On Geometry of Equiform Smarandache Ruled Surfaces Via Equiform Frame in Minkowski 3-Space

Emad Solouma

Department of Mathematics and Statistics
College of Science
Imam Mohammad Ibn Saud Islamic University
Ryiadh, Saudi Arabia
E-mail. emmahmoud@imamu.edu.sa

Department of Mathematics and Computer Science
Faculty of Science
Beni-Suef University
Beni-Suef, Egypt
E-mail. emadms74@gmail.com

Abstract

In this paper, some geometric properties of equiform Smarandache ruled surfaces in Minkowski space E_1^3 using an equiform frame are investigated. Also, we give the sufficient conditions that make these surfaces are equiform developable and equiform minimal related to the equiform curvatures and when the equiform base curve contained in a plane or general helix. Finally, we provide an example, such as these surfaces.

1. Introduction

The theory of ruled surface is a branch of the classical differential geometry which has been developed by several researchers. In general, the rulings of the ruled surface are the set of a family of straight lines that depend on a parameter that is mentioned see Do Carmo (2016); Struik (1988); Barbosa and Colares (1986). One of the most interesting points is to study of ruled surfaces with different moving frames (see, for example, Hu et. al. (2020); Ibrahim Al-Dayel and Solouma

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(2021); Lam (2020); Ouarab et al. (2018); Ouarab et al. (2020); Ouarab (2021); Solouma and Ibrahim AL-Dayel (2021); Emad Solouma and Mohamed Abdelkawy (2022)).

In Euclidean and Lorentzian geometry, the Smarandache curve is the curve whose position vector is made by Frenet frame vectors on another regular curve (Ashbacher (1997); Bishop (1975); Iseri (2002); Mao (2006)). Many researchers (such as Cetin et al. (2014); Emad Solouma (2021); Solouma (2017); Solouma and Mahmoud (2017); Solouma and Mahmoud (2019); Solouma (2021); Turgut and Yılmaz (2008); Taskopru and Tosun (2014); Yılmaz and Turgut (2010)) studied Smarandache curves in Minkowski and the Euclidean spaces.

In this work, we introduce the definitions of a special kind of ruled surfaces called equiform Smarandache ruled surfaces via the equiform frame in Minkowski 3- space. The main results are presented in theorems that we concert the sufficient and necessary conditions for those ruled surfaces to be equiform developable and equiform minimally. Finally, an illustration-based example is provided.

2. Preliminaries

In Minkowski space E_1^3 the Lorentzian product is defined as:

$$\mathcal{H} = -de_1^2 + de_2^2 + de_3^2,$$

where (e_1, e_2, e_3) is the E_1^3 rectilinear coordinate system. An arbitrary $u \in E_1^3$ vector is one of the following; spacelike if $\mathcal{H}(u, u) > 0$ or $u = 0$, timelike if $\mathcal{H}(u, u) < 0$ and zero if $\mathcal{H}(u, u) = 0$ and $u \neq 0$. Likewise, a curve $\xi = \xi(\varrho)$ can be spacelike, timelike or zero if its $\xi'(\varrho)$ is spacelike, timelike or null. Let $\varphi = \varphi(\varrho)$ is a spacelike curve with a timelike principal normal. If $\{t, n, b\}$ denotes the moving Frenet frame of the spacelike curve φ, then $\{t, n, b\}$ has the following properties:

$$\begin{aligned}
\dot{t}(\varrho) &= \kappa(\varrho)n(\varrho), \\
\dot{n}(\varrho) &= \kappa(\varrho)t(\varrho) + \tau(\varrho)b(\varrho), \\
\dot{b}(\varrho) &= \tau(\varrho)n(\varrho),
\end{aligned} \tag{1}$$

where $\left(\cdot = \frac{d}{d\varrho} \right)$, $\mathcal{H}(t, t) = -\mathcal{H}(n, n) = \mathcal{H}(b, b) = 1$ and $\mathcal{H}(t, n) = \mathcal{H}(t, b) = \mathcal{H}(n, b) = 0$.

For a spacelike curve $\zeta: I \to E_1^3$ with a timelike principal normal in E_1^3. The equiform parameter of ζ by $\vartheta = \int \kappa d\varrho$. Then $\sigma = \frac{d\varrho}{d\vartheta}$, where $\sigma = \frac{1}{\kappa}$. We recall that $\{T, N, B\}$ is the moving equiform Frenet frame with the equiform tangent $T(\vartheta) = \sigma\, t(\varrho)$, the equiform principal normal $N(\vartheta) = \sigma\, n(\varrho)$ and the equiform binormal $B(\vartheta) = \sigma\, b(\varrho)$. The equiform curvatures of $\zeta = \zeta(\vartheta)$ are defined by $k_1(\vartheta) = \dot{\sigma} = \frac{d\sigma}{d\varrho}$ and $k_2(\vartheta) = \left(\frac{\tau}{\kappa} \right)$. As a result, the ζ equiform Frenet frame is given as:

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$$T'(\vartheta) = k_1(\vartheta)\, T(\vartheta) + N(\vartheta),$$
$$N'(\vartheta) = -T(\vartheta) + k_1(\vartheta)N(\vartheta) + k_2(\vartheta)B(\vartheta), \tag{2}$$
$$B'(\vartheta) = k_1(\vartheta)\, N(\vartheta) + k_2(\vartheta)\, B(\vartheta),$$

for $\left(' = \frac{d}{d\vartheta}\right)$, $\mathcal{H}(T,T) = -\mathcal{H}(B,B) = \mathcal{H}(N,N) = \sigma^2$, and $\mathcal{H}(T,B) = \mathcal{H}(N,B) = \mathcal{H}(T,N) = 0$.

