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Abstract

In differential geometry, the theory of curves has an important place. The concept of moving frames defined on curves is an important part of this theory. Recently, Özen and Tosun have introduced a new moving frame for the trajectories with non-vanishing angular momentum in 3-dimensional Euclidean space (J. Math. Sci. Model. 4(1), 2021). This frame is denoted by $\{\mathbf{T}, \mathbf{M}, \mathbf{Y}\}$ and called as positional adapted frame. In the present study, we investigate the special trajectories generated by $\mathbf{TM}$, $\mathbf{TY}$ and $\mathbf{MY}-$Smarandache curves according to positional adapted frame in E^3 and we calculate the Serret-Frenet apparatus of these trajectories. Later, we consider a specific curve and obtain the parametric equations of the aforesaid special trajectories for this curve. Finally, we give the graphics of these obtained special trajectories which were drawn with the mathematica program. The results obtained here are new contributions to the field. We expect that these results will be useful in some specific applications of differential geometry and particle kinematics in the future.

Keywords: Angular momentum; Kinematics of a particle; Moving frame; Smarandache curves.

1 Introduction and Preliminaries

The local theory of space curves plays an important role in differential geometry. The concept of moving frames is one of the most important concepts in the theory of curves. Despite its long history, it is still an issue of interest. The discovery of the Serret-Frenet frame was a milestone for the researchers interested in this topic. Until now, many researchers have carried out many interesting studies on the local theory of space curves by using Serret-Frenet frame.

There is a very close relationship between the kinematics of a moving particle and the differential geometry of the trajectory which is the oriented curve traced out by this particle. As a result of this case, Serret-Frenet frame has been used to investigate the kinematics of a moving particle, as well. From past to present, many researchers have developed new moving frames which have a common base vector with the Serret-Frenet frame (see [1, 2, 3] for some examples). One of the newest of them is the study [4] presented by Özen and Tosun. They introduced the Positional Adapted Frame (PAF) for the trajectories with non-vanishing angular momentum in this study.

Let the Euclidean 3-space E^3 be taken into account with the standard scalar product $\langle \mathbf{G}, \mathbf{H} \rangle = g_1h_1 + g_2h_2 + g_3h_3$ where $\mathbf{G} = (g_1\,,\, g_2\,,\, g_3)$, $\mathbf{H} = (h_1\,,\ h_2\,,\ h_3)$ are any vectors in E^3. The norm of $\mathbf{G}$ is given as $\|\mathbf{G}\| = \sqrt{\langle \mathbf{G}\,,\, \mathbf{G} \rangle}$. If a differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ satisfies $\left\| \frac{d\alpha}{ds} \right\| = 1$ for all $s \in I$, it is called a unit speed curve. In that case, s is said to be arc-length parameter of α. A differentiable curve is called as regular curve if its derivative is not equal to zero along the curve. An arbitrary regular curve can be reparameterized by the arc-length of itself [5]. Throughout the paper, the differentiation with respect to the arc-length parameter s will be shown with a dash.

In Euclidean 3-space E^3, assume that a point particle of constant mass moves on a unit speed curve $\alpha = \alpha(s)$. Let $\{\mathbf{T}(s)\,,\, \mathbf{N}(s)\,,\, \mathbf{B}(s)\}$ show the Serret-Frenet frame of the curve $\alpha = \alpha(s)$. $\mathbf{T}(s) = \alpha'(s)$, $\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ and $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$ are called the unit tangent, unit principal normal and unit binormal vectors, respectively. Also, the Serret-Frenet formulas are given as in the following:

$$\begin{pmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix} \begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix} \tag{1.1}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature function and $\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s)\rangle$ is the torsion function [5].

Another thing that can be of importance is the angular momentum vector of the aforesaid particle about origin. It has an important place in Newtonian mechanics. It is determined by vector product of the position vector and linear momentum vector of the particle. It always lies on the instantaneous normal plane $Sp\{\mathbf{N}(s), \mathbf{B}(s)\}$. Suppose that this vector does not equal to zero vector along the trajectory $\alpha = \alpha(s)$. This assumption ensures that the functions $\langle \alpha(s), \mathbf{N}(s)\rangle$ and $\langle \alpha(s), \mathbf{B}(s)\rangle$ do not equal to zero simultaneously during the motion of the aforesaid particle. So, it can be said that the tangent line of $\alpha = \alpha(s)$ never passes through the origin. Then, there exists Positional adapted frame (PAF) shown with $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s)\}$ along $\alpha = \alpha(s)$. Take into consideration the vector whose starting point is the foot of the perpendicular (from origin to instantaneous rectifying plane) and endpoint is the foot of the perpendicular (from origin to instantaneous osculating plane). The equivalent of it at the point $\alpha(s)$ determines the vector $\mathbf{Y}(s)$. Thus, $\mathbf{Y}(s)$ is given as in the following (see [4] for more details):

$$\mathbf{Y}(s) = \frac{\langle -\alpha(s), \mathbf{N}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s)\rangle^2 + \langle \alpha(s), \mathbf{B}(s)\rangle^2}}\mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{B}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s)\rangle^2 + \langle \alpha(s), \mathbf{B}(s)\rangle^2}}\mathbf{B}(s). \tag{1.2}$$

