A New View of Combinatorial Maps by Smarandache's Notion

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Abstract: On a geometrical view, the conception of map geometries is introduced, which is a nice model of the Smarandache geometries, also new kind of and more general intrinsic geometry of surfaces. Some open problems related combinatorial maps with the Riemann geometry and Smarandache geometries are presented.

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1. **What is a combinatorial map**

A *graph* Γ is a 2-tuple (V, E) consists of a finite non-empty set V of vertices together with a set E of unordered pairs of vertices, i.e., $E \subseteq V \times V$. Often denoted by $V(\Gamma)$, $E(\Gamma)$ the vertex set and edge set of the graph Γ([9]).

For example, the graph in the Fig.1 is the complete graph K_4 with vertex set $V = \{1, 2, 3, 4\}$ and edge set $E = \{12, 13, 14, 23, 24, 34\}$.

A map is a connected topological graph cellularly embedded in a surface. In 1973, Tutte gave an algebraic representation for an embedding of a graph on locally orientable surface ([18]), which transfer a geometrical partition of a surface to a kind of permutation in algebra as follows([7][8]).

A *combinatorial map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a basic permutation $\mathcal{P}$, i.e, for any $x \in \mathcal{X}_{\alpha,\beta}$, no integer k exists such that $\mathcal{P}^k x = \alpha x$, acting on $\mathcal{X}_{\alpha,\beta}$, the disjoint union of *quadricells* Kx of $x \in X$ (the base set), where $K = \{1, \alpha, \beta, \alpha\beta\}$ is the *Klein group*, satisfying the following two conditions:

(i) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

(ii) *the group* $\Psi_J =< \alpha, \beta, \mathcal{P} >$ *is transitive on* $\mathcal{X}_{\alpha,\beta}$.

For a given map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, it can be shown that $M^* = (\mathcal{X}_{\beta,\alpha}, \mathcal{P}\alpha\beta)$ is also a map, call it the *dual* of the map M. The vertices of M are defined as the pairs of conjugatcy orbits of $\mathcal{P}$ action on $\mathcal{X}_{\alpha,\beta}$ by the condition (Ci) and edges the orbits of K on $\mathcal{X}_{\alpha,\beta}$, for example,$\forall x \in \mathcal{X}_{\alpha,\beta}$, $\{x, \alpha x, \beta x, \alpha\beta x\}$ is an edge of the map M. Define the faces of M to be the vertices in the dual map M^*. Then the Euler characteristic $\chi(M)$ of the map M is

$$\chi(M) = \nu(M) - \varepsilon(M) + \phi(M)$$

where,$\nu(M), \varepsilon(M), \phi(M)$ are the number of vertices, edges and faces of the map M, respectively. For each vertex of a map M, its valency is defined to be the length of the orbits of $\mathcal{P}$ action on a quadricell incident with u.

For example, the graph K_4 on the tours with one face length 4 and another 8 , can be algebraic represented as follows:

A map $(\mathcal{X}_{\alpha,\beta},\mathcal{P})$ with $\mathcal{X}_{\alpha,\beta}=\{x,y,z,u,v,w,\alpha x,\alpha y,\alpha z,\alpha u,\alpha v,\alpha w,\beta x,\beta y,\beta z,\beta u,\beta v,\beta w,\alpha\beta x,\alpha\beta y,\alpha\beta z,\alpha\beta u,\alpha\beta v,\alpha\beta w\}$ and

$$\begin{aligned}\mathcal{P} &= (x,y,z)(\alpha\beta x,u,w)(\alpha\beta z,\alpha\beta u,v)(\alpha\beta y,\alpha\beta v,\alpha\beta w)\\ &\times (\alpha x,\alpha z,\alpha y)(\beta x,\alpha w,\alpha u)(\beta z,\alpha v,\beta u)(\beta y,\beta w,\beta v)\end{aligned}$$

The four vertices of this map are $\{(x,y,z),(\alpha x,\alpha z,\alpha y)\}$, $\{(\alpha\beta x,u,w),(\beta x,\alpha w,\alpha u)\}$, $\{(\alpha\beta z,\alpha\beta u,v),(\beta z,\alpha v,\beta u)\}$ and $\{(\alpha\beta y,\alpha\beta v,\alpha\beta w),(\beta y,\beta w,\beta v)\}$ and six edges are $\{e,\alpha e,\beta e,\alpha\beta e\}$, where, $e\in\{x,y,z,u,v,w\}$. The Euler characteristic $\chi(M)$ is $\chi(M)=4-6+2=0$.

Geometrically, an embedding M of a graph Γ on a surface is a map and has an algebraic representation. The graph Γ is said the *underlying graph* of the map M and denoted by $\Gamma=\Gamma(M)$. For determining a given map $(\mathcal{X}_{\alpha,\beta},\mathcal{P})$ is orientable or not, the following condition is needed.

(iii) If the group $\Psi_I=<\alpha\beta,\mathcal{P}>$ is transitive on $\mathcal{X}_{\alpha,\beta}$, then M is non-orientable. Otherwise, orientable.

It can be shown that the number of orbits of the group $\Psi_I=<\alpha\beta,\mathcal{P}>$ in the Fig.2 action on $\mathcal{X}_{\alpha,\beta}=\{x,y,z,u,v,w,\alpha x,\alpha y,\ \alpha z,\alpha u,\alpha v,\alpha w,\beta x,\beta y,\beta z,\beta u,\beta v,\beta w,\alpha\beta x,\alpha\beta y,\alpha\beta z,\alpha\beta u,\alpha\beta v,\alpha\beta w\}$ is 2. Whence, it is an orientable map and the genus of the surface is 1. Therefore, the algebraic representation is correspondent with its geometrical mean.

