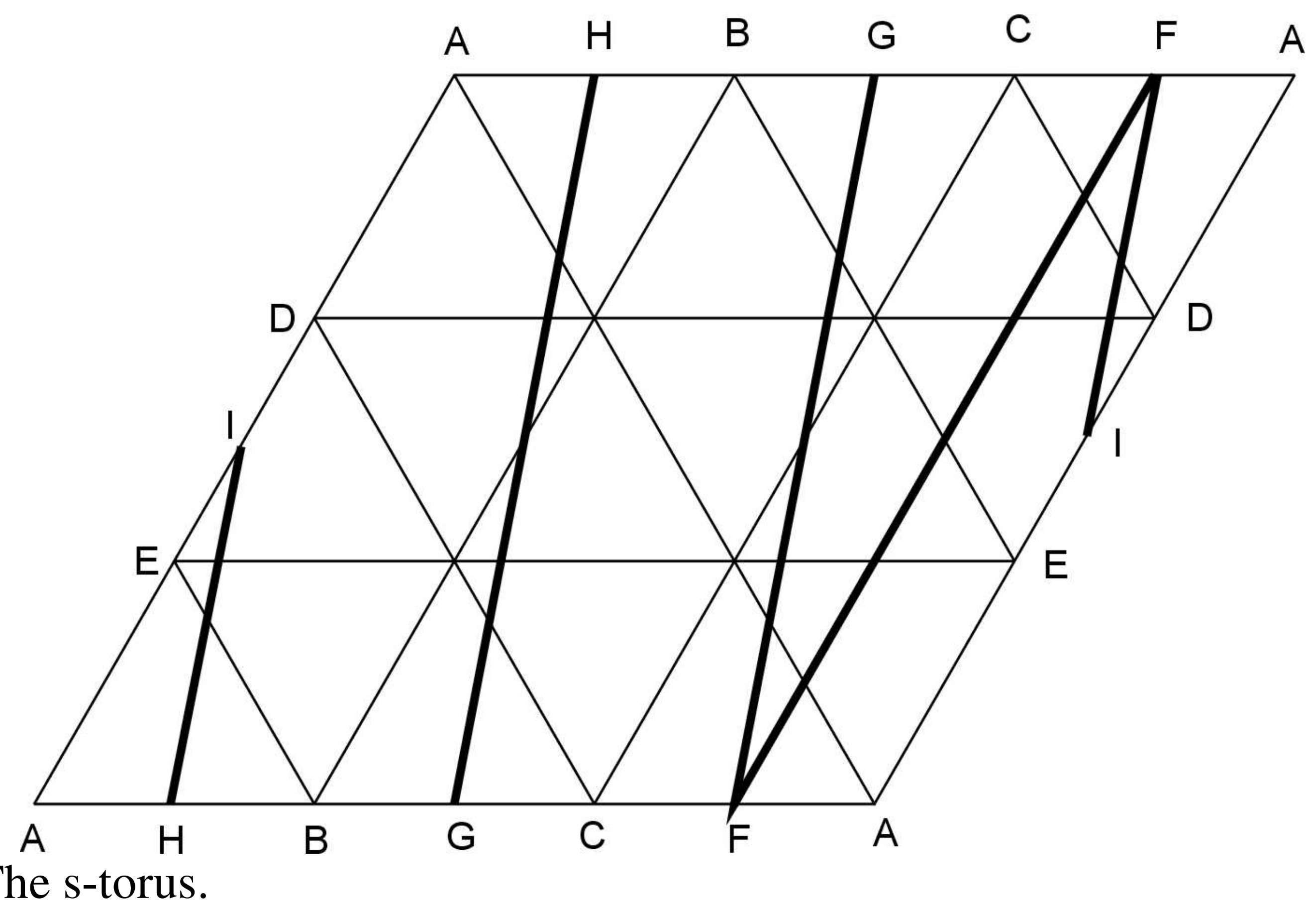


Figure 6. The s-torus.

Since a Euclidean vertex is essentially the same geometrically as any non-vertex point, the s-torus is a perfectly uniform space, and is essentially the flat torus of differential geometry. Each point has precisely the same properties as any other. The s-torus is finite, however, so there is a difference in the properties of large and small objects. For example, the large triangle GHI in Figure 7 has no interior. The s-torus is a Smarandache geometry in this way, and others.

Pasch's axiom

Pasch's axiom can be formulated to say that a line entering a triangle through a vertex will intersect the opposite side. This is equivalent to the formulation given earlier. Pasch's axiom does not hold in the s-torus due to the topology of the s-torus, and in Figure 7, we show an example of a triangle whose inside is connected to its outside.