Let $\zeta = \zeta(\vartheta)$ be a regular equiform spacelike curve in E_1^3 via equiform frame $\{T, N, B\}$. Then TN, TB and NB- equiform Smarandache curves of ζ are defined, respectively, as follows (Solouma (2021)):

$$\varphi(\vartheta^*(\vartheta)) = \frac{1}{\sqrt{2}\sigma}(T(\vartheta) + N(\vartheta)),$$
$$\psi(\vartheta^*(\vartheta)) = \frac{1}{\sqrt{2}\sigma}(T(\vartheta) + B(\vartheta)),$$
$$\omega(\vartheta^*(\vartheta)) = \frac{1}{\sqrt{2}\sigma}(N(\vartheta) + B(\vartheta)).$$

The Lorentzian sphere with the origin center in the E_1^3 space and a radius of $\epsilon > 0$ is defined as

$$S_1^2 = \{x \in E_1^3 : \mathcal{H}(x,x) = \epsilon^2\}.$$

A ruled surface Γ in E_1^3 can be represented as

$$\Gamma(\varrho, v) = \varphi(\varrho) + vX(\varrho), \tag{3}$$

where $\varphi(\varrho)$ is the base curve and $X(\varrho)$ is a space curve that represents the direction of a straight line.

The unit normal vector field $\mathbb{N}$ on Γ can be defined by

$$\mathbb{N} = \frac{\Gamma_\varrho \times \Gamma_v}{\|\Gamma_\varrho \times \Gamma_v\|}, \tag{4}$$

where $\Gamma_\varrho = \frac{\partial \Gamma}{\partial \varrho}$ and $\Gamma_v = \frac{\partial \Gamma}{\partial v}$. The components of Γ's first and second fundamental forms are given by, and respectively,

$$E = \|\Gamma_\varrho\|^2, \quad F = \langle \Gamma_\varrho, \Gamma_v \rangle, \quad G = \|\Gamma_v\|^2,$$

$$e = \langle \Gamma_{\varrho\varrho}, \mathbb{N} \rangle, \quad f = \langle \Gamma_{\varrho v}, \mathbb{N} \rangle, \quad g = \langle \Gamma_{vv}, \mathbb{N} \rangle.$$

The Gaussian and mean curvatures of Γ respectively are given by

$$K = \frac{e\,g - f^2}{EG - F^2}, \tag{5}$$

$$H = \frac{E\,g + G\,e - 2F\,f}{2(EG - F^2)}. \tag{6}$$

A ruled surface is developable if and only if $K = 0$ and minimal if and only if $H = 0$.

3. Main results

In this section, we define the equiform Smarandache ruled surfaces within Mikowski 3-space E_1^3 referring to the equiform frame $\{T, N, B\}$. Also, we investigate the necessary and sufficient conditions that make these surfaces have $K = 0$ and $H = 0$.

3.1 TN-equiform Smarandache ruled surface

Definition 3.1.

For a regular equiform spacelike curve $\zeta = \zeta(\vartheta)$ in E_1^3 via the frame (2). The TN-equiform Smarandache ruled surface is given by

$$\Lambda = \Lambda(\vartheta, v) = \frac{1}{\sqrt{2}\sigma}(T(\vartheta) + N(\vartheta)) + vB(\vartheta). \tag{7}$$

Theorem 3.1.

Let $\Lambda = \Lambda(\vartheta, v)$ is TN-equiform Smarandache ruled surface in E_1^3 defined by (7). Then, we have
1. If $k_1 = 1$, then Λ is equiform developable surface and H_Λ given by the formula

$$H_\Lambda = \frac{\sigma v k_2(\sqrt{2}\sigma v k_2 + 2) + 2k_2(k_2 + \sqrt{2}\sigma v)}{\sqrt{2}(\sqrt{2}\sigma v k_2 + 2)^{\frac{3}{2}}}.$$

2. If $\zeta(\varrho)$ is a plane curve ($k_2 = 0$), then Λ is equiform developable surface and H_Λ satisfying

$$H_\Lambda = \frac{(k_1+1)[k_1' + k_1(k_1-2)] + (k_1-1)(k_1^2 + k_1' + 2k_1 + 1)}{4\sqrt{2}(k_1)^{\frac{3}{2}}}.$$

Proof:

Let $\Lambda(\vartheta, v) = \frac{1}{\sqrt{2}\sigma}(T(\vartheta) + N(\vartheta)) + vB(\vartheta)$ be TN-equiform Smarandache ruled surface recording by the equiform frame $\{T, N, B\}$ in E_1^3. Taking the first derivative of $\Lambda(\vartheta, v)$ with respect to ϑ and v, we get

$$\begin{aligned}
\Lambda_\vartheta &= \left[\frac{k_1 - 1}{\sqrt{2}\sigma}\right]T(\vartheta) + \left[\frac{k_1 + 1}{\sqrt{2}\sigma} + vk_2\right]N(\vartheta) + \left[\frac{k_2}{\sqrt{2}\sigma} + vk_1\right]B(\vartheta), \\
\Lambda_v &= B(\vartheta).
\end{aligned} \tag{8}$$

From (8), The components of Λ's first fundamental form and the unit normal vector field are given by:

$$\begin{aligned}
E_\Lambda &= \frac{1}{2}\left[(k_1 - 1)^2 - \left(k_1 + \sqrt{2}\sigma v k_2\right)^2 + \left(k_2 + \sqrt{2}\sigma v k_1\right)^2\right], \\
F_\Lambda &= \frac{\sigma}{\sqrt{2}}\left[k_2 + \sqrt{2}\sigma v k_1\right], \\
G_\Lambda &= \sigma^2.
\end{aligned} \tag{9}$$

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$$\mathbb{N}_\Lambda = \frac{(k_1+\sqrt{2}\sigma v k_2+1)T(\vartheta)-(k_1-1)N(\vartheta)}{\sigma\sqrt{(k_1+\sqrt{2}\sigma v k_2+1)^2-(k_1-1)^2}}. \tag{10}$$

Another time, we can differentiate (7) with respect to ϑ and v, respectively, and use (2) to get

$$\begin{aligned}
\Lambda_{\vartheta\vartheta} &= \varepsilon_1 T(\vartheta) + \varepsilon_2\, N(\vartheta) + \varepsilon_3\, B(\vartheta), \\
\Lambda_{\vartheta v} &= k_2\, N(\vartheta) + k_1\, B(\vartheta), \\
\Lambda_{vv} &= 0.
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{\sqrt{2}\sigma}\big[k_1' + k_1(k_1-2) + \sqrt{2}\sigma v k_2 + 1\big], \\
\varepsilon_2 &= \frac{1}{\sqrt{2}\sigma}\big[k_1^2 + k_2^2 + k_1' + 2k_1 + \sqrt{2}\sigma v(k_1 k_2 + k_2')\big], \\
\varepsilon_3 &= \frac{1}{\sqrt{2}\sigma}\big[k_2' + k_2(2k_1+1) + \sqrt{2}\sigma v(k_1^2 + k_2^2 + k_2')\big].
\end{aligned} \tag{12}$$

From (10) and (11), the components of Λ's second fundamental form are given by:

$$\begin{aligned}
e_\Lambda &= \frac{\sigma\{\varepsilon_1[k_1+\sqrt{2}\sigma v k_2+1]+\varepsilon_2(k_1-1)\}}{\sqrt{(k_1+\sqrt{2}\sigma v k_2+1)^2-(k_1-1)^2}}, \\
f_\Lambda &= \frac{\sigma k_2(k_1-1)}{\sqrt{(k_1+\sqrt{2}\sigma v k_2+1)^2-(k_1-1)^2}}, \\
g_\Lambda &= 0.
\end{aligned} \tag{13}$$

So, from (9) and (11), the equiform Gaussian and mean curvatures of TN-equiform Smarandache ruled surface Λ given by:

$$\begin{aligned}
K_\Lambda &= \frac{k_2^2(k_1-1)^2}{\big[(k_1+\sqrt{2}\sigma v k_2+1)^2-(k_1-1)^2\big]^2}, \\
H_\Lambda &= \frac{2\sqrt{2}k_2(k_2+\sqrt{2}\sigma v k_2)-2\sigma\{\varepsilon_1[k_1+\sqrt{2}\sigma v k_2+1]+\varepsilon_2(k_1-1)\}}{\big[(k_1+\sqrt{2}\sigma v k_2+1)^2-(k_1-1)^2\big]^{\frac{3}{2}}}.
\end{aligned} \tag{14}$$

Consequently, from (14) we complete our proof.   ∎

Corollary 3.2.

Let $\Lambda = \Lambda(\vartheta, v)$ is TN-equiform Smarandache ruled surface in E_1^3 defined by (7). Then Λ is equiform minimal surface if and only if the equiform curvatures satisfy the following differential equation

$$2\sqrt{2}k_2\big(k_2 + \sqrt{2}\sigma v k_2\big) - 2\sigma\{\varepsilon_1\big[k_1 + \sqrt{2}\sigma v k_2 + 1\big] + \varepsilon_2(k_1 - 1)\} = 0,$$

where ε_1 and ε_2 are given by (12).