2. **What are lost in combinatorial maps**

As we known, mathematics is a powerful tool of sciences for its unity and neatness, without any shade of mankind. On the other hand, it is also a kind of aesthetics deep down in one's mind. There is a famous proverb says that *only the beautiful things can be handed down to today*, which is also true for the mathematics.

Here, the term *unity* and *neatness* is relative and local, also have various conditions. For acquiring the target, many unimportant matters are abandoned in the process. Whether are those matters in this time still unimportant in another time? It is not true. That is why we need to think the question: *what are lost in the classical mathematics?*

For example, a compact surface is topological equivalent to a polygon with even number of edges by identifying each pairs of edges along a given direction on it([17]). If label each pair of edges by a letter $e,e\in\mathcal{E}$, a surface S is also identifying to a cyclic permutation such that each edge $e,e\in\mathcal{E}$ just appears two times in S, one is e and another is e^{-1}. Let $a, b, c, \cdots$ denote the letters in $\mathcal{E}$ and $A, B, C, \cdots$ the sections of successive letters in linear order on a surface S (or a string of letters on S). Then, a surface can be represented as follows:

$$S = (\cdots, A, a, B, a^{-1}, C, \cdots),$$

where$a \in \mathcal{E}$,A, B, C denote a string of letters. Define three elementary transformations as follows:

(O_1) $\quad (A, a, a^{-1}, B) \Leftrightarrow (A, B);$

(O_2) $\quad (i) \quad (A, a, b, B, b^{-1}, a^{-1}) \Leftrightarrow (A, c, B, c^{-1});$
$\quad (ii) \quad (A, a, b, B, a, b) \Leftrightarrow (A, c, B, c);$

(O_3) $\quad (i) \quad (A, a, B, C, a^{-1}, D) \Leftrightarrow (B, a, A, D, a^{-1}, C);$
$\quad (ii) \quad (A, a, B, C, a, D) \Leftrightarrow (B, a, A, C^{-1}, a, D^{-1}).$

If a surface S_0 can be obtained by the elementary transformation O_1-O_3 from a surface S, it is said that S *elementary equivalent* with S_0, denoted by $S \sim_{El} S_0$.

We have known the following formula in [8]:
$(i) \quad (A, a, B, b, C, a^{-1}, D, b^{-1}, E) \sim_{El} (A, D, C, B, E, a, b, a^{-1}, b^{-1});$
$(ii) \quad (A, c, B, c) \sim_{El} (A, B^{-1}, C, c, c);$
$(iii) \quad (A, c, c, a, b, a^{-1}, b^{-1}) \sim_{El} (A, c, c, a, a, b, b).$

Then we can get the classification theorem of compact surface as follows([14]):

Any compact surface is homeomorphic to one of the following standard surfaces:
(P_0) The sphere: aa^{-1};
(P_n) The connected sum of $n, n \geq 1$, tori:

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1};$$

(Q_n) The connected sum of $n, n \geq 1$, projective planes:

$$a_1a_1a_2a_2 \cdots a_na_n.$$

Generally, a combinatorial map is a kind of decomposition of a surface. Notice that all the standard surfaces are just one face map underlying an one vertex graph. By combinatorial view, a combinatorial map is also a surface. But this assertion need more clarifying. For example, see the tetrahedron graph Π_4 in the R^3 and a map K_4 on the sphere. Whether we can say it is the sphere? Certainly NOT. Since any point u on a sphere has a neighborhood $N(u)$ homeomorphic to the open disc, therefore, all angles incident with the point 1 must all be $120°$ degree on a sphere. But in Π_4, they are all $60°$ degree. For making them topologically same, i.e., homeomorphism, we must blow up the Π_4 to a sphere, as shown in the Fig.3. Whence, for getting the classification theorem of compact surfaces, we lose the *angle,area, volume,distance,curvature,$\cdots$*, etc, which are also lost in the combinatorial maps.

Klein Erlanger Program says that *any geometry is finding invariant properties under the transformation group of this geometry.* This is essentially the group action idea and widely used in mathematics today. In the combinatorial maps, we know the following problems are applications of the Klein Erlanger Program:

(i)*to determine isomorphism maps or rooted maps;*
(ii)*to determine equivalent embeddings of a graph;*
(iii)*to determine an embedding whether exists;*
(iv)*to enumerate maps or rooted maps on a surface;*
(v)*to enumerate embeddings of a graph on a surface;*
(vi) $\cdots$, etc.

All the problems are extensively investigated by researches in the last century and papers related those problems are still appearing frequently on the journals today. Then, *what are their importance to classical mathematics?* and *what are their contributions to science?* Those are the central topics of this paper.

3. **The Smarandache geometries**

The *Smarandache geometries* is proposed by Smarandache in 1969 ([16]), which is a generalization of the classical geometries, i.e., the Euclid, Lobachevshy-Bolyai-Gauss and Riemannian geometries may be united altogether in the same space, by some Smarandache geometries. These last geometries can be either partially Euclidean and partially Non-Euclidean, or Non-Euclidean. It seems that the Smarandache geometries are connected with the *Relativity Theory* (because they include the Riemann geometry in a subspace) and with the *Parallel Universes* (because they combine separate spaces into one space) too([5]). For a detail illustration, we need to consider the classical geometries.

The axioms system of *Euclid geometry* are the following:

(A1)*there is a straight line between any two points.*
(A2)*a finite straight line can produce a infinite straight line continuously.*
(A3)*any point and a distance can describe a circle.*
(A4)*all right angles are equal to one another.*
(A5)*if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

The axiom (A5) can be also replaced by:

(A5')*given a line and a point exterior this line, there is one line parallel to this line.*

The Lobachevshy-Bolyai-Gauss geometry, also called hyperbolic geometry, is a geometry with axioms $(A1)-(A4)$ and the following axiom $(L5)$:

(L5) *there are infinitely many line parallels to a given line passing through an* *exterior point.*

The Riemann geometry, also called elliptic geometry, is a geometry with axioms $(A1)-(A4)$ and the following axiom $(R5)$:

there is no parallel to a given line passing through an exterior point.

