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Applications of Smarandache Function, and Prime and Coprime Functions

\[ C_k(n_1, n_2, \ldots, n_k) = \begin{cases} 
0 & \text{if } n_1, n_2, \ldots, n_k \text{ are coprime numbers} \\
1 & \text{otherwise} 
\end{cases} \]

American Research Press
Rehoboth
2002
Applications of Smarandache Function, and Prime and Coprime Functions
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Chapter 1: Smarandache function applied to perfect numbers

The Smarandache function is defined as follows:

\[ S(n) = \text{the smallest positive integer such that } S(n)! \text{ is divisible by } n. \] [1]

In this article we are going to see that the value this function takes when \( n \) is a perfect number of the form \( n = 2^{k-1} \cdot (2^k - 1) \), \( p = 2^k - 1 \) being a prime number.

Lemma 1: Let \( n = 2^i \cdot p \) when \( p \) is an odd prime number and \( i \) an integer such that:

\[
0 \leq i \leq E\left(\frac{p}{2}\right) + E\left(\frac{p}{2^2}\right) + E\left(\frac{p}{2^3}\right) + \cdots + E\left(\frac{p}{2^{E\left(p \log_2 p\right)}}\right) = e_2(p!)
\]

where \( e_2(p!) \) is the exponent of 2 in the prime number decomposition of \( p! \).

\( E(x) \) is the greatest integer less than or equal to \( x \).

One has that \( S(n) = p \).

Demonstration:
Given that \( GCD(2^i, p) = 1 \) (GCD= greatest common divisor) one has that \( S(n) = \max\{S(2^i), S(p)\} \geq S(p) = p \). Therefore \( S(n) \geq p \).

If we prove that \( p! \) is divisible by \( n \) then one would have the equality.

\[ p! = p_1^{e_{p_1}(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_s^{e_{p_s}(p!)} \]

where \( p_i \) is the \( i-th \) prime of the prime number decomposition of \( p! \). It is clear that \( p_1 = 2 \), \( p_s = p \), \( e_{p_s}(p!) = 1 \) for which:

\[ p! = 2^{e_2(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)} \cdot p \]
From where one can deduce that:

\[
\frac{p!}{n} = 2^{e_2(p^b) - 1} \cdot p_2^{e_{p^2}(p^b)} \cdots p_{n-1}^{e_{p^{n-1}}(p^b)}
\]

is a positive integer since \( e_2(p!) - i \geq 0 \).

Therefore one has that \( S(n) = p \).

Proposition 1: If \( n \) is a perfect number of the form \( n = 2^{k-1} \cdot (2^k - 1) \) with \( k \) is a positive integer, \( 2^k - 1 = p \) prime, one has that \( S(n) = p \).

Demonstration:

For the Lemma it is sufficient to prove that \( k - 1 \leq e_2(p!) \).

If we can prove that:

\[
k - 1 \leq 2^{k-1} - \frac{1}{2}
\]  (1)

we will have proof of the proposition since:

\[
k - 1 \leq 2^{k-1} - \frac{1}{2} = \frac{2^k - 1}{2} = \frac{p}{2}
\]

As \( k - 1 \) is an integer one has that \( k - 1 \leq E\left(\frac{p}{2}\right) \leq e_2(p!) \)

Proving (1) is the same as proving \( k \leq 2^{k-1} + \frac{1}{2} \) at the same time, since \( k \) is integer, is equivalent to proving \( k \leq 2^{k-1} \) (2).

In order to prove (2) we may consider the function: \( f(x) = 2^{x-1} - x \) \( x \) real number.

This function may be derived and its derivate is \( f'(x) = 2^{x-1} \ln 2 - 1 \).

\( f \) will be increasing when \( 2^{x-1} \ln 2 - 1 > 0 \) resolving \( x \):

\[
x > 1 - \frac{\ln(\ln 2)}{\ln 2} \equiv 1.5287
\]

In particular \( f \) will be increasing \( \forall x \geq 2 \).