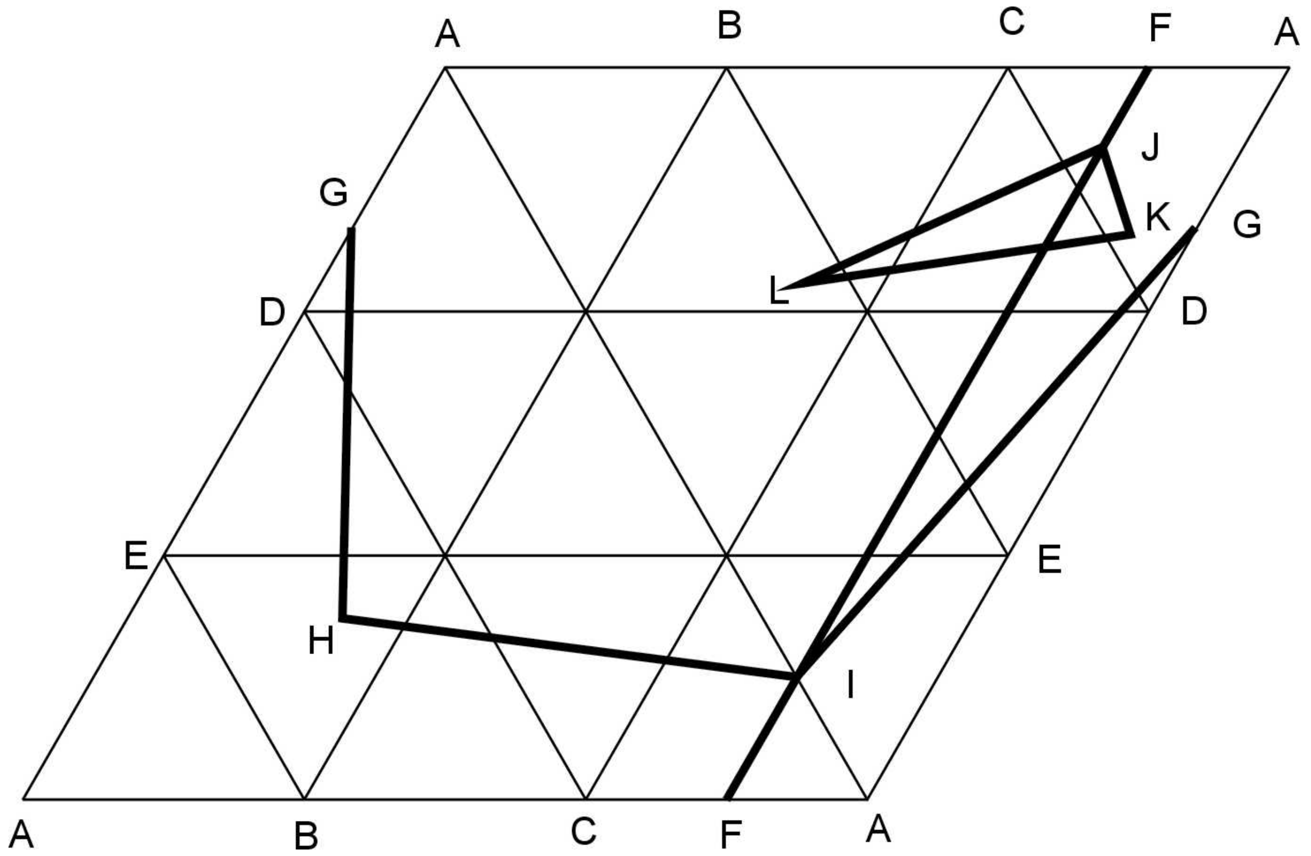


Figure 7. Pasch's axiom is Smarandachely denied in the s-torus.

In Figure 7, the smaller triangle JKL has the s-line through F entering the triangle at the vertex J and intersecting the opposite side KL. Any s-line entering the triangle at J, K, or L will pass through the opposite side of the triangle JKL satisfying Pasch's axiom. The larger triangle GHI, which wraps around the s-torus in the horizontal direction, violates Pasch's axiom. Here we see that the s-line shown passes through the vertex I, but does not intersect the opposite side GH. In fact, while it crosses the boundary formed by the triangle, it intersects the triangle in only this one point. It does not really even make sense to say that the s-line through F enters the triangle GHI, since this triangle does not have an inside or outside. This example also illustrates the property that there are pairs of simple closed curves on the torus that cross at only one point.

The s-Klein bottle

The Klein bottle, like the torus, is commonly constructed out of a square disk by identifying edges. This is not possible for us, since our basic building block is an equilateral triangle. There is no essential difference with the s-torus, using a parallelogram, as we did in Figure 7. Identifying the left and right edges on a parallelogram with a twist, as in Figure 9, still yields a Klein bottle topologically, but gives a geometry that differs from a flat Klein bottle in differential geometry.

There are two s-lines shown in Figure 8. The one through F is similar to the s-line through F in Figure 6 for the torus. Horizontally, the orientation is reversed, and the other s-line passes through G, H, and I. This second s-line has a single self-intersection (or singularity). Both s-lines are closed.

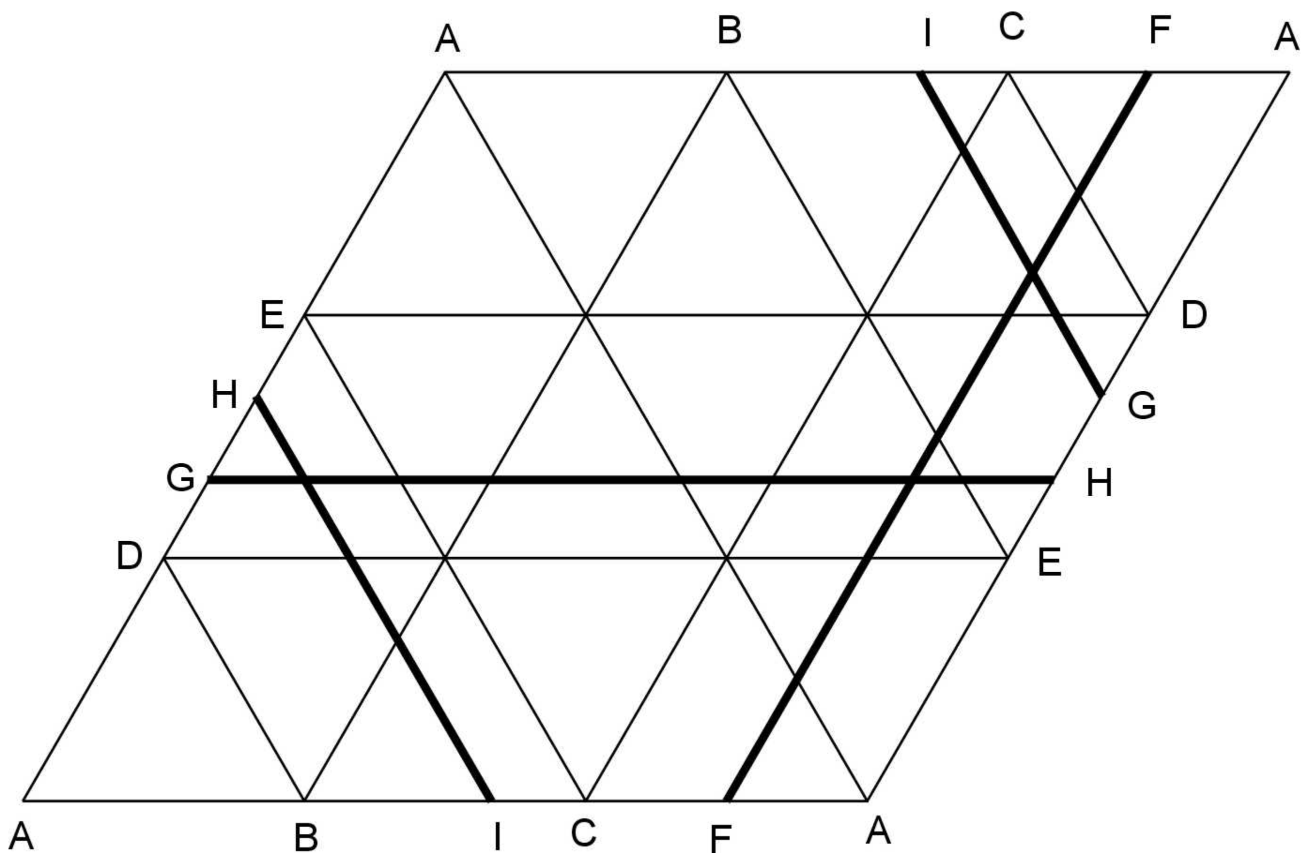


Figure 8. The s-Klein bottle.