By the thought of Anti-Mathematics: not in a nihilistic way, but in a positive one, i.e., banish the old concepts by some new ones: their opposites, Smarandache introduced the *paradoxist geometry, non-geometry, counter-projective geometry and anti-geometry* in [16] by contradicts the axioms $(A1)-(A5)$ in Euclid geometry, generalize the classical geometries.

Paradoxist geometry

In this geometry, its axioms are $(A1)-(A4)$ and with one of the following as the axiom $(P5)$:

(i)*there are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersect the initial line.*

(ii)*there are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line.*

(iii)*there are at least a straight line and a point exterior to it in this space for which only a finite number of lines* $l_1, l_2, \cdots, l_k, k \geq 2$ *pass through the point and do not intersect the initial line.*

(iv)*there are at least a straight line and a point exterior to it in this space for which an infinite number of lines pass through the point (but not all of them) and do not intersect the initial line.*

(v)*there are at least a straight line and a point exterior to it in this space for which any line that passes through the point and does not intersect the initial line.*

Non-Geometry

The non-geometry is a geometry by denial some axioms of $(A1)-(A5)$, such as:

$(A1^-)$*It is not always possible to draw a line from an arbitrary point to another arbitrary point.*

$(A2^-)$*It is not always possible to extend by continuity a finite line to an infinite line.*

$(A3^-)$*It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.*

$(A4^-)$*not all the right angles are congruent.*

$(A5^-)$*if a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.*

Counter-Projective geometry

Denoted by P the point set, L the line set and R a relation included in $P \times L$. A counter-projective geometry is a geometry with the following counter-axioms:

$(C1)$ *There exist: either at least two lines, or no line, that contains two given distinct points.*

$(C2)$ *Let p_1, p_2, p_3 be three non-collinear points, and q_1, q_2 two distinct points. Suppose that $\{p_1.q_1, p_3\}$ and $\{p_2, q_2, p_3\}$ are collinear triples. Then the line containing p_1, p_2 and the line containing q_1, q_2 do not intersect.*

$(C3)$ *Every line contains at most two distinct points.*

Anti-Geometry

A geometry by denial some axioms of the Hilbert's 21 axioms of Euclidean geometry. As shown in [5], there are at least $2^{21} - 1$ anti-geometries.

The Smarandache geometries are defined as follows.

Definition 3.1 *An axiom is said Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(*1969*).*

A nice model for the Smarandache geometries, called s-manifolds, is found by Iseri in [3][4], which is defined as follows:

An s-manifold is any collection $\mathcal{C}(T, n)$ of these equilateral triangular disks $T_i, 1 \leq i \leq n$ satisfying the following conditions:

(i) *Each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks $T_i, T_j, 1 \leq i, j \leq n$ and $i \neq j$;*

(ii) *Each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks.*

The vertices are classified by the number of the disks around them. A vertex around five, six or seven triangular disks is called an *elliptic vertex*, a *Euclid vertex* or a *hyperbolic vertex*, respectively.

An s-manifold is called closed if each edge is shared by exactly two triangular disks. An elementary classification for closed s-manifolds by triangulation are made in the reference [11]. The closed s-manifolds are classified into 7 classes in [11], as follows:

Classical Type:

(1) $\Delta_1 = \{5 - \textit{regular triangular maps}\}$ (*elliptic*);
(2) $\Delta_2 = \{6 - \textit{regular triangular maps}\}$(*euclidean*);
(3) $\Delta_3 = \{7 - \textit{regular triangular maps}\}$(*hyperbolic*).

Smarandache Type:

(4) $\Delta_4 = \{$*triangular maps with vertex valency* 5 *and* $6\}$ (*euclid-elliptic*);
(5) $\Delta_5 = \{$*triangular maps with vertex valency* 5 *and* $7\}$ (*elliptic-hyperbolic*);
(6) $\Delta_6 = \{$*triangular maps with vertex valency* 6 *and* $7\}$ (*euclid-hyperbolic*);
(7) $\Delta_7 = \{$*triangular maps with vertex valency* $5, 6$ *and* $7\}$ (*mixed*).

It is proved in [11] that $|\Delta_1| = 2$, $|\Delta_5| \geq 2$ and $|\Delta_i|, i = 2, 3, 4, 6, 7$ are infinite. Isier proposed a question in [3]: *Do the other closed* 2*-manifolds correspond to s-manifolds with only hyperbolic vertices?*. Since there are infinite Hurwitz maps, i.e., $|\Delta_3|$ is infinite, the answer is affirmative.

4. **The map geometries**

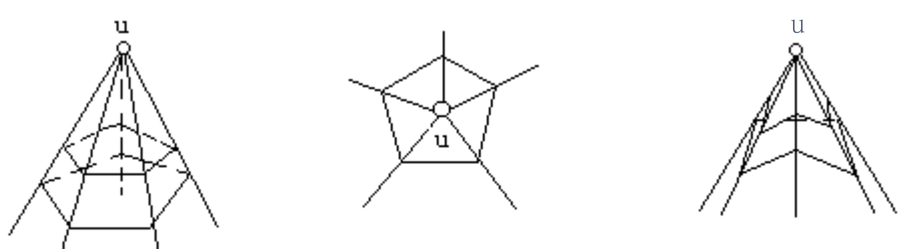
Combinatorial maps can be used to construct new geometries, which are nice models for the Smarandache geometries, also a generalization of Isier's model and Poincaré's model for hyperbolic geometry.

