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N^*C^*- Smarandache Curves of Mannheim Curve Couple According to Frenet Frame

Süleyman ŞENYURT and Abdussamet ÇALIŞKAN

(Faculty of Arts and Sciences, Department of Mathematics, Ordu University, 52100, Ordu/Turkey)

E-mail: senyurtsuleyman@hotmail.com

Abstract: In this paper, when the unit Darboux vector of the partner curve of Mannheim curve are taken as the position vectors, the curvature and the torsion of Smarandache curve are calculated. These values are expressed depending upon the Mannheim curve. Besides, we illustrate example of our main results.

Key Words: Mannheim curve, Mannheim partner curve, Smarandache Curves, Frenet invariants.

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§1. Introduction

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve ([12]). Special Smarandache curves have been studied by some authors .

Melih Turgut and Süha Yılmaz studied a special case of such curves and called it Smarandache TB_2 curves in the space E_1^4 ([12]). Ahmad T.Ali studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case ([1]). Muhammed Çetin , Yılmaz Tunçer and Kemal Karacan investigated special Smarandache curves according to Bishop frame in Euclidean 3-Space and they gave some differential goematric properties of Smarandache curves, also they found the centers of the osculating spheres and curvature spheres of Smarandache curves ([5]). Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves ([4]). Özcan Bektaş and Salim Yüce studied some special Smarandache curves according to Darboux Frame in E^3 ([2]). Nurten Bayrak, Özcan Bektaş and Salim Yüce studied some special Smarandache curves in E_1^3 [3]. Kemal Taşköprü, Murat Tosun studied special Smarandache curves according to Sabban frame on S^2 ([11]).

In this paper, special Smarandache curve belonging to α^* Mannheim partner curve such as N^*C^* drawn by Frenet frame are defined and some related results are given.

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§2. Preliminaries

The Euclidean 3-space E^3 be inner product given by

$$\langle , \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame . For an arbitrary curve $\alpha \in E^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by ([6], [9])

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \tag{2.1}$$

For any unit speed $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by

$$W = \tau(s)T(s) + \kappa(s) + B(s).$$

If consider the normalization of the Darboux $C = \frac{1}{\|W\|}W$, we have

$$\cos\varphi = \frac{\kappa(s)}{\|W\|}, \quad \sin\varphi = \frac{\tau(s)}{\|W\|},$$

$$C = \sin\varphi T(s) + \cos\varphi B(s) \tag{2.2}$$

where $\angle(W, B) = \varphi$. Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed two curves and let $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector N of the curve α is linearly dependent on the binormal vector B of the curve α^*, then (α) is called a Mannheim curve and (α^*) a Mannheim partner curve of (α). The pair (α, α^*) is said to be Mannheim pair ([7], [8]). The relations between the Frenet frames $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ are as follows:

$$\begin{cases} T^* = \cos\theta T - \sin\theta B \\ N^* = \sin\theta T + \cos\theta B \\ B^* = N \end{cases} \tag{2.3}$$

$$\begin{cases} \cos\theta = \frac{ds*}{ds} \\ \sin\theta = \lambda\tau^* \frac{ds*}{ds} \end{cases} . \tag{2.4}$$

where $\angle(T, T^*) = \theta$ ([8]).

Theorem 2.1([7]) *The distance between corresponding points of the Mannheim partner curves in $\mathbb{E}^3$ is constant.*

Theorem 2.2 *Let (α,α^*) be a Mannheim pair curves in $\mathbb{E}^3$. For the curvatures and the torsions of the Mannheim curve pair (α,α^*) we have,*

$$\begin{cases} \kappa = \tau^* \sin\theta \dfrac{ds*}{ds} \\ \\ \tau = -\tau^* \cos\theta \dfrac{ds*}{ds} \end{cases} \tag{2.5}$$

and

$$\begin{cases} \kappa^* = \dfrac{d\theta}{ds*} = \theta' \dfrac{\kappa}{\lambda\tau\sqrt{\kappa^2+\tau^2}} \\ \\ \tau^* = (\kappa\sin\theta - \tau\cos\theta)\dfrac{ds*}{ds} \end{cases} \tag{2.6}$$

Theorem 2.3 *Let (α,α^*) be a Mannheim pair curves in $\mathbb{E}^3$. For the torsions τ^* of the Mannheim partner curve α^* we have*

$$\tau^* = \frac{\kappa}{\lambda\tau}$$

Theorem 2.4([10]) *Let (α,α^*) be a Mannheim pair curves in $\mathbb{E}^3$. For the vector C^* is the direction of the Mannheim partner curve α^* we have*

$$C^* = \frac{1}{\sqrt{1+\left(\dfrac{\theta'}{\|W\|}\right)^2}}C + \frac{\dfrac{\theta'}{\|W\|}}{\sqrt{1+\left(\dfrac{\theta'}{\|W\|}\right)^2}}N \tag{2.7}$$

where the vector C is the direction of the Darboux vector W of the Mannheim curve α.