Therefore \( x \geq 2 \) \( f(x) \geq f(2) = 0 \) that is to say \( 2^{x-1} - x \geq 0 \) \( \forall x \geq 2 \).
Therefore:  \(2^{k-1} \geq k \forall k \geq 2\) integer.

And thus is proved the proposition.

EXAMPLES:

\[
\begin{align*}
6 &= 2 \cdot 3 & S(6) &= 3 \\
28 &= 2^2 \cdot 7 & S(28) &= 7 \\
496 &= 2^4 \cdot 31 & S(496) &= 31 \\
8128 &= 2^6 \cdot 127 & S(8128) &= 127
\end{align*}
\]

References:

Chapter 2: A result obtained using the Smarandache Function

Smarandache Function is defined as followed: 
S(m) = The smallest positive integer so that S(m)! is divisible by m. \[1\] 
Let’s see the value which such function takes for \( m = p^n \) with \( n \) integer, \( n \geq 2 \) and \( p \) prime number. To do so a Lemma required.

Lemma 1 \( \forall \ m, n \in \mathbb{N}, \ m, n \geq 2 \)

\[
m^n = E \left[ \frac{m^{n+1} - m^n + m}{m} \right] + E \left[ \frac{m^{n+1} - m^n + m}{m^2} \right] + \cdots + E \left[ \frac{m^{n+1} - m^n + m}{E \left[ \log_m \left( m^{n+1} - m^n + m \right) \right]} \right]
\]

where \( E(x) \) gives the greatest integer less than or equal to \( x \).

Proof:

Let’s see in the first place the value taken by \( E \left[ \log_m \left( m^{n+1} - m^n + m \right) \right] \).

If \( n \geq 2 \): \( m^{n+1} - m^n + m < m^{n+1} \) and therefore

\[
\log_m \left( m^{n+1} - m^n + m \right) < \log_m m^{n+1} = n + 1.
\]

And if \( m \geq 2 \):

\[
mm^n \geq 2m^n \Rightarrow m^{n+1} \geq 2m^n \Rightarrow m^{n+1} + m \geq 2m^n \Rightarrow m^{n+1} - m^n + m \geq m^n \Rightarrow \log_m \left( m^{n+1} - m^n + m \right) \geq \log_m m^n = n \Rightarrow E \left[ \log_m \left( m^{n+1} - m^n + m \right) \right] \geq n
\]

As a result: \( n \leq E \left[ \log_m \left( m^{n+1} - m^n + m \right) \right] < n + 1 \) therefore:

\[
E \left[ \log_m \left( m^{n+1} - m^n + m \right) \right] = n \quad \text{if} \quad n, m \geq 2
\]

Now let’s see the value which it takes for \( 1 \leq k \leq n \):

\[
E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = E \left[ m^{n+1-k} - m^{n-k} + \frac{1}{m^{k-1}} \right]
\]
If $k=1$: \[ E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^n - m^{n-1} + 1 \]

If $1 < k \leq n$: \[ E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^{n+1-k} - m^{n-k} \]

Let’s see what is the value of the sum:

\[
\begin{align*}
\text{k=1} & \quad m^n \quad -m^{n-1} \quad \ldots \quad \ldots \quad \ldots \quad +1 \\
\text{k=2} & \quad m^{n-1} \quad -m^{n-2} \\
\text{k=3} & \quad m^{n-2} \quad -m^{n-3} \\
& \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\text{k=n-1} & \quad m^2 \quad -m \\
\text{k=n} & \quad m \quad -1
\end{align*}
\]

Therefore:

\[
\sum_{k=1}^{n} E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^n \quad m,n \geq 2
\]

**Proposition:** \( \forall \quad p \text{ prime number} \quad \forall n \geq 2 : \)

\[
S(p^{p^n}) = p^{n+1} - p^n + p
\]

**Proof:**

Having \( e_p(k) = \text{exponent of the prime number p in the prime decomposition of k} \).