On the other hand, the vector $\mathbf{M}(s)$ is obtained by vector product $\mathbf{Y}(s) \wedge \mathbf{T}(s)$ as follows:

$$\mathbf{M}(s) = \frac{\langle \alpha(s), \mathbf{B}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s)\rangle^2 + \langle \alpha(s), \mathbf{B}(s)\rangle^2}}\mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{N}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s)\rangle^2 + \langle \alpha(s), \mathbf{B}(s)\rangle^2}}\mathbf{B}(s). \tag{1.3}$$

Because $\mathbf{T}(s)$ is mutual in both PAF and Serret-Frenet frame, $\mathbf{N}(s)$, $\mathbf{B}(s)$, $\mathbf{M}(s)$ and $\mathbf{Y}(s)$ lie on the same plane. Therefore, there is a relation between the Serret-Frenet frame and PAF as in the following:

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\Omega(s) & -\sin\Omega(s)\\ 0 & \sin\Omega(s) & \cos\Omega(s)\end{pmatrix} \begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix} \tag{1.4}$$

where $\Omega(s)$ is the angle between the vectors $\mathbf{B}(s)$ and $\mathbf{Y}(s)$ which is positively oriented from $\mathbf{B}(s)$ to $\mathbf{Y}(s)$(see Figure 1). Also, the derivative formulas of PAF are given by

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{M}'(s)\\ \mathbf{Y}'(s)\end{pmatrix} = \begin{pmatrix}0 & k_1(s) & k_2(s)\\ -k_1(s) & 0 & k_3(s)\\ -k_2(s) & -k_3(s) & 0\end{pmatrix} \begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix} \tag{1.5}$$

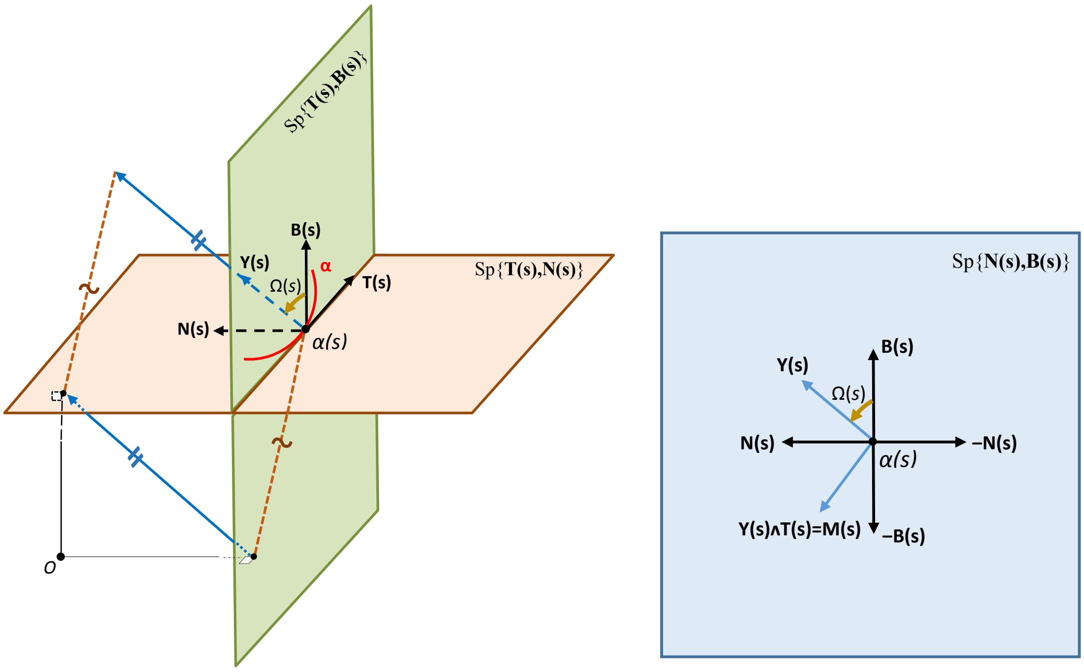
where

$$\begin{aligned} k_1(s) &= \kappa(s)\cos\Omega(s)\\ k_2(s) &= \kappa(s)\sin\Omega(s)\\ k_3(s) &= \tau(s) - \Omega'(s)\\ \kappa^2(s) &= {k_1}^2(s) + {k_2}^2(s)\\ \frac{k_2(s)}{k_1(s)} &= \tan\Omega(s). \end{aligned} \tag{1.6}$$