§3. N^*C^*- Smarandache Curves of Mannheim Curve Couple According to Frenet Frame

Let (α,α^*) be a Mannheim pair curves in E^3 and $\{T^*N^*B^*\}$ be the Frenet frame of the Mannheim partner curve α^* at $\alpha^*(s)$. In this case, N^*C^* - Smarandache curve can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(N^* + C^*). \tag{3.1}$$

Solving the above equation by substitution of N^* and C^* from (2.3) and (2.7), we obtain

$$\beta_1(s) = \frac{\left(\cos\theta\|W\| + \sin\theta\sqrt{\theta'^2+\|W\|^2}\right)T + \theta' N + \left(\cos\theta\sqrt{\theta'^2+\|W\|^2} - \sin\theta\|W\|\right)B}{\sqrt{\theta'^2+\|W\|^2}}. \tag{3.2}$$

The derivative of this equation with respect to s is as follows,

$$T_{\beta_1}(s) = \frac{\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\cos\theta - \frac{\theta'\kappa\cos\theta}{\lambda\tau\|W\|}\right]T + \left[\frac{\kappa}{\lambda\tau} - \left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\|W\|}{\theta'}\right]N}{\sqrt{\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^2 + \frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|}\left[\frac{\kappa}{\lambda\tau\|W\|} - 2\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{1}{\theta'}\right]}}$$

$$+\frac{\left[\frac{\theta'\kappa\sin\theta}{\lambda\tau\|W\|} - \left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\sin\theta\right]B}{\sqrt{\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^2 + \frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|}\left[\frac{\kappa}{\lambda\tau\|W\|} - 2\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{1}{\theta'}\right]}}. \quad (3.3)$$

In order to determine the first curvature and the principal normal of the curve $\beta_1(s)$, we formalize

$$T'_{\beta_1}(s) = \frac{\sqrt{2}\Big[(\bar{r_1}\cos\theta + \bar{r_2}\sin\theta)T + \bar{r_3}N + (-\bar{r_1}\sin\theta + \bar{r_2}\cos\theta)B\Big]}{\left(\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^2 + \frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|}\left[\frac{\kappa}{\lambda\tau\|W\|} - 2\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{1}{\theta'}\right]\right)^2}$$

where

$$\begin{aligned}
\bar{r}_1 = {} & 2\left(\frac{\kappa}{\lambda\tau}\right)^2\left(\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]'\right)\left(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\right) \\
& -\left(\frac{\theta'\kappa}{\lambda\tau\|W\|}\right)\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]' \\
& \left(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\right)^2 - \left(\frac{\kappa}{\lambda\tau}\right)\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]\left(\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\right. \\
& \left.\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]'\right)\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)\left(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\right) - \left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\right. \\
& \left.\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^4\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right) - \left(\frac{\kappa}{\lambda\tau}\right)^2\left(\frac{\theta'\kappa}{\lambda\tau\|W\|}\right)' - \left(\frac{\kappa}{\lambda\tau}\right)^2\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\right.
\end{aligned}$$

$$\times\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)+2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3$$

$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+2\Big(\frac{\kappa}{\lambda\tau}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3$$

$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)$$

$$-2\kappa^*\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'$$

$$\Big(\frac{\kappa}{\lambda\tau}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\tau^*\Big(\frac{\kappa}{\lambda\tau}\Big)'$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2$$

$$+\Big(\frac{\kappa}{\lambda\tau}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)$$

$$\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\bigg(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\bigg)$$

$$+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\kappa}{\lambda\tau}\Big)\Big(\frac{\kappa}{\lambda\tau}\Big)'-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\kappa}{\lambda\tau}\Big)\bigg(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\bigg)$$

$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\kappa}{\lambda\tau}\Big)'\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big),$$

$$\begin{aligned}\bar{r}_2 &= \Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^3\\ &\quad\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+3\Big(\frac{\kappa}{\lambda\tau}\Big)^2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\end{aligned}$$

$$
\begin{aligned}
& \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2 \\
& \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2 \\
& \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2-(\frac{\kappa}{\lambda\tau})^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2 \\
& +3(\frac{\kappa}{\lambda\tau})^3\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)+(\frac{\kappa}{\lambda\tau}) \\
& \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)-2(\frac{\kappa}{\lambda\tau})^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' \\
& \frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-4\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' \\
& \frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^4-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2 \\
& (\frac{\kappa}{\lambda\tau})^2+3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\bar{r}_3 \;=\;& 2(\frac{\kappa}{\lambda\tau})'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})'-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big) \\
& (\frac{\kappa}{\lambda\tau})'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2 \\
& \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' \\
& \frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big) \\
& +(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^4\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)\\
&-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-(\frac{\kappa}{\lambda\tau})'\\
&\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-(\frac{\kappa}{\lambda\tau})^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\\
&\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\\
&\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2+2(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\\
&\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\\
&\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'-(\frac{\kappa}{\lambda\tau})\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'+(\frac{\kappa}{\lambda\tau})\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\\
&\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\\
&\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\\
&\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\\
&\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big).
\end{aligned}
$$