The s-line in Figure 8 through G, H, and I wraps around the space in two directions, so it may seem that it would be difficult to find an s-line that does not intersect it. This is in fact the case. We can see this easily by considering what a topologist would call a lift. The parallelogram enclosing Figure 8 along with an infinite number of copies can tile the plane as in Figure 9. Instead of viewing the identifications as joining the top and bottom edges, or the left and right edges, each edge of the parallelogram is identified with an edge of an adjacent parallelogram. The s-line through G, H, and I in Figure 8 is shown as a dotted line in one of the parallelograms in Figure 9. This s-line is also shown extended as if it were a line in the plane. This is a **lift** of the line GHI. The parallelograms that are shaded have orientations that are the reverse of the non-shaded ones. If we start at the point G at the left of Figure 8, the s-line runs across the parallelogram to the point H on the right. It continues from H on the left down to the point I. In the lift, the line continues to the right to a copy of the point I. Note that this segment of the lift corresponds exactly to the segment HI in Figure 8. The same can be said of the segment from I to G.

Clearly, if the lifts of two s-lines intersect, then the corresponding s-lines in the s-Klein bottle must also intersect. It is also clear that if two s-lines in the s-Klein bottle intersect, then there are lifts of the two s-lines that intersect (although not any two lifts will intersect). Therefore, if an s-line *m* in the s-Klein bottle is to be parallel to the line GHI, it must have a lift that is parallel to the one shown in Figure 9. A lift based on the segment HI in Figure 8 will intersect this lift of the line *m*, however, so *m* cannot be parallel to GHI.

In the s-Klein bottle, every point is elliptic relative to the s-line through G, H, and I. This shows that we can have a point that is elliptic relative to some s-line without there being an elliptic vertex. Every point is Euclidean relative to the s-line through F. Euclid's parallel postulate, which would require that every point be Euclidean relative to every s-line, is therefore S-denied, and the s-Klein bottle is a Smarandache geometry in this way.

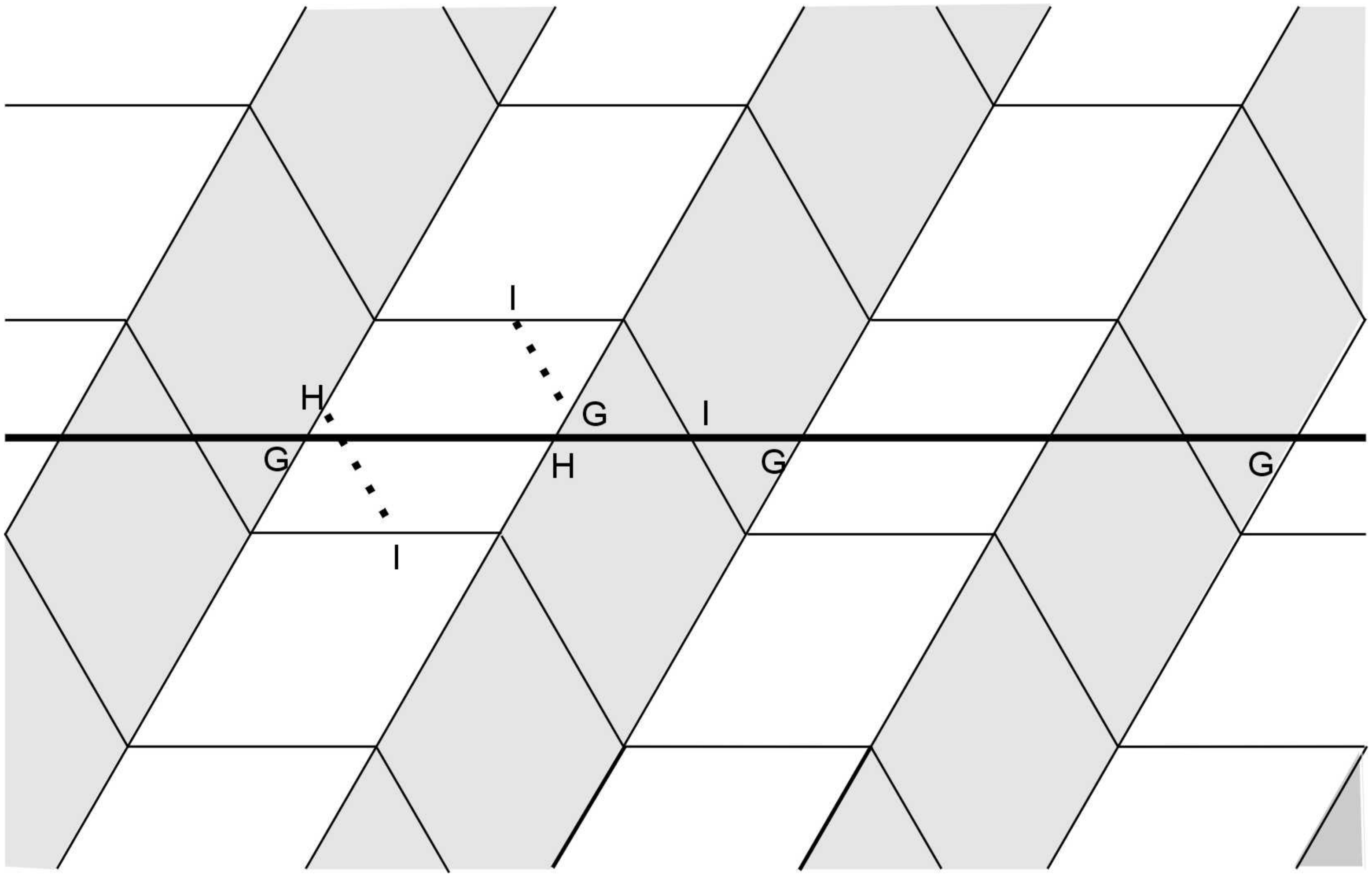


Figure 9. A covering of the s-Klein bottle by the plane.