The aforesaid angle $\Omega(s)$ is calculated as follows:

$$\Omega(s)=\begin{cases} \arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right) \;\; if \;\; \langle\alpha(s),\,\mathbf{B}(s)\rangle>0 \\ \arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right)+\pi \;\; if \;\; \langle\alpha(s),\,\mathbf{B}(s)\rangle<0 \\ -\frac{\pi}{2} \;\; if \;\; \langle\alpha(s),\,\mathbf{B}(s)\rangle=0\;, \;\; \langle\alpha(s),\,\mathbf{N}(s)\rangle>0 \\ \frac{\pi}{2} \;\; if \;\; \langle\alpha(s),\,\mathbf{B}(s)\rangle=0\;, \;\; \langle\alpha(s),\,\mathbf{N}(s)\rangle<0. \end{cases} \tag{1.7}$$

Any element of the set $\{\mathbf{T}(s),\mathbf{M}(s),\mathbf{Y}(s),k_1(s),k_2(s),k_3(s)\}$ is called PAF apparatus of $\alpha=\alpha\,(s)$ [4].



(a) (b)

Figure 1: An illustration for the Positional Adapted Frame (PAF)

This paper is organized as follows. In Section 2, we study the special trajectories generated by $\mathbf{TM}$, $\mathbf{TY}$ and $\mathbf{MY}-$Smarandache curves according to PAF in three-dimensional Euclidean space and we calculate the Serret-Frenet apparatus of them. In Section 3, we provide an example involving illustrative figures for the obtained results.

2 Some Special Trajectories Generated by Smarandache Curves According to PAF

In the study [6], A. T. Ali defined special Smarandache curves in the Euclidean space. He took into consideration a unit speed regular curve $\gamma=\gamma(s)$ with its Serret-Frenet frame $\{\mathbf{T},\,\mathbf{N},\,\mathbf{B}\}$ and expressed $\mathbf{TN}$, $\mathbf{NB}$,

$\mathbf{TNB}-$ Smarandache curves as in the following:

$$\begin{aligned}\beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{N})\\ \beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{N}+\mathbf{B})\\ \beta(s^*) &= \frac{1}{\sqrt{3}}(\mathbf{T}+\mathbf{N}+\mathbf{B}),\end{aligned}$$

respectively. Some studies [6, 7, 8, 9, 10, 11, 12] on Smarandache curves can be found in the literature.

In this section, we continue to consider any moving point particle satisfying the aforesaid assumption and to denote the unit speed parameterization of the trajectory by $\alpha = \alpha(s)$. We will investigate special trajectories generated by Smarandache curves according to PAF in E^3.

We must emphasize that $\{\mathbf{T}_\alpha(s),\, \mathbf{M}_\alpha(s),\, \mathbf{Y}_\alpha(s),\, k_1(s),\, k_2(s),\, k_3(s)\}$ will show the PAF apparatus of $\alpha = \alpha\,(s)$ throughout the paper. Finally, note that we will follow similar steps given in [13] in this section.

Definition 1. *The special trajectories generated by $\mathbf{T}_\alpha\mathbf{M}_\alpha-$Smarandache curves may be defined as*

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{T}_\alpha + \mathbf{M}_\alpha\right). \tag{2.1}$$

For convenience, they are said to be $\mathbf{T}_\alpha\mathbf{M}_\alpha-$Smarandache trajectories.

Now, we investigate Serret-Frenet apparatus of $\mathbf{T}_\alpha\mathbf{M}_\alpha-$Smarandache trajectories. Differentiating the equation (2.1) with respect to s, we obtain

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\alpha + k_1\mathbf{M}_\alpha + (k_2+k_3)\,\mathbf{Y}_\alpha\right)$$

and so

$$\mathbf{T}_\sigma\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\alpha + k_1\mathbf{M}_\alpha + (k_2+k_3)\,\mathbf{Y}_\alpha\right). \tag{2.2}$$

From the equation (2.2),

$$\frac{ds^*}{ds} = \sqrt{{k_1}^2 + \frac{(k_2+k_3)^2}{2}} \tag{2.3}$$

can be found. Therefore, the equation (2.2) can be rewritten as

$$\mathbf{T}_\sigma\sqrt{{k_1}^2 + \frac{(k_2+k_3)^2}{2}} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\alpha + k_1\mathbf{M}_\alpha + (k_2+k_3)\,\mathbf{Y}_\alpha\right). \tag{2.4}$$