Proof:

Let $\Lambda(\vartheta, v)$ be TN-equiform Smarandache ruled surface defined by (7) via the equiform frame $\{T, N, B\}$ in E_1^3. From (14), the equiform surface $\Lambda(\vartheta, v)$ is equiform minimal surface if and only if $H_\Lambda = 0$ which mean that

$$2\sqrt{2}k_2\left(k_2 + \sqrt{2}\sigma v k_2\right) - 2\sigma\{\varepsilon_1\left[k_1 + \sqrt{2}\sigma v k_2 + 1\right] + \varepsilon_2(k_1 - 1)\} = 0,$$

where ε_1 and ε_2 are given by (12) which complete our proof. ∎

3.2 TB-equiform Smarandache ruled surface

Definition 3.2.

For a regular equiform spacelike curve $\zeta = \zeta(\vartheta)$ in E_1^3 via the frame (2). The TB-equiform Smarandache ruled surface is given by

$$\Theta = \Theta(\vartheta, v) = \frac{1}{\sqrt{2}\sigma}(T(\vartheta) + B(\vartheta)) + vN(\vartheta). \tag{15}$$

Theorem 3.3.

Let $\Theta = \Theta(\vartheta, v)$ is TB-equiform Smarandache ruled surface in E_1^3 defined by (15). Then, we have
1. If $\zeta(\varrho)$ has non-zero constant curvature $(k_1 = 0)$, then Θ is equiform developable surface and

$$H_\Theta = \frac{-\sqrt{2}k_2(k_2+1)+\sigma v k_2'}{2\sigma^2 v^2(k_2+1)^{\frac{3}{2}}}.$$

2. If $\zeta(\varrho)$ is a general helix $(k_2 = 1)$, then Θ is equiform developable surface and

$$H_\Theta = -\frac{k_1'+k_1(k_1+\sqrt{2}\sigma v)}{(k_1-\sqrt{2}\sigma v)^2}.$$

Proof:

We can study the K_Θ and H_Θ of TB-equiform Smarandache ruled surface via the equiform frame $\{T, N, B\}$. The velocity vectors of (15) are given by

$$\begin{aligned}
\Theta_\vartheta &= \left[\frac{k_1-\sqrt{2}\sigma v}{\sqrt{2}\sigma}\right]T(\vartheta) + \left[\frac{k_2+\sqrt{2}\sigma v k_1+1}{\sqrt{2}\sigma}\right]N(\vartheta) + \left[\frac{k_1+\sqrt{2}\sigma v k_2}{\sqrt{2}\sigma}\right]B(\vartheta), \\
\Theta_v &= N(\vartheta).
\end{aligned} \tag{16}$$

Now, using (16), we get the quantities of the first fundamental form and the unit normal vector field of Θ are given by:

$$\begin{aligned}
E_\Theta &= \frac{1}{2}\left[\left(k_1 - \sqrt{2}\sigma v\right)^2 - \left(k_2 + \sqrt{2}\sigma v k_1 + 1\right)^2 + \left(k_1 + \sqrt{2}\sigma v k_2\right)^2\right], \\
F_\Theta &= -\frac{\sigma}{\sqrt{2}}\left[k_2 + \sqrt{2}\sigma v k_1 + 1\right], \\
G_\Theta &= -\sigma^2.
\end{aligned} \tag{17}$$

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$$\mathbb{N}_\Theta = \frac{(k_1+\sqrt{2}\sigma v k_2)T(\vartheta)+(k_1-\sqrt{2}\sigma v)B(\vartheta)}{\sigma\sqrt{(k_1+\sqrt{2}\sigma v k_2)^2+(k_1-\sqrt{2}\sigma v)^2}}. \qquad (18)$$

Differentiating (16) with respect to ϑ and v respectively and using (2) we get

$$\begin{aligned}
\Theta_{\vartheta\vartheta} &= \mu_1 T(\vartheta) + \mu_2\, N(\vartheta) + \mu_3\, B(\vartheta), \\
\Theta_{\vartheta v} &= -T(\vartheta) + k_1\, N(\vartheta) + k_2\, B(\vartheta), \\
\Theta_{vv} &= 0,
\end{aligned} \qquad (19)$$

where

$$\mu_1 = \frac{1}{\sqrt{2}\sigma}\left[k_1' - k_2 + k_1(k_1 - \sqrt{2}\sigma v) - \sqrt{2}\sigma v k_1 + 1\right],$$

$$\mu_2 = \frac{1}{\sqrt{2}\sigma}\left[k_1' + k_2' + k_1(k_2 + \sqrt{2}\sigma v k_1 + 2) + k_2(k_1 + \sqrt{2}\sigma v k_2) + \sqrt{2}\sigma v(k_1' - 1)\right], \qquad (20)$$

$$\mu_3 = \frac{1}{\sqrt{2}\sigma}\left[k_1' + k_2(k_2 + \sqrt{2}\sigma v k_1 + 1) + k_1(k_1 + \sqrt{2}\sigma v k_2) + \sqrt{2}\sigma v k_2'\right].$$