Topological 2-Manifolds

Topological considerations

For the s-projective plane in Figure 5, the five triangular disks containing the s-lines through E and F form a Möbius band. The remaining five triangular disks form a topological disk. The s-projective plane is obtained from these by identifying the boundary of the Möbius band with the boundary of the disk. Both boundaries are circles topologically (five adjacent edges).

In the study of the topology of closed 2-dimensional manifolds, or 2-manifolds (all of our s-manifolds are 2-manifolds), it is convenient to view the projective plane as being obtained from the sphere by removing a disk (leaving a disk) and gluing in a Möbius band as we have just described. Removing two disks and gluing in two Möbius bands results in a Klein bottle. All non-orientable closed 2-manifolds are obtainable in this way, and they are classified by the number of Möbius bands (see **[20,21]**). We will discuss this later in the chapter.

The orientable closed 2-manifolds are obtained in a similar way. A torus with a disk removed is called a handle. A sphere is an orientable closed 2-manifold. Removing a disk and gluing in a handle gives the torus. Removing two disks and gluing in two handles results in a 2-holed torus. All orientable closed 2-manifolds can be obtained in this way, and these are classified by the number of handles (see **[20,21]**). This will also be discussed later.

One question for us is whether there is an s-manifold with a topology corresponding to each of the closed 2-manifolds. It is known that each of these topological 2-manifolds has a Riemannian manifold structure with constant curvature. The projective plane and sphere can have constant positive curvature, the torus and Klein bottle can have constant zero curvature, and every other closed 2-manifold can have constant negative curvature. We have already seen that the sphere and projective plane can manifest themselves as s-manifolds with only elliptic vertices, which have positive impulse curvature, and the torus and Klein bottle exist as s-manifolds with only Euclidean vertices, which have zero impulse curvature.

Question. Do the other closed 2-manifolds correspond to s-manifolds with only hyperbolic vertices?

Euler characteristic

Preliminary to this investigation, we will introduce the Euler characteristic (see **[1, 11, 17, 18, 20, 21]**). Our s-manifolds are a special case of a class of 2-manifolds called piecewise linear 2-manifolds. These correspond to the triangulations of 2-manifolds, which are the decompositions of 2-manifolds into triangular disks (not necessarily flat or with straight edges) such that the triangular disks meet edge to edge. The Euler characteristic is a topological invariant of these triangulations. That is, the Euler characteristics for any triangulations of two 2-manifolds that are topologically equivalent are the same. The Euler characteristic is defined as follows. Let f be the number of triangular disks or faces, let e be the number of edges, and let v be the number of vertices. Then the Euler characteristic is $\chi = f - e + v$.

For the s-sphere, there are 20 faces, 30 edges, and 12 vertices, so $\chi = 20 - 30 + 12 = 2$. Topologically, any triangulation of a sphere will give $\chi = 2$. For example, a tetrahedron is a topological sphere, and $\chi = 4 - 6 + 4 = 2$ (although a tetrahedron is not an s-manifold, since there are only three triangular disks around each vertex).

For the s-projective plane, there are 10 faces, 15 edges, and 6 vertices, so $\chi = 10 - 15 + 6 = 1$. Again, $\chi = 1$ for any triangulation of a projective plane.

In general, the Euler characteristic for any non-orientable 2-manifold is $\chi = 2 - m$, where m is the number of Möbius bands. The Euler characteristic for any orientable 2-manifold is $\chi = 2 - 2h$, where h is the number of handles. We should get, therefore, $\chi = 0$ for both the torus and Klein bottle, and χ is negative for all remaining closed 2-manifolds. Note that the sign of the Euler characteristic corresponds inversely with the constant curvature geometries. Checking χ for the s-torus and s-Klein bottle, we get $\chi = 18 - 27 + 9 = 0$ for both.

The s-Euler characteristic

The Euler characteristic can be formulated nicely in terms of the number of elliptic and hyperbolic vertices. In an s-manifold, there are five triangular disks around each elliptic vertex, six around each Euclidean vertex, and seven around each hyperbolic vertex. Let v_e be the number of elliptic vertices, v_E the number Euclidean vertices, and v_h the number of hyperbolic vertices. Then $v = v_e + v_E + v_h$. Each triangular disk has three edges, and each edge is shared by two triangular disks, so $3f = 2e$. Each triangular disk has three vertices, and these are shared by five, six, or seven triangular disks, so $3f = 5v_e + 6v_E + 7v_h$.

The Euler characteristic is then $\chi = f - e + v = (5v_e + 6v_E + 7v_h)/3 - (5v_e + 6v_E + 7v_h)/2 + (v_e + v_E + v_h) = [(10–15+6)v_e + (12–18+6)v_E + (14–21+6)v_h]/6 = [v_e - v_h]/6$. This can be rewritten as $6\,\chi = v_e - v_h$.

Theorem. For a closed s-manifold, the number of elliptic vertices minus the number of hyperbolic vertices is equal to six times the Euler characteristic.

The Euler characteristic for a torus is $\chi = 0$, so any s-manifold torus must have an equal number of elliptic and hyperbolic vertices. The s-torus has zero elliptic and zero hyperbolic vertices, for example. Another s-manifold torus is shown in Figure 10. This model is made out of Polydron® blocks, but does not represent a true embedding, since the model needed to be bent slightly to close a 7° gap.

Of course, an s-manifold does not need to be embeddable in 3-space, so this does represent an actual s-manifold torus. A close inspection of the picture in Figure 10 shows that the model consists of five identical sections around the points of the inside star. These contain 14 triangular disks with two elliptic vertices on the outer edge and two hyperbolic vertices, one in front and one behind. At the junction between adjacent sections are three Euclidean vertices. All total, there are 10 elliptic vertices and 10 hyperbolic vertices, an equal number of each, as expected. There are also 15 Euclidean vertices, and there are 70 faces and 105 edges. This agrees with the Euler characteristic $\chi = 0 = 70 - 105 + 35$.

Figure 10. A Polydron® model of an s-manifold torus.