The equation (2.4) yields the tangent vector of σ:

$$\mathbf{T}_\sigma = \frac{1}{\sqrt{2{k_1}^2 + (k_2+k_3)^2}}\left(-k_1\mathbf{T}_\alpha + k_1\mathbf{M}_\alpha + (k_2+k_3)\,\mathbf{Y}_\alpha\right). \tag{2.5}$$

Differentiating the last equation with respect to s, we get

$$\frac{d\mathbf{T}_\sigma}{ds^*}\frac{ds^*}{ds} = \left(2{k_1}^2 + (k_2+k_3)^2\right)^{-3/2}\left(\mu_1\mathbf{T}_\alpha + \mu_2\mathbf{M}_\alpha + \mu_3\mathbf{Y}_\alpha\right) \tag{2.6}$$

where

$$\begin{aligned}
\mu_1 &= -2k_1{}^4 + \left[k_1k'_2 + k_1k'_3 - k_1{}^2k_2 - k_1{}^2k_3 - k'_1\left(k_2+k_3\right) - k_2\left(2k_1{}^2 + \left(k_2+k_3\right)^2\right)\right]\left(k_2+k_3\right) \\
\mu_2 &= -2k_1{}^4 + \left[-k_1k'_2 - k_1k'_3 - k_1{}^2k_2 - k_1{}^2k_3 + k'_1\left(k_2+k_3\right) - k_3\left(2k_1{}^2 + \left(k_2+k_3\right)^2\right)\right]\left(k_2+k_3\right) \\
\mu_3 &= 2k_1{}^2\left[k'_2 + k'_3 + k_1k_3 - k_1k_2\right] + \left[-2k'_1 - k_2{}^2 + k_3{}^2\right]k_1\left(k_2+k_3\right).
\end{aligned}$$

Taking into consideration the equation (2.3) in the equation (2.6), we obtain

$$\frac{d\mathbf{T}_\sigma}{ds^*} = \sqrt{2}\left(2k_1{}^2 + \left(k_2+k_3\right)^2\right)^{-2}\left(\mu_1\mathbf{T}_\alpha + \mu_2\mathbf{M}_\alpha + \mu_3\mathbf{Y}_\alpha\right).$$

In this case, the curvature and principal normal vector of σ are obtained as follows:

$$\kappa_\sigma = \left\|\frac{d\mathbf{T}_\sigma}{ds^*}\right\| = \frac{\sqrt{2\left(\mu_1{}^2 + \mu_2{}^2 + \mu_3{}^2\right)}}{\left(2k_1{}^2 + \left(k_2+k_3\right)^2\right)^2}$$

and

$$\mathbf{N}_\sigma = \frac{1}{\sqrt{\mu_1{}^2 + \mu_2{}^2 + \mu_3{}^2}}\left(\mu_1\mathbf{T}_\alpha + \mu_2\mathbf{M}_\alpha + \mu_3\mathbf{Y}_\alpha\right).$$

Finally, we can immediately find the binormal vector of σ as

$$\mathbf{B}_\sigma = \frac{1}{\sqrt{\left(2k_1{}^2 + \left(k_2+k_3\right)^2\right)\left(\mu_1{}^2 + \mu_2{}^2 + \mu_3{}^2\right)}}\left[\left(k_1\mu_3 - k_2\mu_2 - k_3\mu_2\right)\mathbf{T}_\alpha + \left(k_2\mu_1 + k_3\mu_1 + k_1\mu_3\right)\mathbf{M}_\alpha - \left(k_1\mu_2 + k_1\mu_1\right)\mathbf{Y}_\alpha\right]$$

by vector product $\mathbf{T}_\sigma \wedge \mathbf{N}_\sigma$.

Definition 2. *The special trajectories generated by $\mathbf{T}_\alpha\mathbf{Y}_\alpha-$Smarandache curves may be defined by*

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{T}_\alpha + \mathbf{Y}_\alpha\right). \tag{2.7}$$

For convenience, they are called as $\mathbf{T}_\alpha\mathbf{Y}_\alpha-$Smarandache trajectories.