4.1 **Map geometries without boundary**

For a given map on a surface, the map geometries without boundary are defined as follows.

Definition 4.1 *For a combinatorial map M with each vertex valency≥ 3, associates a real number $\mu(u), 0 < \mu(u) < \pi$, to each vertex $u, u \in V(M)$. Call (M, μ) a map geometry with out boundary, $\mu(u)$ the angle factor of the vertex u and to be orientablle or non-orientable if M is orientable or not.*

The realization of each vertex $u, u \in V(M)$ in R^3 space is shown in the Fig.1 for each case of $\rho(u)\mu(u) > 2\pi$, $= 2\pi$ or $< 2\pi$.



$\rho(u)\mu(u) < 2\pi$ $\rho(u)\mu(u) = 2\pi$ $\rho(u)\mu(u) > 2\pi$

Fig.1

As pointed out in the Section 2, this kind of realization is not a surface, but it is homeomorphic to a surface. We classify points in a map geometry (M, μ) with out boundary as follows.

Definition 4.2 *A point u in a map geometry (M, μ) is called elliptic, euclidean or hyperbolic if $\rho(u)\mu(u) \ < \ 2\pi$, $\rho(u)\mu(u) = 2\pi$ or $\rho(u)\mu(u) \ > \ 2\pi$.*

Then we have the following results.

Proposition 4.1 *Let M be a map with $\forall u \in V(M), \rho(u) \geq 3$. Then for $\forall u \in V(M)$, there is a map geometries (M, μ) without boundary such that u is elliptic, euclidean or hyperbolic in this geometry.*

Proof Since $\rho(u) \geq 3$, we can choose the angle factor $\mu(u)$ such that $\mu(u)\rho(u) \ < \ 2\pi$, $\mu(u)\rho(u) = 2\pi$ or $\mu(u)\rho(u) \ > \ 2\pi$. Notice that

$$0 \ < \frac{2\pi}{\rho(u)} \ < \ \pi.$$

Whence, we can also choose $\mu(u)$ satisfying that $0 \ < \mu(u) \ < \pi$ $\natural$

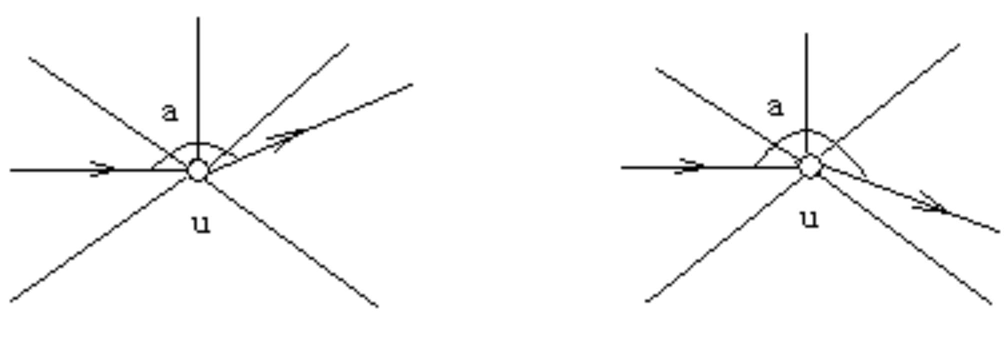
Proposition 4.2 *Let M be a map of order≥ 3 and $\forall u \in V(M), \rho(u) \geq 3$. Then there exists a map geometry (M, μ) with out boundary, in which all points are one of the elliptic vertices, euclidean vertices and hyperbolic vertices or their mixed.*

Proof According to the Proposition 4.1, we can choose an angle factor μ such that a vertex $u, u \in V(M)$ to be elliptic, or euclidean, or hyperbolic. Since $|V(M)| \geq 3$, we can also choose the angle factor μ such that any two vertices $v, w \in V(M) \setminus \{u\}$ to be elliptic, or euclidean, or hyperbolic according to our wish. Then the map geometry (M, μ) makes the assertion hold. $\natural$

A *geodesic* in a manifold is a curve as straight as possible. Similarly, in a map geometry, its m-lines and m-points are defined as follows.

Definition 4.3 *Let (M, μ) be a map geometry without boundary. An m-line in (M, μ) is a curve with a constant curvature and points in it are called m-points.*

If an m-line pass through an elliptic point or a hyperbolic point u, it must has the angle $\frac{\mu(u)\rho(u)}{2}$ with the entering line, not $180°$, which are explained in the Fig.2.



a $= \frac{\mu(u)\rho(u)}{2} \ < \ \pi$ a $= \frac{\mu(u)\rho(u)}{2} \ > \ \pi$

Fig.2

The following proposition asserts that all map geometries without boundary are Smarandache geometries.

Proposition 4.3 *For a map M on a locally orientable surface with order≥ 3 and vertex valency≥ 3, there is an angle factor μ such that (M, μ) is a Smarandache geometry by denial the axiom (A5) with the axioms (A5),(L5) and (R5).*

Proof According to the Proposition 4.1, we know that there exist an angle factor μ such that there are elliptic vertices, euclidean vertices and hyperbolic vertices in (M, μ) simultaneously. The proof is divided into three cases.

Case 1. M **is a planar map**

Notice that for a given line L not pass through the vertices in the map M and a point u on its left side in (M, μ), if u is an euclidean point, then there is one and only one line passes through u not intersect with L, and if u is an elliptic point, then there are infinite lines pass through u not intersect with L, but if u is a hyperbolic point, then each line passes through u will intersect with L. Therefore, (M, μ) is a Smarandache geometry by denial the axiom (A5) with the axioms (A5), (L5) and (R5).