If an s-manifold is topologically equivalent to a 2-holed torus, which has $\chi = -2$, then there must be 12 more hyperbolic vertices than elliptic vertices. In particular, if there is a 2-holed closed s-manifold with only hyperbolic vertices, then it would have 12 vertices. Since $3f = 7v_h$, there must be 28 faces or triangular disks.

Question. Is it possible to construct a 2-holed torus s-manifold with twelve hyperbolic vertices?

Closed topological 2-manifolds

The class of closed 2-manifolds is nicely classified from a topological point of view. By this we mean that this classification is up to homeomorphism, or roughly, without regard to continuous deformations (see **[20, 21]**). This classification comes in two main categories, the orientable and the non-orientable, as was mentioned earlier.

The orientable closed 2-manifolds are the sphere, the torus, the 2-holed torus, the 3-holed torus, etc. Representations of these are shown in Figure 11.

These 2-manifolds are conveniently described in terms of a topological operation called the **connect sum**. Here two 2-manifolds are joined by a tube. Using T for the torus, and #

for the connect sum, we would write T# T for the connect sum of two tori. This can be represented as a picture like Figure 12.

The class of orientable closed 2-manifolds is countable and can be listed out as S (for sphere), T (for torus), T#T, T#T#T, etc.

The non-orientable closed 2-manifolds can be arranged similarly. If P is the projective plane, then P#P is the Klein bottle, and the rest of this class can be listed out as P#P#P, P#P#P#P, etc. We could think of the projective plane as having a non-orientable hole (a Möbius band glued in), and the Klein bottle would then have two of these. The **genus** of a closed 2-manifold corresponds to the number of holes, so we would have one orientable and one non-orientable closed 2-manifold of every possible positive genus and one, the sphere, which has genus 0.

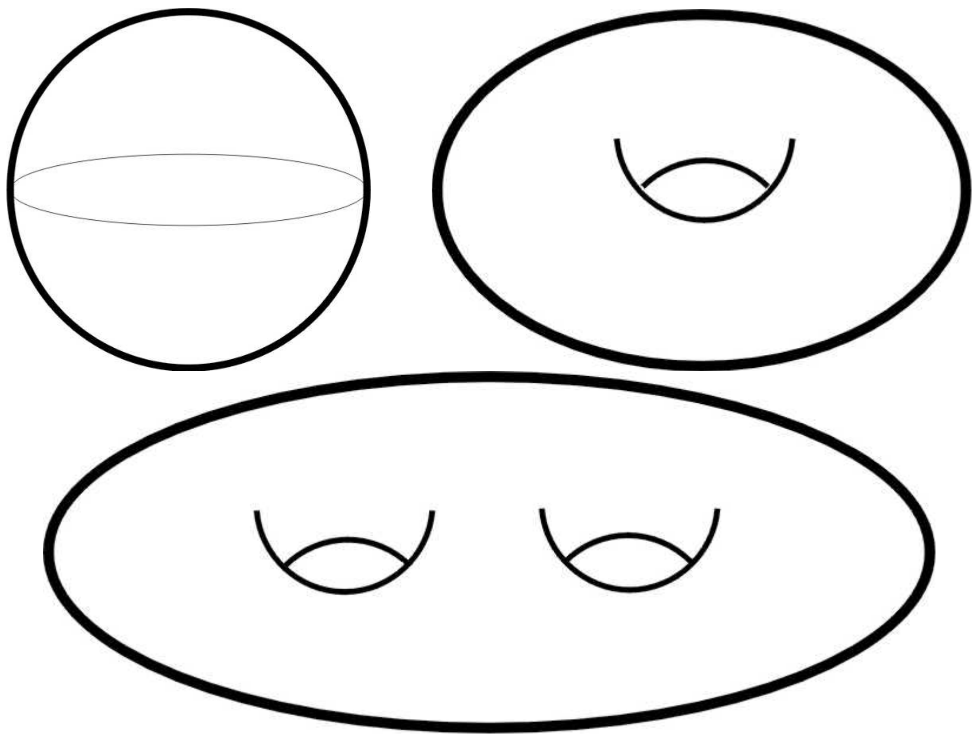


Figure 11. Representations of the sphere, the torus, and the 2-holed torus.

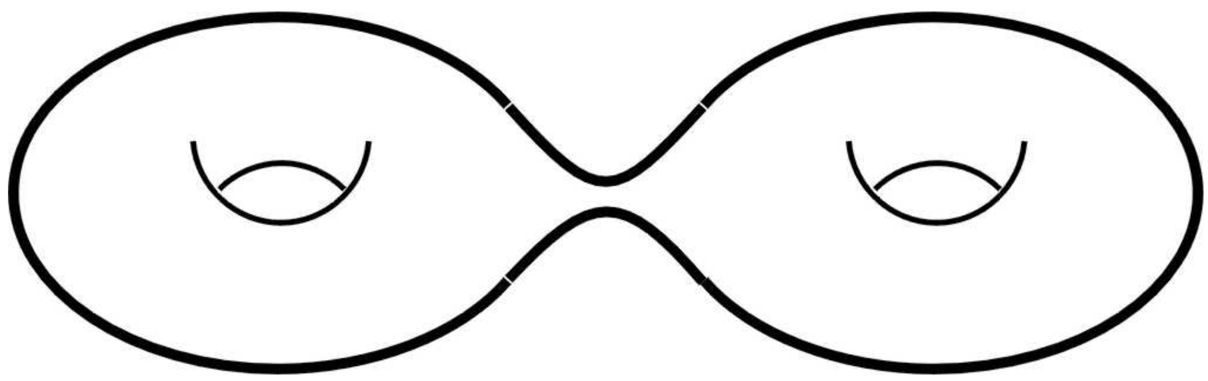


Figure 12. A representation of the connect sum of two tori, which yields a 2-holed torus.

It should be noted that other closed 2-manifolds might arise from a connect sum like P#T. This 2-manifold is topologically equivalent to P#P#P, however, and in general, the connect sum of a torus with a non-orientable 2-manifold is equivalent to the connect sum with a Klein bottle (see **[20,21]**).

Existence of closed s-manifolds

The idea of a connect sum can be extended to s-manifolds. Using this idea we will show that there is a closed s-manifold of every possible topological type. It will remain to be seen, however, whether this can be done more simply or with a more uniform structure. The main difficulty in applying the connect sum to s-manifolds is in ensuring that the s-manifold structure is maintained. In particular, we need to make sure that every vertex has 5, 6, or 7 triangular disks around it.