Now, we discuss the Serret-Frenet apparatus of $\mathbf{T}_\alpha\mathbf{Y}_\alpha-$Smarandache trajectories. Differentiating the equation (2.7) with respect to s, we get

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\alpha + \left(k_1 - k_3\right)\mathbf{M}_\alpha + k_2\mathbf{Y}_\alpha\right)$$

and so

$$\mathbf{T}_\sigma\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\alpha + \left(k_1 - k_3\right)\mathbf{M}_\alpha + k_2\mathbf{Y}_\alpha\right). \tag{2.8}$$

From the equation (2.8), we can find

$$\frac{ds^*}{ds} = \sqrt{k_2{}^2 + \frac{\left(k_1 - k_3\right)^2}{2}}. \tag{2.9}$$

Therefore, the equation (2.8) can be rewritten as follows:

$$\mathbf{T}_\sigma\sqrt{k_2{}^2 + \frac{\left(k_1 - k_3\right)^2}{2}} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\alpha + \left(k_1 - k_3\right)\mathbf{M}_\alpha + k_2\mathbf{Y}_\alpha\right). \tag{2.10}$$

The equation (2.10) gives us the tangent vector of σ:

$$\mathbf{T}_\sigma = \frac{1}{\sqrt{2{k_2}^2 + (k_1 - k_3)^2}} \left(-k_2\mathbf{T}_\alpha + (k_1 - k_3)\mathbf{M}_\alpha + k_2\mathbf{Y}_\alpha\right). \tag{2.11}$$

If we differentiate the equation (2.11) with respect to s, then we find

$$\frac{d\mathbf{T}_\sigma}{ds^*}\frac{ds^*}{ds} = \left(2{k_2}^2 + (k_1 - k_3)^2\right)^{-3/2} \left(\upsilon_1\mathbf{T}_\alpha + \upsilon_2\mathbf{M}_\alpha + \upsilon_3\mathbf{Y}_\alpha\right) \tag{2.12}$$

where

$$\begin{aligned}
\upsilon_1 &= -2{k_2}^4 + \left[-k_2\left({k'}_1 - {k'}_3\right) + 2k_1{k_2}^2 - {k'}_2\left(k_3 - k_1\right) - {k_2}^2\left(k_3 - k_1\right) + k_1(k_3 - k_1)^2\right]\left(k_3 - k_1\right) \\
\upsilon_2 &= 2{k_2}^2\left[{k'}_1 - {k'}_3 - k_1k_2 - k_2k_3\right] + \left[-2{k'}_2 - {k_1}^2 + {k_3}^2k_2\left({k'}_1 - {k'}_3\right)\right]k_2\left(k_1 - k_3\right) \\
\upsilon_3 &= -2{k_2}^4 + \left[k_2\left({k'}_1 - {k'}_3\right) - 2k_3{k_2}^2 + {k'}_2\left(k_3 - k_1\right) - {k_2}^2\left(k_3 - k_1\right) - k_3(k_3 - k_1)^2\right]\left(k_3 - k_1\right).
\end{aligned}$$

Considering the equation (2.9) in the equation (2.12), we get

$$\frac{d\mathbf{T}_\sigma}{ds^*} = \sqrt{2}\left(2{k_2}^2 + (k_1 - k_3)^2\right)^{-2}\left(\upsilon_1\mathbf{T}_\alpha + \upsilon_2\mathbf{M}_\alpha + \upsilon_3\mathbf{Y}_\alpha\right).$$

Then, the curvature and principal normal vector of σ are found as in the following:

$$\kappa_\sigma = \left\|\frac{d\mathbf{T}_\sigma}{ds^*}\right\| = \frac{\sqrt{2\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}}{\left(2{k_2}^2 + (k_1 - k_3)^2\right)^2}$$

and

$$\mathbf{N}_\sigma = \frac{1}{\sqrt{{\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2}}\left(\upsilon_1\mathbf{T}_\alpha + \upsilon_2\mathbf{M}_\alpha + \upsilon_3\mathbf{Y}_\alpha\right).$$

We can easily obtain the binormal vector of σ as

$$\mathbf{B}_\sigma = \frac{1}{\sqrt{\left(2{k_2}^2 + (k_1 - k_3)^2\right)\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}}\left[(k_1\upsilon_3 - k_3\upsilon_3 - k_2\upsilon_2)\mathbf{T}_\alpha + (k_2\upsilon_1 + k_2\upsilon_3)\mathbf{M}_\alpha - (k_2\upsilon_2 - k_3\upsilon_1 + k_1\upsilon_1)\mathbf{Y}_\alpha\right]$$

by vector product of $\mathbf{T}_\sigma$ and $\mathbf{N}_\sigma$.

Definition 3. *The special trajectories generated by $\mathbf{M}_\alpha\mathbf{Y}_\alpha-$Smarandache curves can be given by*

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{M}_\alpha + \mathbf{Y}_\alpha\right). \tag{2.13}$$

For convenience, they are said to be $\mathbf{M}_\alpha\mathbf{Y}_\alpha-$Smarandache trajectories.