Case 2. M **is an orientable map**

According to the classification theorem of compact surfaces, We only need to prove this result for the torus. Notice that on the torus, an m-line has the following properties ([15]):

If the slope ς of m-line L is a rational number, then L is a closed line on the torus. Otherwise, L is infinite, and moreover L passes arbitrarily close to every point of the torus.

Whence, if L_1 is an m-line on the torus, not passes through an elliptic or hyperbolic point, then for any point u exterior L_1, we know that if u is an euclidean point, then there is only one m-line passes through u not intersect with L_1, and if u is elliptic or hyperbolic, then any m-line passes through u will intersect with L_1.

Now let L_2 be an m-line passes through an elliptic or hyperbolic point, such as the m-line in the Fig.3 and v an euclidean point.

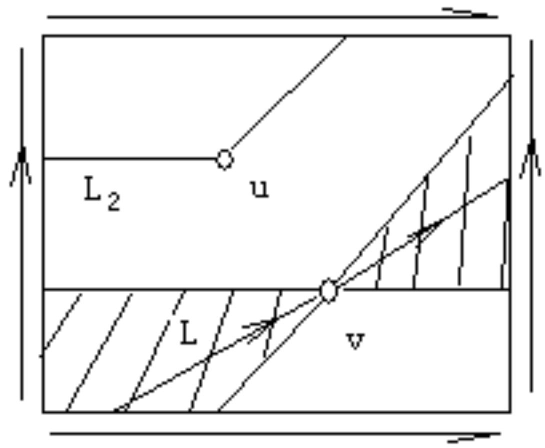


Fig.3

Then any m-line L in the shade filed passes through the point v will not intersect with L_2. Therefore, (M, μ) is a Smarandache geometry by denial the axiom (A5) with the axioms (A5),(L5) and (R5).

Case 3. M **is a non-orientable map**

Similar to the Case 2, by the classification theorem of the compact surfaces, we only need to prove this result for the projective plane. Now let the m-line passes through the center in the circle. Then if u is an euclidean point, there is only one m-line passes through u, see (a) in the Fig.4. If v is an elliptic point and there is an m-line passes through it and intersect with L, see (b) in the Fig.4, assume the point 1 is a point such that the m-line $1v$ passes through 0, then any m-line in the shade of (b) passes through the point v will intersect with L.

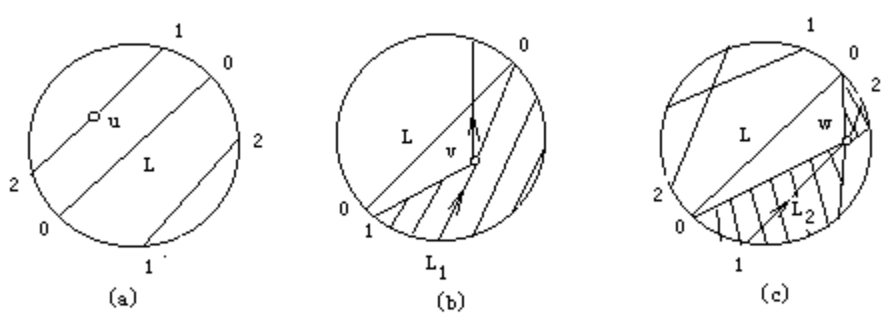


Fig.4

If w is a point and there is an m-line passes through it and does not intersect with L, see (c) in the Fig.4, then any m-line in the shade of (c) passes through the point w will not intersect with L. Since the position of the vertices of the map M on the projective plane can be choose as our wish, the proof is complete. $\natural$.

4.2 Map geometries with boundary

The Poincaré's model for hyperbolic geometry hints us to introduce the map geometries with boundary, which is defined as follows.

Definition 4.4 *For a map geometry (M,μ) without boundary and faces $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M)-1$, if $(M,\mu)\setminus\{f_1, f_2, \cdots, f_l\}$ is connected, then call $(M,\mu)^{-l} = (M,\mu)\setminus\{f_1, f_2, \cdots, f_l\}$ a map geometry with boundary $f_1, f_2, \cdots, f_l$ and orientable or not if (M,μ) is orientable or not.*

A connected curve with constant curvature in $(M,\mu)^{-l}$ is called an m^--line and points m^--points.

The map geometries with boundary also are Smarandache geometries, which is convince by the following result.

Proposition 4.4 *For a map M on a locally orientable surface with order≥ 3, vertex valency≥ 3 and a face $f \in F(M)$, there is an angle factor μ such that $(M,\mu)^{-1}$ is a Smarandache geometry by denial the axiom (A5) with the axioms (A5),(L5) and (R5).*

Proof Similar to the proof of the Proposition 4.3, consider the map M being a planar map, an orientable map on a torus or a non-orientable map on a projective plane, respectively. We get the assertion. $\natural$

Notice that for an one face map geometry $(M,\mu)^{-1}$ with boundary, if we choose all points being euclidean, then $(M,\mu)^{-1}$ is just the Poincaré's model for hyperbolic geometry.

4.3 Classification of map geometries

For the classification of the map geometries, we introduce the following definition.

Definition 4.5 *Two map geometries (M_1,μ_1) and (M_2,μ_2) or $(M_1,\mu_1)^{-l}$ and $(M_2,\mu_2)^{-l}$ are equivalent if there is a bijection $\theta : M_1 \rightarrow M_2$ such that for $\forall u \in V(M)$, $\theta(u)$ is euclidean, elliptic or hyperbolic iff u is euclidean, elliptic or hyperbolic.*

The relation of the numbers of unrooted maps with the map geometries is the following.