The first curvature is

$$
\kappa_{\beta_1}=\frac{\sqrt{2}(\sqrt{\bar{r_1}^2+\bar{r_2}^2+\bar{r_3}^2})}{\Bigg(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2+\frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|}\Big[\frac{\kappa}{\lambda\tau\|W\|}-2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{1}{\theta'}\Big]\Bigg)^2}.
$$

The principal normal vector field and the binormal vector field are respectively given by

$$N_{\beta_1}=\frac{(\bar{r_1}\cos\theta+\bar{r_2}\sin\theta)T+\bar{r_3}N+(-\bar{r_1}\sin\theta+\bar{r_2}\cos\theta)B}{\sqrt{\bar{r_1}^2+\bar{r_2}^2+\bar{r_3}^2}}, \tag{3.4}$$

$$B_{\beta_1}(s)=\frac{\xi_1}{\xi_4}T+\frac{\xi_2}{\xi_4}N+\frac{\xi_3}{\xi_4}B, \tag{3.5}$$

where

$$\begin{cases}\xi_1=\bar{r_2}\cos\theta\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}-\bar{r_2}\cos\theta\frac{\kappa}{\lambda\tau}-\Big[\bar{r_1}\frac{\kappa}{\lambda\tau}-\bar{r_1}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'} \\ \quad -\bar{r_3}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'+\bar{r_3}\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big]\sin\theta \\ \xi_2=\bar{r_1}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'-\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big] \\ \xi_3=\bar{r_2}\sin\theta\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}-\frac{\kappa}{\lambda\tau}\Big]+\Big[\bar{r_1}\frac{\kappa}{\lambda\tau}-\bar{r_1}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'} \\ \quad -\bar{r_3}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'+\bar{r_3}\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big]\cos\theta \\ \xi_4=\sqrt{\begin{array}{l}\Big((\bar{r_1}^2+\bar{r_2}^2+\bar{r_3}^2)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2+(\bar{r_1}^2+\bar{r_2}^2+\bar{r_3}^2)\frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|} \\ \Big[\frac{\kappa}{\lambda\tau\|W\|}-2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{1}{\theta'}\Big]\Big)\end{array}}\ .\end{cases}$$

In order to calculate the torsion of the curve β_1, we differentiate

$$\begin{aligned}\beta_1''=\;&\frac{1}{\sqrt{2}}\Bigg(\Bigg[\cos\theta\Bigg(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}} \\ &-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Bigg)+ \\ &+\sin\theta\Bigg(\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\Big(\frac{\kappa}{\lambda\tau}\Big) \\ &\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2-\Big(\frac{\kappa}{\lambda\tau}\Big)^2\Bigg)\Bigg]\mathbf{T} \\ &+\Bigg[\Big(\frac{\kappa}{\lambda\tau}\Big)-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\end{aligned}$$

$$-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big]\mathbf{N}$$

$$\Big[-\sin\theta\Big(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}$$

$$-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big)+$$

$$+\cos\theta\Big(\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+(\frac{\kappa}{\lambda\tau})$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2-(\frac{\kappa}{\lambda\tau})^2\Big)\Big]\mathbf{B}\Big).$$

and thus

$$\beta_1''' = \frac{(t_1\cos\theta+t_2\sin\theta+t_3)T+t_3N+(t_2\cos\theta-t_1\sin\theta+t_3)T}{\sqrt{2}},$$

where

$$\begin{aligned} t_1 \;=\;& \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]''\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-3\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' \\ & \frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}} \\ & -\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'' \\ & -\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau}) \\ & \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^3+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau})^2 \end{aligned}$$

$$\begin{aligned} t_2 \;=\;& 2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big) \\ & \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)' \end{aligned}$$

$$+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+(\frac{\kappa}{\lambda\tau})'$$
$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}+2(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}+2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2$$
$$\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-3(\frac{\kappa}{\lambda\tau})(\frac{\kappa}{\lambda\tau})'$$

$$t_3 = \Bigg(\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]-3\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\Bigg)\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\Bigg((\frac{\kappa}{\lambda\tau})^2$$
$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]''$$
$$+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\Bigg)\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})$$
$$-(\frac{\kappa}{\lambda\tau})^3+(\frac{\kappa}{\lambda\tau})''$$