Now, we investigate Serret-Frenet apparatus of $\mathbf{M}_\alpha\mathbf{Y}_\alpha-$Smarandache trajectories. Differentiating the equation (2.13) with respect to the arc-length parameter s, we find

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(\left(-k_1 - k_2\right)\mathbf{T}_\alpha - k_3\mathbf{M}_\alpha + k_3\mathbf{Y}_\alpha\right)$$

and so

$$\mathbf{T}_\sigma\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(\left(-k_1 - k_2\right)\mathbf{T}_\alpha - k_3\mathbf{M}_\alpha + k_3\mathbf{Y}_\alpha\right). \tag{2.14}$$

From the equation (2.14),

$$\frac{ds^*}{ds}=\sqrt{{k_3}^2+\frac{(k_1+k_2)^2}{2}} \tag{2.15}$$

can be easily obtained. Thus, we can rewrite the equation (2.14) as follows:

$$\mathbf{T}_\sigma\sqrt{{k_3}^2+\frac{(k_1+k_2)^2}{2}}=\frac{1}{\sqrt{2}}\left(\left(-k_1-k_2\right)\mathbf{T}_\alpha-k_3\mathbf{M}_\alpha+k_3\mathbf{Y}_\alpha\right). \tag{2.16}$$

The equation (2.16) yields $\mathbf{T}_\sigma$

$$\mathbf{T}_\sigma=\frac{1}{\sqrt{2{k_3}^2+(k_1+k_2)^2}}\left(\left(-k_1-k_2\right)\mathbf{T}_\alpha-k_3\mathbf{M}_\alpha+k_3\mathbf{Y}_\alpha\right). \tag{2.17}$$

If we differentiate the equation (2.17) with respect to s, we find

$$\frac{d\mathbf{T}_\sigma}{ds^*}\frac{ds^*}{ds}=\left(2{k_3}^2+(k_1+k_2)^2\right)^{-3/2}\left(\xi_1\mathbf{T}_\alpha+\xi_2\mathbf{M}_\alpha+\xi_3\mathbf{Y}_\alpha\right) \tag{2.18}$$

where

$$\begin{aligned}
\xi_1 &= 2{k_3}^2\left[k_1k_3-k_2k_3-k'_1-k'_2\right]+\left[2k'_3+{k_1}^2-{k_2}^2\right]k_3\left(k_1+k_2\right)\\
\xi_2 &= -2{k_3}^4+\left[k_3\left(k'_1+k'_2\right)-2k_1{k_3}^2-k'_3\left(k_1+k_2\right)-{k_3}^2\left(k_1+k_2\right)-k_1(k_1+k_2)^2\right]\left(k_1+k_2\right)\\
\xi_3 &= -2{k_3}^4+\left[-k_3\left(k'_1+k'_2\right)-2k_2{k_3}^2+k'_3\left(k_1+k_2\right)-{k_3}^2\left(k_1+k_2\right)-k_2(k_1+k_2)^2\right]\left(k_1+k_2\right).
\end{aligned}$$

Taking into account of the equation (2.15) in the equation (2.18), we get

$$\frac{d\mathbf{T}_\sigma}{ds^*}=\sqrt{2}\left(2{k_3}^2+(k_1+k_2)^2\right)^{-2}\left(\xi_1\mathbf{T}_\alpha+\xi_2\mathbf{M}_\alpha+\xi_3\mathbf{Y}_\alpha\right).$$

Then, κ_σ and $\mathbf{B}_\sigma$ are obtained as

$$\begin{aligned}
\kappa_\sigma &= \frac{\sqrt{2\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right)}}{\left(2{k_3}^2+(k_1+k_2)^2\right)^2}\\
\mathbf{N}_\sigma &= \frac{1}{\sqrt{{\xi_1}^2+{\xi_2}^2+{\xi_3}^2}}\left(\xi_1\mathbf{T}_\alpha+\xi_2\mathbf{M}_\alpha+\xi_3\mathbf{Y}_\alpha\right).
\end{aligned}$$

By vector product $\mathbf{T}_\sigma\wedge\mathbf{N}_\sigma$, we can immediately find $\mathbf{B}_\sigma$ as

$$\mathbf{B}_\sigma=\frac{1}{\sqrt{\left(2{k_3}^2+(k_1+k_2)^2\right)\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right)}}\left[-(k_3\xi_3+k_3\xi_2)\mathbf{T}_\alpha+(k_3\xi_1+k_2\xi_3+k_1\xi_3)\mathbf{M}_\alpha-(k_1\xi_2+k_2\xi_2-k_3\xi_1)\mathbf{Y}_\alpha\right].$$

We must note that the torsions of $\mathbf{T}_\alpha\mathbf{M}_\alpha$, $\mathbf{T}_\alpha\mathbf{Y}_\alpha$, $\mathbf{M}_\alpha\mathbf{Y}_\alpha$-Smarandache trajectories can be obtained similarly. We leave that to the readers.