From (18) and (19), the quantities of the second fundamental form of Θ are given by:

$$e_\Theta = \frac{\sigma\{\mu_1(k_1+\sqrt{2}\sigma v k_2)+\mu_2(k_1+\sqrt{2}\sigma v)\}}{\sqrt{(k_1+\sqrt{2}\sigma v k_2)^2+(k_1-\sqrt{2}\sigma v)^2}},$$

$$f_\Theta = \frac{\sigma k_1(k_2-1)}{\sqrt{(k_1+\sqrt{2}\sigma v k_2)^2+(k_1-\sqrt{2}\sigma v)^2}}, \qquad (21)$$

$$g_\Theta = 0.$$

Then, from (17) and (21), the equiform K_Θ and H_Θ of TB-equiform Smarandache ruled surface Θ given by

$$K_\Theta = \frac{2k_1^2(k_2-1)^2}{\left[(k_1+\sqrt{2}\sigma v k_2)^2+(k_1-\sqrt{2}\sigma v)^2\right]^2},$$

$$H_\Theta = \frac{2\sqrt{2}k_1(k_1-1)(k_2+\sqrt{2}\sigma v k_1+1)-2\sigma\{\mu_1[k_1+\sqrt{2}\sigma v k_2]+\mu_3(k_1+\sqrt{2}\sigma v)\}}{\left[(k_1+\sqrt{2}\sigma v k_2)^2+(k_1-\sqrt{2}\sigma v)^2\right]^{\frac{3}{2}}}, \qquad (22)$$

which complete our proof.   ∎

Corollary 3.4.

Let $\Theta = \Theta(\vartheta, v)$ is TB-equiform Smarandache ruled surface in E_1^3 defined by (15). Then Θ is equiform minimal surface if and only if the equiform curvatures satisfy the following differential equation

$$2\sqrt{2}k_1(k_1 - 1)(k_2 + \sqrt{2}\sigma v k_1 + 1) - 2\sigma\{\mu_1[k_1 + \sqrt{2}\sigma v k_2] + \mu_3(k_1 + \sqrt{2}\sigma v)\} = 0,$$

where μ_1 and μ_3 are given by (20).

Proof:

Let $\Theta = \Theta(\vartheta, \upsilon)$ be TB-equiform Smarandache ruled surface defined by (15) in E_1^3 via the equiform frame $\{T, N, B\}$. Then, $H_\Theta = 0$ implies that $\Theta(\vartheta, \upsilon)$ is equiform minimal surface. From (22), we have

$$2\sqrt{2}k_1(k_1 - 1)(k_2 + \sqrt{2}\sigma \upsilon k_1 + 1) - 2\sigma\{\mu_1[k_1 + \sqrt{2}\sigma \upsilon k_2] + \mu_3(k_1 + \sqrt{2}\sigma \upsilon)\} = 0,$$

for μ_1 and μ_3 are given by (20). This complete the proof. ∎

3.3 NB-equiform Smarandache ruled surface

Definition 3.2.

For a regular equiform spacelike curve $\zeta = \zeta(\vartheta)$ in E_1^3 via the frame (2). The NB-equiform Smarandache ruled surface is given by

$$\Upsilon = \Upsilon(\vartheta, \upsilon) = \frac{1}{\sqrt{2}\sigma}(N(\vartheta) + B(\vartheta)) + \upsilon T(\vartheta). \tag{23}$$

Theorem 3.5.

Let $\Upsilon = \Upsilon(\vartheta, \upsilon)$ is NB-equiform Smarandache ruled surface in E_1^3 defined by (23). If $k_1 + k_2 = 0$, then Υ is equiform developable surface satisfying

$$H_\Upsilon = \frac{k_2}{2\sigma^2 \upsilon^2}.$$

Proof:

We compute the equiform Gaussian and the equiform mean curvatures of NB-equiform Smarandache ruled surface given by (23) via the equiform frame $\{T, N, B\}$. The Υ's velocity vectors are given by

$$\begin{aligned}
\Upsilon_\vartheta &= \left[\frac{\sqrt{2}\sigma \upsilon k_1 - 1}{\sqrt{2}\sigma}\right] T(\vartheta) + \left[\frac{k_1 + k_2 + \sqrt{2}\sigma \upsilon}{\sqrt{2}\sigma}\right] N(\vartheta) + \left[\frac{k_1 + k_2}{\sqrt{2}\sigma}\right] B(\vartheta), \\
\Upsilon_\upsilon &= T(\vartheta).
\end{aligned} \tag{24}$$

By using (24), we get the components of the first fundamental form and the unit normal vector field of Υ are given by:

$$\begin{aligned}
E_\Upsilon &= \frac{1}{2}\left[(\sqrt{2}\sigma \upsilon k_1 - 1)^2 - (k_1 + k_2 + \sqrt{2}\sigma \upsilon)^2 + (k_1 + k_2)^2\right], \\
F_\Upsilon &= \frac{\sigma}{\sqrt{2}}[\sqrt{2}\sigma \upsilon k_1 - 1], \\
G_\Upsilon &= \sigma^2,
\end{aligned} \tag{25}$$

$$\mathbb{N}_\Upsilon = \frac{(k_1 + k_2)N(\vartheta) - (k_1 + k_2 + \sqrt{2}\sigma \upsilon)B(\vartheta)}{\sqrt{2}\sigma\sqrt{\sigma \upsilon(k_1 + k_2 + \sigma \upsilon)}}. \tag{26}$$