Three basic structures are required, a projective plane, a torus, and a tube needed to perform the connect sum. The s-projective plane and s-torus that we have already are too small for the process we will illustrate here, so we will introduce a **big s-projective plane**, which we will denote by **P**, and a **big s-torus**, which we will denote by **T**. Each of the triangular disks in the s-projective plane and the s-torus will be replaced by four triangular disks increasing the area and the number of triangles by a factor of four. The big s-projective plane is shown in Figure 13, and the big s-torus is shown in Figure 14.

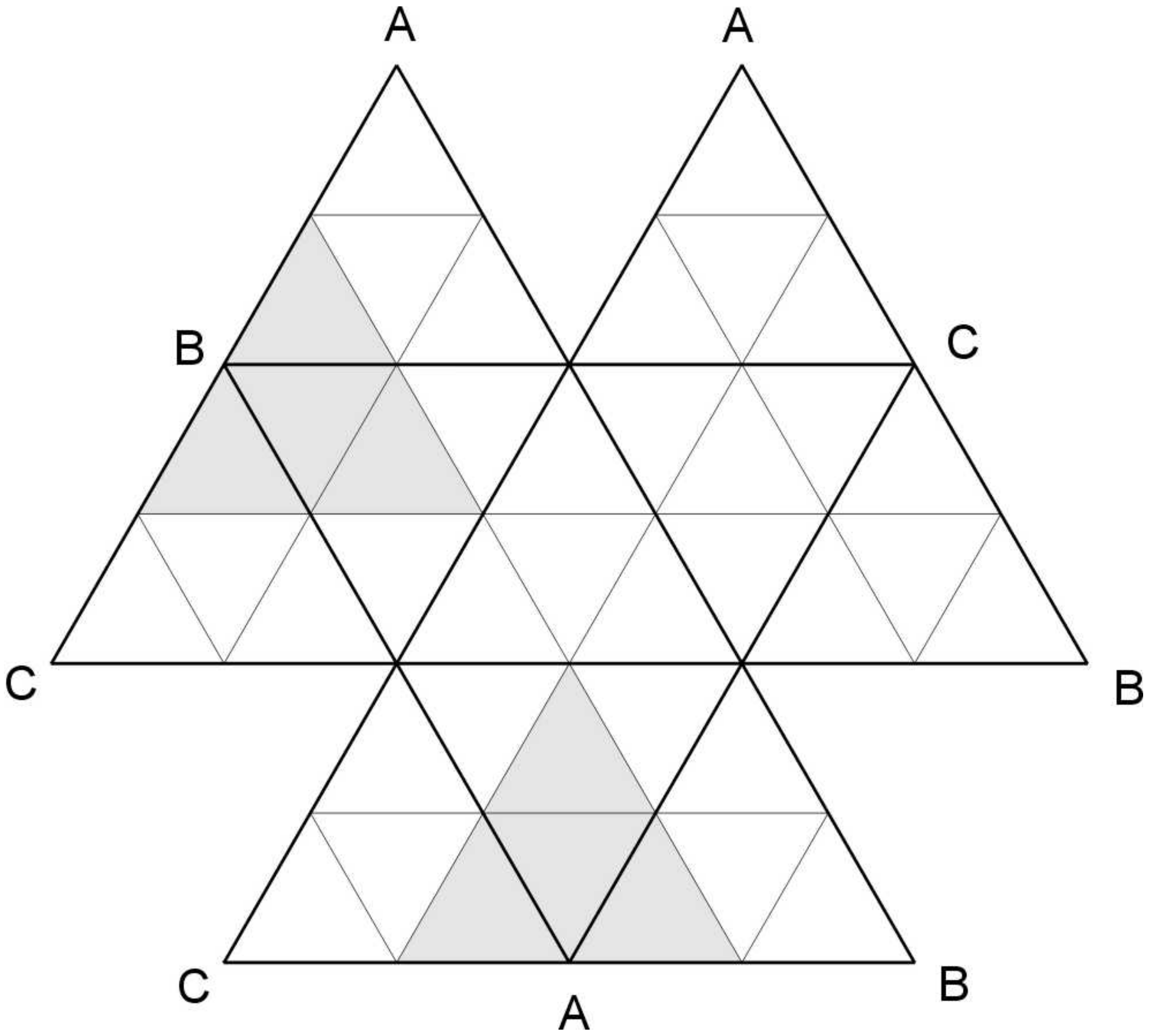


Figure 13. The big s-projective plane, **P**.

The connect sum will be performed by removing one of the quartets of shaded triangles in Figure 13 or 14 and replacing it with a **half-tube**, which is shown in Figure 15. The triangle UVW attaches to **P** or **T**, and the triangle XYZ will attach to the same triangle from another copy of the half-tube.

At the vertices U, V, and W in Figure 15, there are two triangular disks. After removing the shaded triangles in either Figure 13 or 14, there are five triangular disks around the corners of the holes. After attaching a half-tube, U, V, and W will have seven triangular disks around them, and so they will be hyperbolic. Around the midpoints of triangle UVW, three triangular disks are replaced by three, so these vertices will be elliptic or Euclidean, as they were originally in **P** and **T**. In particular, the vertices A and B in Figure 13 are elliptic, and they will remain elliptic after attaching a half-tube. The vertices R, S, and T in Figure 15 are hyperbolic, so attaching a half-tube adds six hyperbolic vertices. Each connect sum, therefore, will add twelve hyperbolic vertices.