The torsion is then given by

$$\tau_{\beta_1}=\frac{det(\beta_1',\beta_1'',\beta_1''')}{\|\beta_1'\wedge\beta_1''\|^2}\ ,$$

$$\tau_{\beta_1}=\sqrt{2}\frac{\Omega_1}{\Omega_2}$$

where

$$\Omega_1=-2t_1(\frac{\kappa}{\lambda\tau})^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}+t_1\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2\frac{\|W\|^2}{\theta'^2}-t_1\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2$$
$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)t_2(\frac{\kappa}{\lambda\tau})+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\frac{\theta'^2}{\theta'^2+\|W\|^2}t_2$$

$$+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2 t_3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)-2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'t_3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2-\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'t_3(\frac{\kappa}{\lambda\tau})^2$$

$$-\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'t_2(\frac{\kappa}{\lambda\tau})-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)t_2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)$$

$$t_2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+t_2\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}$$

$$-t_2\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2 t_3\frac{\kappa}{\lambda\tau}\frac{\|W\|}{\theta'}+t_2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'$$

$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)t_3\frac{\kappa}{\lambda\tau}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}-t_1\frac{\kappa}{\lambda\tau}\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$+t_1\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2+t_1\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})+t_2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\frac{\|W\|^2}{\theta'^2+\|W\|^2}$$

$$-t_2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'(\frac{\kappa}{\lambda\tau})+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)t_3(\frac{\kappa}{\lambda\tau})^2+t_3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^3+t_1(\frac{\kappa}{\lambda\tau})^3,$$

$$\Omega_2=\Bigg(\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2+\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]-2(\frac{\kappa}{\lambda\tau})^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})+(\frac{\kappa}{\lambda\tau})^3\Bigg)^2$$

$$+\Bigg(\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'-(\frac{\kappa}{\lambda\tau})'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]$$

$$-\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}$$

$$-\frac{\kappa}{\lambda\tau}\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau})')\Bigg)^2+\Bigg(\Bigg[\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}$$

$$-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]-(\frac{\kappa}{\lambda\tau})^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Bigg]$$

$$\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^3+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau})^2-\Big(\frac{\kappa}{\lambda\tau}\Big)\frac{\kappa}{\lambda\tau}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Bigg)^2. \qquad \square$$

Example 3.1 Let us consider the unit speed Mannheim curve and Mannheim partner curve:

$$\alpha(s)=\frac{1}{\sqrt{2}}\big(-\cos s,-\sin s,s\big),\ \ \alpha^*(s)=\frac{1}{\sqrt{2}}(-2\cos s,-2\sin s,s)\cdot$$

The Frenet invariants of the partner curve, $\alpha^*(s)$ are given as following

$$T^*(s)\ \ =\ \ \frac{1}{\sqrt{5}}\big(2\sin s,-2\cos s,1\big),$$

$$
\begin{aligned}
N^*(s) &= \frac{1}{\sqrt{5}}\big(\sin s, \cos s, -2\big)\\
B^*(s) &= \big(\cos s, \sin s, 0\big)\\
C^*(s) &= \big(\frac{2}{5}\sin s + \frac{2}{\sqrt{5}}\cos s, -\frac{2}{5}\cos s + \frac{2}{\sqrt{5}}\sin s, \frac{1}{5}\big)\\
\kappa^*(s) &= \frac{2\sqrt{2}}{5}\\
\tau^*(s) &= \frac{\sqrt{2}}{5}.
\end{aligned}
$$

In terms of definitions, we obtain special Smarandache curve, see Figure 1.

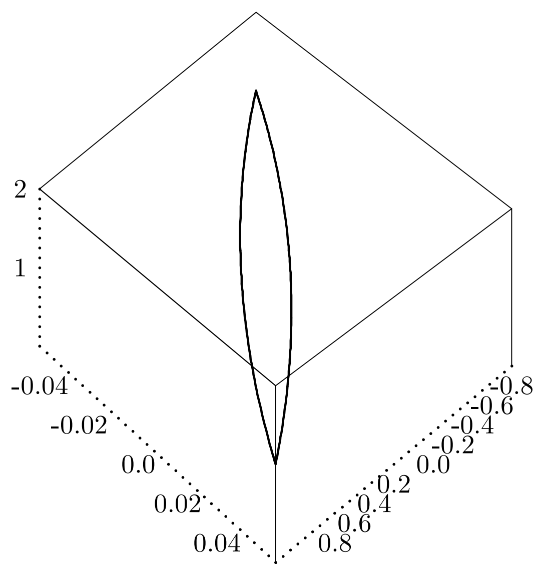

Figure 1 $\beta_1 = \frac{1}{5\sqrt{5}}\big((5+2\sqrt{5})\sin s + 10\cos s, (5-2\sqrt{5})\cos s + 10\sin s, -9\sqrt{5}\big)$

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