We get:

\[
e_p(k) = E \left( \frac{k}{p} \right) + E \left( \frac{k}{p^2} \right) + E \left( \frac{k}{p^3} \right) + \ldots + E \left( \frac{k}{E(\log_p k)} \right)
\]
And using the lemma we have

$$e_p \left( \left( p^{n+1} - p^n + p \right)! \right) = E \left[ \frac{p^{n+1} - p^n + p}{p} \right] + E \left[ \frac{p^{n+1} - p^n + p}{p^2} \right] + \cdots + E \left[ \frac{p^{n+1} - p^n + p}{p^{\log_p \left( p^{n+1} - p^n + p \right)}} \right] = p^n$$

Therefore:

$$\frac{(p^{n+1} - p^n + p)!}{p^n} \in \mathbb{N} \quad \text{and} \quad \frac{(p^{n+1} - p^n + p - 1)!}{p^n} \notin \mathbb{N}$$

And:

$$S\left(p^n\right) = p^{n+1} - p^n + p$$

References:

Chapter 3: A Congruence with the Smarandache function

Smarandache’s function is defined thus:

$S(n) =$ the smallest integer such that $S(n)!$ is divisible by n. \[1\]

In this article we are going to look at the value that has $S(2^k - 1) \pmod k$ for all integer, $2 \leq k \leq 97$.

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One can see from the table that there are only 4 exceptions for $2 \leq k \leq 97$
We can see in detail the 4 exceptions in a table:

- $k=28=2^2\cdot7$ \quad $S(2^{28}-1)\equiv 15 \pmod{28}$
- $k=52=2^2\cdot13$ \quad $S(2^{52}-1)\equiv 27 \pmod{52}$
- $k=68=2^2\cdot17$ \quad $S(2^{68}-1)\equiv 35 \pmod{68}$
- $k=92=2^2\cdot23$ \quad $S(2^{92}-1)\equiv 47 \pmod{92}$

One can observe in these 4 cases that $k=2^p$ with $p$ is a prime and more over $S(2^k-1) \equiv \frac{k}{2} + 1 \pmod{k}$

**UNSOLVED QUESTION:**

One can obtain a general formula that gives us, in function of $k$ the value $S(2^k-1) \pmod{k}$ for all positive integer values of $k$.

Reference:

Chapter 4: A functional recurrence to obtain the prime numbers using the Smarandache prime function.

Theorem: We are considering the function:

For \( n \) integer:

\[
F(n) = n + 1 + \sum_{m=2}^{2n} \prod_{i=1}^{m} \left[ 1 + \left( \sum_{j=1}^{\left\lfloor \frac{i}{j} \right\rfloor} \frac{\left\lfloor \frac{i}{j} \right\rfloor - 1}{j} \right) - 2 \right]
\]

one has: \( p_{k+1} = F(p_k) \) for all \( k \geq 1 \) where \( \{p_k\}_{k \geq 1} \) are the prime numbers and \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \).

Observe that the knowledge of \( p_{k+1} \) only depends on knowledge of \( p_k \) and the knowledge of the fore primes is unnecessary.

Proof:

Suppose that we have found a function \( P(i) \) with the following property:

\[
P(i) = \begin{cases} 
1 & \text{if } i \text{ is composite} \\
0 & \text{if } i \text{ is prime}
\end{cases}
\]

This function is called Smarandache prime function.(Ref.)

Consider the following product:

\[
\prod_{i=p_k+1}^{m} P(i)
\]

If \( p_k < m < p_{k+1} \), \( \prod_{i=p_k+1}^{m} P(i) = 1 \) since \( i : p_k + 1 \leq i \leq m \) are all composites.
If \( m \geq p_{k+1} \),
\[
\prod_{i = p_k + 1}^{m} P(i) = 0 \quad \text{since } P(p_{k+1}) = 0
\]

Here is the sum:
\[
\sum_{m=p_k+1}^{2p_k} \prod_{i = p_k + 1}^{m} P(i) = \sum_{m=p_k+1}^{p_{k+1} - 1} \prod_{i = p_k + 1}^{m} P(i) + \sum_{m=p_k+1}^{2p_k} \prod_{i = p_k + 1}^{m} P(i) = \sum_{m=p_k+1}^{p_{k+1} - 1} 1 = p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1
\]

The second sum is zero since all products have the factor \( P(p_{k+1}) = 0 \).