Using (2) and differentiating (24) again with respect to ϑ and υ respectively, we get

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$$Y_{\vartheta\vartheta} = \alpha_1 T(\vartheta) + \alpha_2 N(\vartheta) + \alpha_3 B(\vartheta),$$
$$Y_{\vartheta v} = k_1 T(\vartheta) + N(\vartheta), \qquad\qquad (27)$$
$$Y_{vv} = 0,$$

where

$$\alpha_1 = \frac{1}{\sqrt{2}\sigma}\left[k_1(\sqrt{2}\sigma v k_1 - 1) + \sqrt{2}\sigma v(k_1' - 1) - k_1 - k_2\right],$$
$$\alpha_2 = \frac{1}{\sqrt{2}\sigma}\left[k_1' + k_2' + k_1(k_1 + k_2 + \sqrt{2}\sigma v) + k_2(k_1 + k_2) + \sqrt{2}\sigma v k_1 - 1\right], \qquad (28)$$
$$\alpha_3 = \frac{1}{\sqrt{2}\sigma}\left[k_1' + k_2' + k_1(k_1 + k_2 + \sqrt{2}\sigma v) + k_2(k_1 + k_2)\right].$$

From (26) and (27), the quantities of the second fundamental form of Y are given by:

$$e_Y = -\frac{\sigma\{\alpha_2(k_1+k_2)+\alpha_3(k_1+k_2+\sqrt{2}\sigma v)\}}{\sqrt{2}\sqrt{\sigma v(k_1+k_2+\sigma v)}},$$
$$f_Y = -\frac{\sigma(k_1+k_2)}{\sqrt{2}\sigma\sqrt{\sigma v(k_1+k_2+\sigma v)}}, \qquad\qquad (29)$$
$$g_Y = 0.$$

Then, from (25) and (29), the equiform Gaussian curvature K_Y and the equiform mean curvature H_Y of Y are given by

$$K_Y = \frac{(k_1+k_2)^2}{2\sigma^2 v^2(k_1+k_2+\sigma v)^2},$$
$$H_Y = \frac{\sqrt{2}\sigma\{\alpha_2(k_1+k_2)+\alpha_3(k_1+k_2+\sqrt{2}\sigma v)\}+2(k_1+k_2)(\sqrt{2}\sigma v k_1-1)}{2[\sigma v(k_1+k_2+\sigma v)]^{\frac{3}{2}}}. \qquad (30)$$

As a consequence of the above results, we complete the proof. ∎

Corollary 3.4.

Let $Y = Y(\vartheta, v)$ is NB-equiform Smarandache ruled surface in E_1^3 defined by (23). Then Y is equiform minimal surface if and only if the equiform curvatures satisfy the following differential equation

$$\sigma\{\alpha_2(k_1 + k_2) + \alpha_3(k_1 + k_2 + \sqrt{2}\sigma v)\} + \sqrt{2}(k_1 + k_2)(\sqrt{2}\sigma v k_1 - 1) = 0,$$

where α_2 and α_3 are given by (28).

Proof:

Let $Y = Y(\vartheta, v)$ be NB-equiform Smarandache ruled surface defined by (23) in E_1^3 via the equiform frame $\{T, N, B\}$. As the above way, the equiform mean curvature H_Y of Y is given by (30). Then, $H_Y = 0$ means the equiform surface $Y(\vartheta, v)$ is equiform minimal surface. Then, from (30), we have

$$\sigma\{\alpha_2(k_1 + k_2) + \alpha_3(k_1 + k_2 + \sqrt{2}\sigma v)\} + \sqrt{2}(k_1 + k_2)(\sqrt{2}\sigma v k_1 - 1) = 0,$$

where α_2 and α_3 are given by (28) which complete the proof. ∎

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3.4 Example

Consider the case of a regular spacelike curve $\varphi(\varrho)$ with a timelike principal normal in E_1^3 (see Figure 1)

$$\varphi(\varrho) = \left(\frac{\varrho}{\sqrt{2}} \cosh(\sqrt{2} \ln \varrho), \frac{\varrho}{\sqrt{2}} \sinh(\sqrt{2} \ln \varrho), \frac{\varrho}{\sqrt{2}} \right). \tag{31}$$
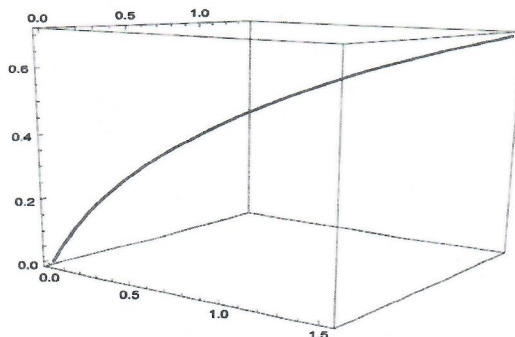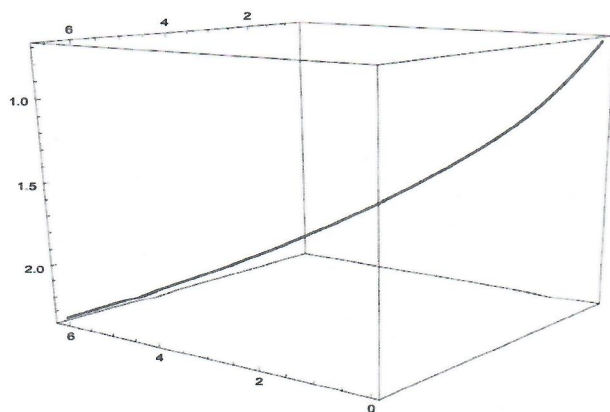
Then, the Frenet apparatus are given as the following