3 Applications

In this section, we will consider a point particle P moving on a specific right handed circular helix $\alpha=\alpha\left(s\right)$ and will provide examples to $\mathbf{T}_\alpha\mathbf{M}_\alpha$, $\mathbf{T}_\alpha\mathbf{Y}_\alpha$, $\mathbf{M}_\alpha\mathbf{Y}_\alpha$-Smarandache trajectories.

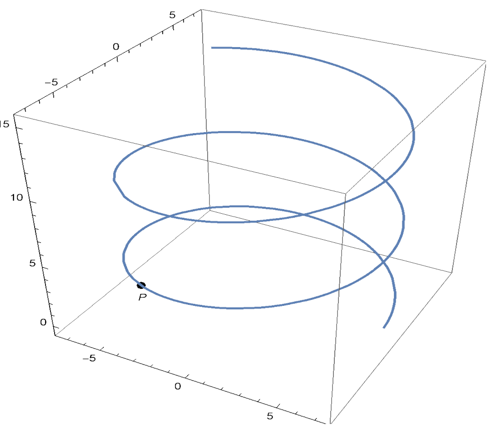


Figure 2: The trajectory of the moving point particle P in Example 1

Example 1. *In E^3, suppose that a point particle P moves on the trajectory*

$$\alpha : \left(0,\, 15\sqrt{50}\right) \to E^3, \;\; \alpha(s) = \left(7\cos\frac{s}{\sqrt{50}}, 7\sin\frac{s}{\sqrt{50}}, \frac{s}{\sqrt{50}}\right)$$

which is a unit speed curve.
In the light of the information given in the first section, PAF apparatus of this trajectory are obtained as follows:

$$\begin{aligned}
\mathbf{T}_\alpha(s) &= \left(\frac{-7}{\sqrt{50}}\sin\frac{s}{\sqrt{50}}, \frac{7}{\sqrt{50}}\cos\frac{s}{\sqrt{50}}, \frac{1}{\sqrt{50}}\right) \\
\mathbf{M}_\alpha(s) &= \begin{pmatrix} -\cos\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}} - \frac{1}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}}, \\ -\cos\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}} + \frac{1}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}}, \\ -\frac{7}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right) \end{pmatrix} \\
\mathbf{Y}_\alpha(s) &= \begin{pmatrix} -\sin\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}} + \frac{1}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}}, \\ -\sin\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}} - \frac{1}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}}, \\ \frac{7}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right) \end{pmatrix} \qquad (3.1) \\
k_1(s) &= \frac{7}{50}\cos\left(\arctan\left(\frac{50}{s}\right)\right) \\
k_2(s) &= \frac{7}{50}\sin\left(\arctan\left(\frac{50}{s}\right)\right) \\
k_3(s) &= \frac{1}{50} + \frac{50}{2500 + s^2}.
\end{aligned}$$

Let us show $\mathbf{T}_\alpha\mathbf{M}_\alpha$, $\mathbf{T}_\alpha\mathbf{Y}_\alpha$, $\mathbf{M}_\alpha\mathbf{Y}_\alpha-$Smarandache trajectories with σ_1, σ_2, σ_3, respectively. In that case, the parametric equation of σ_1 can be easily given as follows:

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{-7}{\sqrt{50}}\sin\frac{s}{\sqrt{50}} - \cos\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}} - \frac{1}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}}, \\ \frac{7}{\sqrt{50}}\cos\frac{s}{\sqrt{50}} - \cos\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}} + \frac{1}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}}, \\ \frac{1}{\sqrt{50}} - \frac{7}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right) \end{pmatrix}.$$

See the curve σ_1 in Figure 3.