Proposition 4.5 *If $\mathcal{M}$ is a set of non-isomorphisc maps with order n and m faces, then the number of map geometries without boundary is $3^n|\mathcal{M}|$ and the number of map geometries with one face being its boundary is $3^n m|\mathcal{M}|$.*

Proof By the definition, for a map $M \in \mathcal{M}$, there are 3^n map geometries without boundary and $3^n m$ map geometries with one face being its boundary by the Proposition 4.3. Whence, we get $3^n|\mathcal{M}|$ map geometries without boundary and $3^n m|\mathcal{M}|$ map geometries with one face being its boundary from $\mathcal{M}$. $\natural$.

We have the following enumeration result for the non-equivalent map geometries without boundary.

Proposition 4.6 *The numbers $n^O(\Gamma, g)$, $n^N(\Gamma, g)$ of non-equivalent orientable, non-orientable map geometries without boundary underlying a simple graph Γ by denial the axiom (A5) by (A5), (L5) or (R5) are*

$$n^O(\Gamma, g) = \frac{3^{|\Gamma|} \prod_{v \in V(\Gamma)} (\rho(v) - 1)!}{2|\mathrm{Aut}\Gamma|},$$

and

$$n^N(\Gamma, g) = \frac{(2^{\beta(\Gamma)} - 1)3^{|\Gamma|} \prod_{v \in V(\Gamma)} (\rho(v) - 1)!}{2|\mathrm{Aut}\Gamma|},$$

where $\beta(\Gamma) = \varepsilon(\Gamma) - \nu(\Gamma) + 1$ is the Betti number of the graph Γ.

Proof Denote by $\mathcal{M}(\Gamma)$ the set of non-isomorphic maps underlying the graph Γ on locally orientable surfaces and by $\mathcal{E}(\Gamma)$ the set of embeddings of the graph Γ on the locally orientable surfaces. For a map $M, M \in \mathcal{M}(\Gamma)$, there are $\frac{3^{|M|}}{|\mathrm{Aut}M|}$ different map geometries without boundary by choosing the angle factor μ on a vertex u such that u is euclidean, elliptic or hyperbolic. From permutation groups, we know that

$$|\mathrm{Aut}\Gamma \times < \alpha > | = |(\mathrm{Aut}\Gamma)_M||M^{\mathrm{Aut}\Gamma \times <\alpha>}| = |\mathrm{Aut}M||M^{\mathrm{Aut}\Gamma \times <\alpha>}|.$$

Therefore, we get that

$$\begin{aligned} n^O(\Gamma, g) &= \sum_{M \in \mathcal{M}(\Gamma)} \frac{3^{|M|}}{|\mathrm{Aut}M|} \\ &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma \times < \alpha > |} \sum_{M \in \mathcal{M}(\Gamma)} \frac{|\mathrm{Aut}\Gamma \times < \alpha > |}{|\mathrm{Aut}M|} \\ &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma \times < \alpha > |} \sum_{M \in \mathcal{M}(\Gamma)} |M^{\mathrm{Aut}\Gamma \times <\alpha>}| \\ &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma \times < \alpha > |} |\mathcal{E}^O(\Gamma)| \\ &= \frac{3^{|\Gamma|} \prod_{v \in V(\Gamma)} (\rho(v) - 1)!}{2|\mathrm{Aut}\Gamma|}. \end{aligned}$$

Similarly, we get that

$$\begin{aligned} n^N(\Gamma, g) &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma \times <\alpha>|}|\mathcal{E}^N(\Gamma)| \\ &= \frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|}. \end{aligned}$$

This completes the proof. ♮

For the classification of map geometries with boundary, we have the following result.

Proposition 4.7 *The numbers $n^O(\Gamma,-g)$, $n^N(\Gamma,-g)$ of non-equivalent orientable, non-orientable map geometries with one face being its boundary and underlying a simple graph Γ by denial the axiom (A5) by (A5), (L5) or (R5) are*

$$n^O(\Gamma,-g) = \frac{3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{2d(g[\Gamma](x))}{dx}|_{x=1}]$$

and

$$n^N(\Gamma,-g) = \frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{2d(g[\Gamma](x))}{dx}|_{x=1}],$$

where $g[\Gamma](x)$ is the genus polynomial of the graph Γ (see [12]*), i.e., $g[\Gamma](x) = \sum\limits_{k=\gamma(\Gamma)}^{\gamma_m(\Gamma)} g_k[\Gamma]x^k$ with $g_k[\Gamma]$ being the number of embeddings of Γ on the orientable surface of genus k.*

Proof Notice that $\nu(M)-\varepsilon(M)+\phi(M) = 2-2g(M)$ for an orientable map M by the Euler characteristic. Similar to the proof of the Proposition 4.6 with the notation $\mathcal{M}(\Gamma)$, by the Proposition 4.5 we know that

$$\begin{aligned} n^O(\Gamma,-g) &= \sum_{M\in\mathcal{M}(\Gamma)}\frac{\phi(M)3^{|M|}}{|\mathrm{Aut}M|} \\ &= \sum_{M\in\mathcal{M}(\Gamma)}\frac{(2+\varepsilon(\Gamma)-\nu(\Gamma)-2g(M))3^{|M|}}{|\mathrm{Aut}M|} \\ &= \sum_{M\in\mathcal{M}(\Gamma)}\frac{(2+\varepsilon(\Gamma)-\nu(\Gamma))3^{|M|}}{|\mathrm{Aut}M|} - \sum_{M\in\mathcal{M}(\Gamma)}\frac{2g(M)3^{|M|}}{|\mathrm{Aut}M|} \\ &= \frac{(2+\varepsilon(\Gamma)-\nu(\Gamma))3^{|M|}}{|\mathrm{Aut}\Gamma\times<\alpha>|}\sum_{M\in\mathcal{M}(\Gamma)}\frac{|\mathrm{Aut}\Gamma\times<\alpha>|}{|\mathrm{Aut}M|} \end{aligned}$$