Figure 1: Spacelike curve $\varphi = \varphi(\varrho)$

$$t(\varrho) = \left(\frac{1}{\sqrt{2}} \cosh(\sqrt{2} \ln \varrho) + \sinh(\sqrt{2} \ln \varrho), \frac{1}{\sqrt{2}} \sinh(\sqrt{2} \ln \varrho) + \cosh(\sqrt{2} \ln \varrho), \frac{1}{\sqrt{2}} \right),$$

$$n(\varrho) = \left(\sqrt{2} \cosh(\sqrt{2} \ln \varrho) + \sinh(\sqrt{2} \ln \varrho), \sqrt{2} \sinh(\sqrt{2} \ln \varrho) + \cosh(\sqrt{2} \ln \varrho), 0 \right),$$

$$\kappa = \frac{1}{\varrho}, \quad \sigma = \varrho, \quad k_1 = 1,$$

$$b(\varrho) = \left(\frac{1}{\sqrt{2}} \cosh(\sqrt{2} \ln \varrho) + \sinh(\sqrt{2} \ln \varrho), \frac{1}{\sqrt{2}} \sinh(\sqrt{2} \ln \varrho) + \cosh(\sqrt{2} \ln \varrho), \frac{-1}{\sqrt{2}} \right),$$

$$\tau = \frac{1}{\varrho}, \quad k_2 = 1.$$

Then, the equiform parameter is $\vartheta = \int_0^\varrho \kappa \, d\varrho = \ln \varrho$, so we have $\varrho = \sigma = e^\vartheta$. Now, the equiform spacelike curve $\zeta(\vartheta)$ is define as (see Figure 2)

$$\zeta(\vartheta) = \left(\frac{e^\vartheta}{\sqrt{2}} \cosh(\sqrt{2}\vartheta), \frac{e^\vartheta}{\sqrt{2}} \sinh(\sqrt{2}\vartheta), \frac{e^\vartheta}{\sqrt{2}} \right). \tag{32}$$

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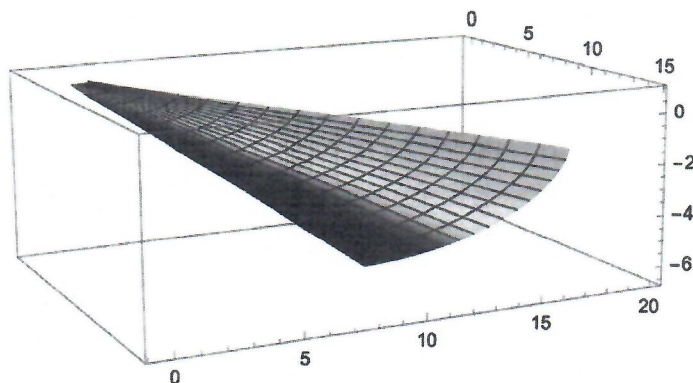
Figure 2: Equiform spacelike curve $\zeta = \zeta(\vartheta)$

It is easy to show that the vectors of equiform Frenet frame are given as:

$$T(\varrho) = e^{\vartheta}\left(\frac{1}{\sqrt{2}}\cosh(\sqrt{2}\vartheta) + \sinh(\sqrt{2}\vartheta), \frac{1}{\sqrt{2}}\sinh(\sqrt{2}\vartheta) + \cosh(\sqrt{2}\vartheta), \frac{1}{\sqrt{2}}\right),$$
$$N(\varrho) = e^{\vartheta}\left(\sqrt{2}\cosh(\sqrt{2}\vartheta) + \sinh(\sqrt{2}\vartheta), \sqrt{2}\sinh(\sqrt{2}\vartheta) + \cosh(\sqrt{2}\vartheta), 0\right),$$
$$B(\varrho) = e^{\vartheta}\left(\frac{1}{\sqrt{2}}\cosh(\sqrt{2}\vartheta) + \sinh(\sqrt{2}\vartheta), \frac{1}{\sqrt{2}}\sinh(\sqrt{2}\vartheta) + \cosh(\sqrt{2}\vartheta), \frac{-1}{\sqrt{2}}\right).$$

Thus, the equiform Smarandache ruled surfaces $\Lambda(\vartheta, \upsilon)$, $\Theta(\vartheta, \upsilon)$ and $\Upsilon(\vartheta, \upsilon)$ are respectively given as (see Figures 3, 4 and 5)