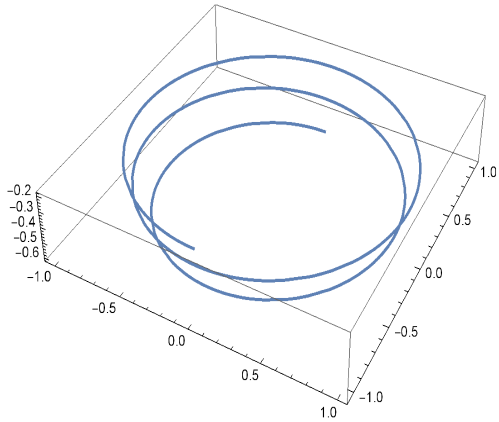


Figure 3: $\mathbf{T}_\alpha \mathbf{M}_\alpha$ Smarandache trajectory

Similarly, the parametric equation of σ_2 can be immediately given as

$$\sigma_2 = \frac{1}{\sqrt{2}} \left(\begin{array}{r} \frac{-7}{\sqrt{50}} \sin \frac{s}{\sqrt{50}} - \sin\left(\arctan\left(\frac{50}{s}\right)\right) \cos \frac{s}{\sqrt{50}} + \frac{1}{\sqrt{50}} \cos\left(\arctan\left(\frac{50}{s}\right)\right) \sin \frac{s}{\sqrt{50}}, \\ \frac{7}{\sqrt{50}} \cos \frac{s}{\sqrt{50}} - \sin\left(\arctan\left(\frac{50}{s}\right)\right) \sin \frac{s}{\sqrt{50}} - \frac{1}{\sqrt{50}} \cos\left(\arctan\left(\frac{50}{s}\right)\right) \cos \frac{s}{\sqrt{50}}, \\ \frac{1}{\sqrt{50}} + \frac{7}{\sqrt{50}} \cos\left(\arctan\left(\frac{50}{s}\right)\right) \end{array} \right).$$

See the curve σ_2 in Figure 4.

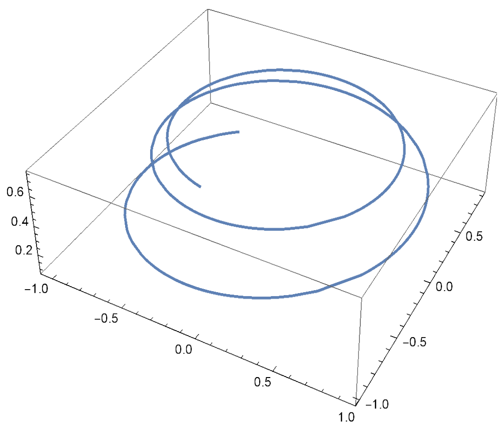


Figure 4: $\mathbf{T}_\alpha \mathbf{Y}_\alpha$ Smarandache trajectory

Finally, we obtain the parametric equation of σ_3 as

$$\sigma_3 = \frac{1}{\sqrt{2}} \left(\begin{array}{r} \cos\left(\arctan\left(\frac{50}{s}\right)\right)\left(\frac{1}{\sqrt{50}} \sin \frac{s}{\sqrt{50}} - \cos \frac{s}{\sqrt{50}}\right) - \sin\left(\arctan\left(\frac{50}{s}\right)\right)\left(\frac{1}{\sqrt{50}} \sin \frac{s}{\sqrt{50}} + \cos \frac{s}{\sqrt{50}}\right), \\ \sin\left(\arctan\left(\frac{50}{s}\right)\right)\left(\frac{1}{\sqrt{50}} \cos \frac{s}{\sqrt{50}} - \sin \frac{s}{\sqrt{50}}\right) - \cos\left(\arctan\left(\frac{50}{s}\right)\right)\left(\frac{1}{\sqrt{50}} \cos \frac{s}{\sqrt{50}} + \sin \frac{s}{\sqrt{50}}\right), \\ -\frac{7}{\sqrt{50}} \sin\left(\arctan\left(\frac{50}{s}\right)\right) + \frac{7}{\sqrt{50}} \cos\left(\arctan\left(\frac{50}{s}\right)\right) \end{array} \right).$$

See the curve σ_3 in Figure 5.

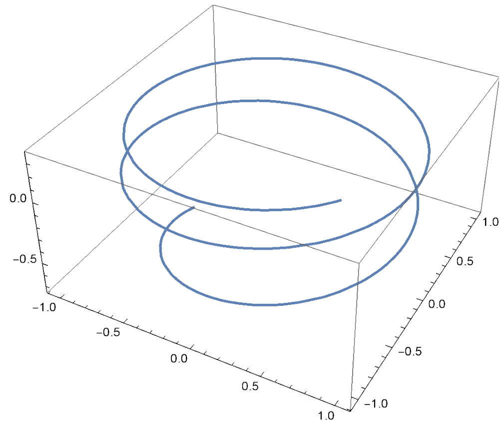


Figure 5: $\mathbf{M}_\alpha \mathbf{Y}_\alpha$ Smarandache trajectory

In the light of the above information, one can immediately see $\mathbf{T}_{\sigma_i}$, $\mathbf{N}_{\sigma_i}$, $\mathbf{B}_{\sigma_i}$, κ_{σ_i}, $(i = 1, 2, 3)$ *by using the equation* (3.1).

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