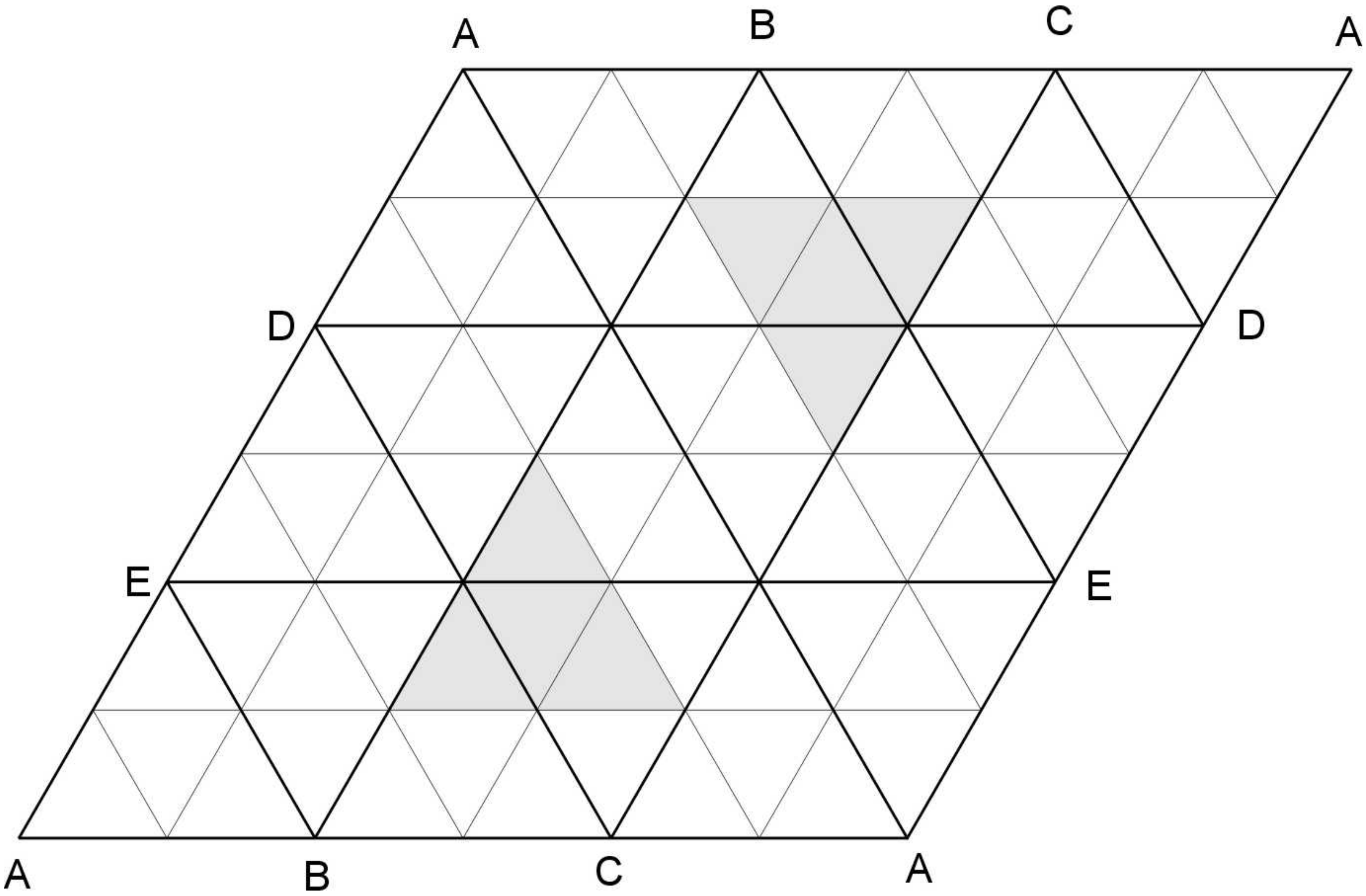


Figure 14. The big s-torus, **T**.

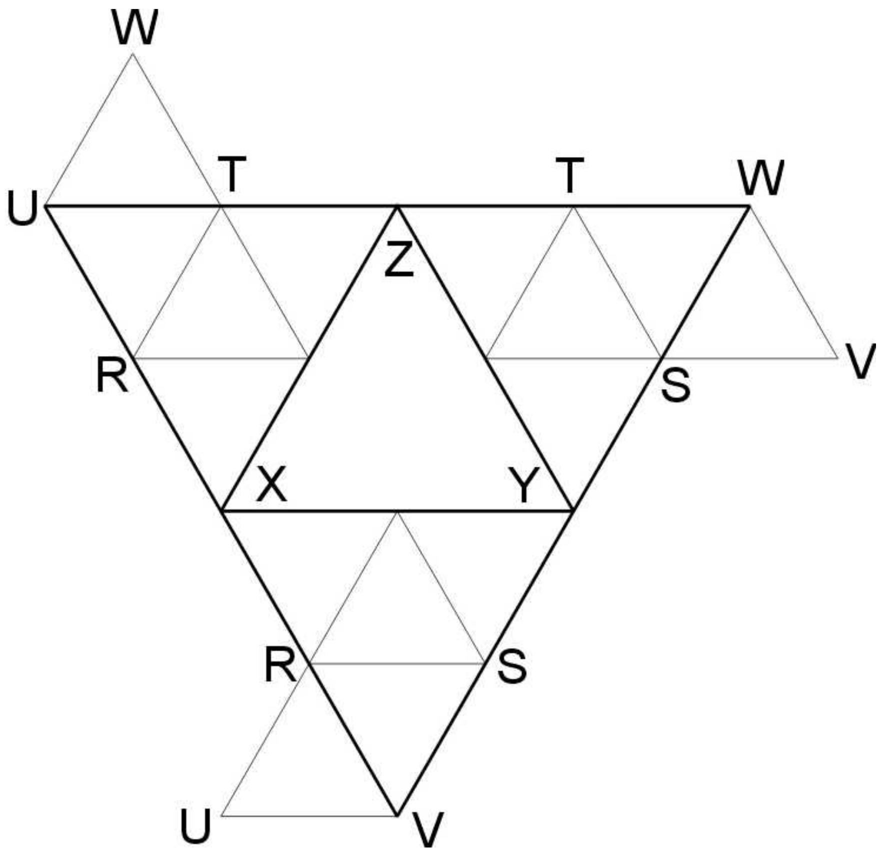


Figure 15. The half-tube.

The big s-torus has only Euclidean vertices, so **T#T** has 12 hyperbolic vertices, and the rest are Euclidean. Each additional connect sum of a big s-torus adds 12 more hyperbolic vertices, so each genus *n* orientable closed 2-manifold obtained this way will have 12(*n* – 1) hyperbolic vertices. The big s-projective plane has six elliptic vertices, so **P#P** will have 12 hyperbolic vertices and 12 elliptic vertices. Each additional connect sum of a big s-projective plane adds 12 hyperbolic and 6 elliptic vertices, so the genus *n* non-orientable 2-manifold obtained this way has $12(n-1)$ hyperbolic and $6n$ elliptic vertices. There are, therefore, $6n-12$ more hyperbolic vertices than elliptic. This corresponds with the s-Euler characteristic, which is $6\chi = 6(2n-2)$ in the orientable case and $6\chi = 6(n-2)$ in the non-orientable case.