Figure 3: TN-equiform Smarandache developable ruled surface $\Lambda(\vartheta, \upsilon)$

$$\Lambda(\vartheta, \upsilon) = \left(\left(\frac{3+\sqrt{2}\upsilon e^{\vartheta}}{2}\right)\cosh(\sqrt{2}\vartheta) + \left(\frac{1+\sqrt{2}\upsilon e^{\vartheta}}{\sqrt{2}}\right)\sinh(\sqrt{2}\vartheta), \left(\frac{3+\sqrt{2}\upsilon e^{\vartheta}}{2}\right)\sinh(\sqrt{2}\vartheta)\right.$$
$$\left. + \left(\frac{1+\sqrt{2}\upsilon e^{\vartheta}}{\sqrt{2}}\right)\cosh(\sqrt{2}\vartheta), \frac{1-\upsilon e^{\vartheta}}{\sqrt{2}}\right).$$
$$(33)$$

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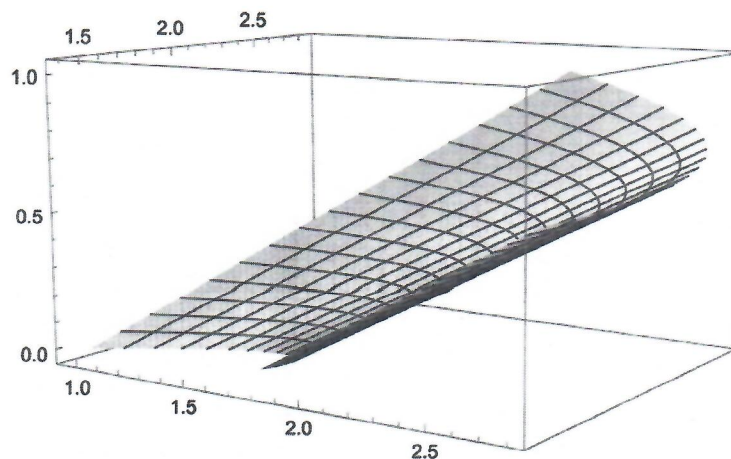**Figure 4:** TB-equiform Smarandache developable ruled surface $\Theta(\vartheta, v)$

$$\Theta(\vartheta,v) = \left(1 + \sqrt{2}ve^{\vartheta}\right)\left(\cosh(\sqrt{2}\vartheta) + \sinh(\sqrt{2}\vartheta), \sinh(\sqrt{2}\vartheta) + \left(\frac{1+\sqrt{2}ve^{\vartheta}}{\sqrt{2}}\right)\cosh(\sqrt{2}\vartheta), 0\right).$$

$$(34)$$

$$\Upsilon(\vartheta,v) = \left(\left(\frac{3+\sqrt{2}ve^{\vartheta}}{2}\right)\cosh(\sqrt{2}\vartheta) + \left(\frac{1+\sqrt{2}ve^{\vartheta}}{\sqrt{2}}\right)\sinh(\sqrt{2}\vartheta), \left(\frac{3+\sqrt{2}ve^{\vartheta}}{2}\right)\sinh(\sqrt{2}\vartheta)\right.$$

$$\left. + \left(\frac{1+\sqrt{2}ve^{\vartheta}}{\sqrt{2}}\right)\cosh(\sqrt{2}\vartheta), \frac{ve^{\vartheta}-1}{\sqrt{2}}\right).$$

$$(35)$$

4. Conclusion

Using an equiform frame, various geometric characteristics of equiform Smarandache ruled surfaces in Minkowski space Minkowski 3-space are studied. We also provide the necessary requirements for these surfaces to be equiform developable and equiform minimal in relation to equiform curvatures, as well as when the equiform base curve is located in a plane or general helix.

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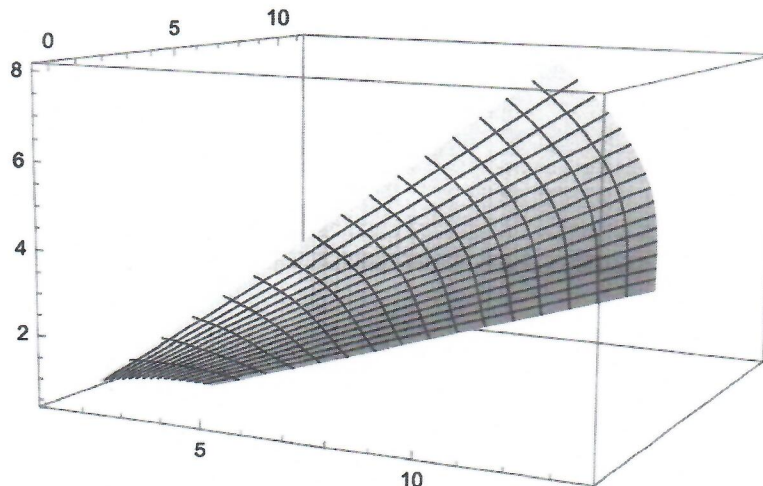
Figure 5: NB-equiform Smarandache ruled surface $Y(\vartheta, v)$

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