$$\begin{aligned}
&- \frac{2\times 3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times <\alpha>|}\sum_{M\in\mathcal{M}(\Gamma)}\frac{g(M)|\mathrm{Aut}\Gamma\times <\alpha>|}{|\mathrm{Aut}M|}\\
&= \frac{(\beta(\Gamma)+1)3^{|M|}}{|\mathrm{Aut}\Gamma\times <\alpha>|}\sum_{M\in\mathcal{M}}(\Gamma)|M^{\mathrm{Aut}\Gamma\times<\alpha>}|\\
&- \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma|}\sum_{M\in\mathcal{M}(\Gamma)} g(M)|M^{\mathrm{Aut}\Gamma\times<\alpha>}|\\
&= \frac{(\beta(\Gamma)+1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}\prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma|}\sum_{k=\gamma(\Gamma)}^{\gamma_m(\Gamma)} kg_k[\Gamma]\\
&= \frac{3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{2d(g[\Gamma](x))}{dx}|_{x=1}].
\end{aligned}$$

Notice that $n^L(\Gamma,-g) = n^O(\Gamma,-g)+n^N(\Gamma,-g)$ and the number of re-embeddings of an orientable map M on surfaces is $2^{\beta(M)}$ (see also [13]). We have that

$$\begin{aligned}
n^L(\Gamma,-g) &= \sum_{M\in\mathcal{M}(\Gamma)}\frac{2^{\beta(M)}\times 3^{|M|}\phi(M)}{|\mathrm{Aut}M|}\\
&= 2^{\beta(M)}n^O(\Gamma,-g).
\end{aligned}$$

Whence, we get that

$$\begin{aligned}
n^N(\Gamma,-g) &= (2^{\beta(M)}-1)n^O(\Gamma,-g)\\
&= \frac{(2^{\beta(M)}-1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{2d(g[\Gamma](x))}{dx}|_{x=1}].
\end{aligned}$$

This completes the proof. $\natural$

4.4 Polygons in a map geometry

A k-polygon in a map geometry is a k-polygon with each line segment being m-lines or m^{-}-lines. For the sum of the internal angles in a k-polygon, we have the following result.

Proposition 4.8 *Let P be a k-polygon in a map geometry with each line segment passes through at most one elliptic or hyperbolic point. If H is the set of elliptic points and hyperbolic points on the line segment of P, then the sum of the internal angles in P is*

$$(k+|H|-2)\pi - \frac{1}{2}\sum_{u\in H}\rho(u)\mu(u).$$

Proof Denote by U, V the sets of elliptic points and hyperbolic points in H and $|U| = p, |V| = q$. If an m-line segment passes through an elliptic point u, add a straight line segment in the plane as the Fig.6(1). Then we get that

$$\text{angle a} = \text{angle1} + \text{angle2} = \pi - \frac{\rho(u)\mu(u)}{2}.$$

If an m-line passes through an hyperbolic point v, also add a straight line segment in the plane as the Fig.6(2). Then we get that

$$\text{angle } b = \text{angle3} + \text{angle4} = \frac{\rho(v)\mu(v)}{2} - \pi.$$

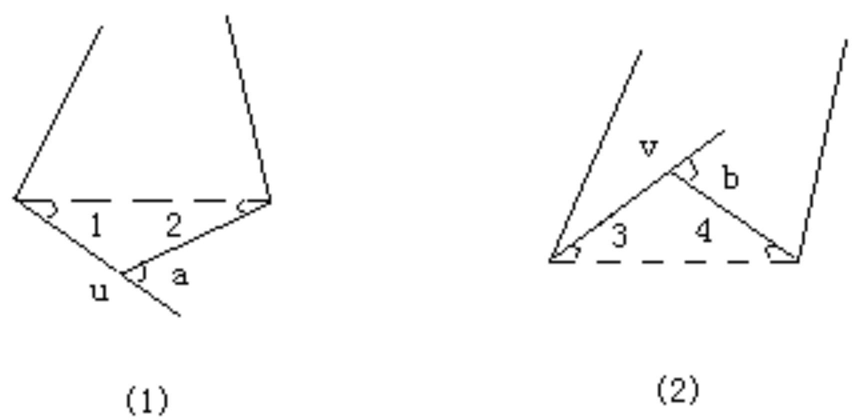


Fig.5

Since the sum of the internal angles of a k-polygon in the plane is $(k-2)\pi$, we know that the sum of the internal angles in P is

$$\begin{aligned}(k & -2)\pi + \sum_{u \in U}(\pi - \frac{\rho(u)\mu(u)}{2}) - \sum_{v \in V}(\frac{\rho(u)\mu(u)}{2} - \pi) \\ &= (k + p + q - 2)\pi - \frac{1}{2}\sum_{u \in H}\rho(u)\mu(u) \\ &= (k + |H| - 2)\pi - \frac{1}{2}\sum_{u \in H}\rho(u)\mu(u).\end{aligned}$$

This completes the proof. $\natural$

As a corollary, we get the sum of the internal angles of a triangle in a map geometry as follows, which is consistent with the classical results.

Corollary 4.1 *Let $\triangle$ be a triangle in a map geometry. Then*

(i) if $\triangle$ is euclidean, then then the sum of its internal angles is equal to π;

(ii) if $\triangle$ is elliptic, then the sum of its internal angles is less than π;

(iii) if $\triangle$ is hyperbolic, then the sum of its internal angles is more than π.

5. **Open problems for applying maps to classical geometries**

Here is a collection of open problems concerned combinatorial maps with the Riemann geometry and Smarandache geometries. Although they are called open problems, in fact, any solution for one of these problems needs to establish a new mathematical system first.

5.1 **The uniformization theorem for simple connected Riemann surfaces**

The *uniformization theorem* for simple connected Riemann surfaces is one of those beautiful results in the Riemann surface theory, which is stated as follows([2]).