In particular, we now have at least one s-manifold corresponding to every possible closed 2-manifold topology. We have the orientable closed s-manifolds: the s-sphere, the s-torus **T**, **T#T**, **T#T#T**, etc., and the non-orientable closed s-manifolds: the s-projective plane **P**, the s-Klein bottle **P#P**, **P#P#P**, etc. Of course there are many more possible. It would be especially interesting to know if there are any closed s-manifolds that have only hyperbolic vertices.

Question. Are there any closed s-manifolds with only hyperbolic vertices?

Embeddings

Especially with closed s-manifolds, a natural question is whether they *exist* in $\mathbf{R}^3$ or $\mathbf{R}^4$. We will consider several levels of interpretation of what this can mean. We will say that a **topological embedding** of an s-manifold M in $\mathbf{R}^n$ is a surface S in $\mathbf{R}^n$ without self-intersections such that there is a function f: M → S that is one-to-one, onto, continuous, and has a continuous inverse (i.e., f is a homeomorphism). We may refer to both the surface S and the function f as the embedding. It is relatively easy to imagine that the s-torus can be mapped onto the middle object in Figure 11 as a topological embedding, since any sort of stretching and bending is consistent with a continuous function. All of the orientable closed s-manifolds can be embedded topologically in $\mathbf{R}^3$, and all of the non-orientable closed s-manifolds can be embedded topologically in $\mathbf{R}^4$.

A topological embedding will be called a **flexible embedding**, if distances and angles are preserved. Intuitively, this means that we allow bending, but not stretching. For example, the s-cylinder can be flexibly embedded. This corresponds to taking a piece of paper and rolling it into a cylinder. The paper is bent, but distances and angles remain the same on the surface.

A topological embedding will be called a **rigid embedding**, if each triangular disk in the s-manifold is mapped to an equilateral triangular disk with sides of length 1. In other words, a flexible embedding is rigid if the bending takes place only along the edges and vertices. This would correspond roughly to a Polydron® model, such as the one shown in Figure 10. Again, this model does not really represent a rigid embedding, since the model had to be forced into place.

Among the closed s-manifolds we have considered, only the s-sphere has an obvious rigid embedding, since it is essentially an icosahedron. Paper models of the s-torus can be made with considerable crumpling, so it seems that the s-torus and any of the orientable connect sum s-manifolds have flexible embeddings. They clearly do not have rigid embeddings, however.

Question. Are there other s-manifold structures on the torus that have rigid embeddings? If this is the case, then all of the orientable closed 2-manifolds should also have rigid embeddings using some sort of connect sum. Can any of the non-orientable closed s-manifolds be rigidly embedded in $\mathbf{R}^4$?

There is clearly much that is not known about closed s-manifolds. The structure of the s-lines is very complicated as is the structure of the hyperbolic closed s-manifolds. Advances in either of these areas would be very interesting. It would seem that the most important kind of result for closed s-manifolds, and s-manifolds in general, would be a strong positive connection between the structure of s-lines and some structure outside of Smarandache geometry. For example, a nice relationship between s-lines and the elements of the fundamental group would be an indication of the importance of s-manifolds in general.

Suggestions For Future Study

We have touched on many concepts basic to the current study of geometry and topology of manifolds. Seeing these in the context of an s-manifold can be viewed as an introduction to further study of manifolds in general. A good place to start would be **[21, 20]**.

The study of polyhedral surfaces is well developed (see **[1]**), and its study today is probably most active in the area of computational geometry. There is a lot of information available on the internet, and a good place to start a search is with **[18]**.

A relatively new, and very important, concept is that of an orbifold. Orbifolds share with s-manifolds a focus on singularities. The Geometry Center has information on orbifolds at the websites **[7, 6]**.

The focus of this book, of course, is Smarandache geometry, so let us finish with a discussion of future research in this area.

There is much analysis that can be done on the theorems of Euclidean, hyperbolic, and elliptic geometry. Pairing any proposition or theorem from **[12]**, for example, and an s-manifold, or group of s-manifolds, presents a problem to be explored. Something along these lines might make a good undergraduate research project.

Any theorem of Riemannian geometry should have an s-manifold analog. The same should be true for any theorem involving polyhedral surfaces or orbifolds. Generalizations of s-manifolds to higher dimensions and alternate configurations are also possible. These would likely be research projects at the level of this book.

Most interesting, of course, would be results peculiar to s-manifolds or Smarandache geometries in general. These might include properties that induce a non-trivial categorization of s-manifolds (non-trivial in the sense of there not being too few or too many categories). Perhaps there are interesting consequences of certain combinations of S-denials.

There might also be certain s-manifold structures that correspond to more mainstream areas of geometry and topology. A strong connection between s-lines and the elements of the fundamental group might provide insight into the topology of manifolds, for example. Anything along these lines would be very exciting.

Thank you for reading my book. Good luck in your studies, and I would be very happy to hear about your findings.

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About the Author

Howard Iseri is an associate professor of mathematics at Mansfield University in Pennsylvania, where he lives with his wife Linda and daughter Zoe. Howard's current interests involve the study of Smarandache geometry and the geometry and topology of manifolds. He has a PhD in mathematics from the University of California, Davis, where he wrote a thesis on minimal surfaces under the supervision of Joel Hass. He also has an MA and BA in mathematics from the California State University, Sacramento.

A **Smarandache geometry** (1969) is a geometric space (i.e., one with points, lines) such that some "axiom" is false in at least two different ways, or is false and also sometimes true. Such an axiom is said to be **Smarandachely denied** (or **S-denied** for short).
In Smarandache geometry, the intent is to study non-uniformity, so we require it in a very general way.

A manifold that supports a such geometry is called **Smarandache manifold** (or **s-manifold**). As a special case, in this book Dr. Howard Iseri studies the s-manifold formed by any collection of (equilateral) triangular disks joined together such that each edge is the identification of one edge each from two distinct disks and each vertex is the identification of one vertex from each of five, six, or seven distinct disks.

Thus, as a particular case, Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries may be united altogether, in the same space, by certain Smarandache geometries. These last geometries can be partially Euclidean and partially Non-Euclidean.