If $\mathcal{S}$ is a simple connected Riemann surface, then $\mathcal{S}$ is conformally equivalent to one and only one of the following three:
(a) $\mathcal{C}\bigcup\infty$;
(b) $\mathcal{C}$;
(c) $\triangle = \{z \in \mathcal{C} | |z| < 1\}$.

We have proved in [11] that any automorphism of a map is conformal. Therefore, we can also introduced the conformal mapping between maps. Then, *how can we define the conformal equivalence for maps enabling us to get the uniformization theorem of maps? What is the correspondence class maps with the three type $(a)-(c)$ Riemann surfaces?*

5.2 **Combinatorial construction of an algebraic curve of genus**

A *complex plane algebraic curve $\mathcal{C}_l$* is a homogeneous equation $f(x,y,z) = 0$ in $P_2\mathcal{C} = (C^2 \setminus (0,0,0))/\sim$, where $f(x,y,z)$ is a polynomial in x, y and z with coefficients in $\mathcal{C}$. The degree of $f(x,y,z)$ is said the *degree of the curve $\mathcal{C}_l$*. For a Riemann surface S, a well-known result is ([2])*there is a holomorphic mapping $\varphi : S \to P_2\mathcal{C}$ such that $\varphi(S)$ is a complex plane algebraic curve and*

$$g(S) = \frac{(d(\varphi(S)) - 1)(d(\varphi(S)) - 2)}{2}.$$

By map theory, we know a combinatorial map also is on a surface with genus. Then *whether we can get an algebraic curve by all edges in a map or by make operations on the vertices or edges of the map to get plane algebraic curve with given k-multiple points?* and *how do we find the equation $f(x,y,z) = 0$?*

5.3 **Classification of s-manifolds by maps**

We present an elementary classification for the closed s-manifolds in the Section 3. For the general s-manifolds, their correspondence combinatorial model is the maps on surfaces with boundary, founded by Bryant and Singerman in 1985 ([1]). The later is also related to the modular groups of spaces and need to investigate further itself. The questions are

(i) *how can we combinatorially classify the general s-manifolds by maps with boundary?*

(ii) *how can we find the automorphism group of an s-manifold?*

(iii) *how can we know the numbers of non-isomorphic s-manifolds, with or without root?*

(iv) *find rulers for drawing an s-manifold on a surface, such as, the torus, the projective plane or Klein bottle, not the plane.*

The s-manifolds only using the triangulations of surfaces with vertex valency in $\{5, 6, 7\}$. Then *what are the geometrical mean of the other maps, such as, the* 4-*regular maps on surfaces.* It is already known that the later is related to the Gauss cross problem of curves([9]).

5.4 Map geometries

As we have seen in the previous section, map geometries are the nice model of the Smarandache geometries. More works should be dong for them.

(i) *For a given graph, determine properties of the map geometries underlying this graph.*

(ii) *For a given locally orientable surface, determine the properties of map geometries on this surface.*

(iii) *Classify the map geometries on a locally orientable surface.*

(iv) *Enumerate non-equivalent map geometries underlying a graph or on a locally orientable surface.*

(v) *Establish the surface geometry by map geometries.*

5.5 Gauss mapping among surfaces

In the classical differential geometry, a *Gauss mapping* among surfaces is defined as follows([10]):

Let $\mathcal{S} \subset R^3$ be a surface with an orientation $\mathbf{N}$. *The mapping $N : \mathcal{S} \to R^3$ takes its value in the unit sphere*

$$S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}$$

along the orientation $\mathbf{N}$. *The map $N : \mathcal{S} \to S^2$, thus defined, is called the Gauss mapping.*

we know that for a point $P \in \mathcal{S}$ such that the Gaussian curvature $K(P) \neq 0$ and V a connected neighborhood of P with K does not change sign,

$$K(P) = \lim_{A \to 0} \frac{N(A)}{A},$$

where A is the area of a region $B \subset V$ and $N(A)$ is the area of the image of B by the Gauss mapping $N : \mathcal{S} \to S^2$. The questions are

(i) *what is its combinatorial meaning of the Gauss mapping? How to realizes it by maps?*

(ii) *how we can define various curvatures for maps and rebuilt the results in the classical differential geometry?*

5.6 **The Gauss-Bonnet theorem**

Let $\mathcal{S}$ be a compact orientable surface. Then

$$\int\int_{\mathcal{S}} K d\sigma = 2\pi\chi(\mathcal{S}),$$

where K is Gaussian curvature on $\mathcal{S}$.

This is the famous *Gauss-Bonnet theorem* for compact surface ([2], [6]). The questions are

(i) *what is its combinatorial mean of the Gauss curvature?*

(ii) *how can we define the angle, area, volume, curvature, $\cdots$, of a map?*

(iii) can we rebuilt the Gauss-Bonnet theorem by maps? or can we get a generalization of the classical Gauss-Bonnet theorem by maps?

5.7 **Riemann manifolds**

A Riemann surface is just a Riemann 2-manifold, which has become a source of the mathematical creative power. A *Riemann n-manifold* (M, g) is a n-manifold M with a Riemann metric g. Many important results in Riemann surfaces are generalized to Riemann manifolds with higher dimension ([6]). For example, let $\mathcal{M}$ be a complete, simple-connected Riemann n-manifold with constant sectional curvature c, then we know that $\mathcal{M}$ *is isometric to one of the model spaces* $\mathcal{R}^n$, $S_{\mathcal{R}^n}$ *or* $H_{\mathcal{R}^n}$. *Whether can we systematically rebuilt the Riemann manifold theory by combinatorial maps? or can we make a combinatorial generalization of results in the Riemann geometry, for example, the Chern-Gauss-Bonnet theorem (*[6]*)?*

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