Therefore we have the following recurrence relation:
\[
p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} \prod_{i = p_k + 1}^{m} P(i)
\]

Let’s now see we can find \( P(i) \) with the asked property.

Consider:
\[
\begin{bmatrix} \lfloor \frac{i}{j} \rfloor - \lfloor \frac{i-1}{j} \rfloor \end{bmatrix} = \begin{cases} 1 & \text{if } j \mid i \\ 0 & \text{if } j \not{\mid} i \end{cases} \quad j = 1,2,\ldots,i \quad i \geq 1
\]

We deduce of this relation:
\[
d(i) = \sum_{j=1}^{i} \left[ \lfloor \frac{i}{j} \rfloor - \lfloor \frac{i-1}{j} \rfloor \right]
\]

where \( d(i) \) is the number of divisors of \( i \).
If \( i \) is prime \( d(i) = 2 \) therefore:

\[
- \left[ - \frac{d(i) - 2}{i} \right] = 0
\]

If \( i \) is composite \( d(i) > 2 \) therefore:

\[
0 < \frac{d(i) - 2}{i} < 1 \Rightarrow - \left[ - \frac{d(i) - 2}{i} \right] = 1
\]

Therefore we have obtained the Smarandache Prime Function \( P(i) \) which is:

\[
P(i) = \left[ - \sum_{j=1}^{i} \left( \left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor \right) \right] - 2 \quad i \geq 2 \quad \text{integer}
\]

With this, the theorem is already proved.

References:


Chapter 5: The general term of the prime number sequence and the Smarandache prime function.

Let is consider the function \( d(i) = \) number of divisors of the positive integer number \( i \). We have found the following expression for this function:

\[
d(i) = \sum_{k=1}^{i} E\left(\frac{i}{k}\right) - E\left(\frac{i-1}{k}\right)
\]

“E(x) = Floor[x]”

We proved this expression in the article “A functional recurrence to obtain the prime numbers using the Smarandache Prime Function”.

We deduce that the following function:

\[
G(i) = -E\left[ \frac{d(i) - 2}{i} \right]
\]

This function is called the Smarandache Prime Function (Reference)
It takes the next values:

\[
G(i) = \begin{cases} 
0 & \text{if } i \text{ is prime} \\
1 & \text{if } i \text{ is composite}
\end{cases}
\]

Let us consider now \( \pi(n) = \) number of prime numbers smaller or equal than \( n \).
It is simple to prove that:

\[
\pi(n) = \sum_{i=2}^{n} (1 - G(i))
\]
Let is have too:

\[
\begin{align*}
\text{If} & \quad 1 \leq k \leq p_n - 1 \quad \Rightarrow \quad E\left(\frac{\pi(k)}{n}\right) = 0 \\
\text{If} & \quad C_n \geq k \geq p_n \quad \Rightarrow \quad E\left(\frac{\pi(k)}{n}\right) = 1 
\end{align*}
\]

We will see what conditions have to carry \( C_n \).

Therefore we have the following expression for \( p_n \), \( n \)-th prime number:

\[
p_n = 1 + \sum_{k=1}^{C_n} (1 - E\left(\frac{\pi(k)}{n}\right))
\]

If we obtain \( C_n \) that only depends on \( n \), this expression will be the general term of the prime numbers sequence, since \( \pi \) is in function with \( G \) and \( G \) does with \( d(i) \) that is expressed in function with \( i \) too. Therefore the expression only depends on \( n \).

Let is consider \( C_n = 2(E(n\log n) + 1) \)

Since \( p_n \approx n\log n \) from of a certain \( n_0 \) it will be true that

\[(1) \quad p_n \leq 2(E(n\log n) + 1)\]

If \( n_0 \) it is not too big, we can prove that the inequality is true for smaller or equal values than \( n_0 \).

It is necessary to that:

\[
E\left[\frac{\pi(2(E(n\log n) + 1))}{n}\right] = 1
\]

If we check the inequality:

\[(2) \quad \pi(2(E(n\log n) + 1)) < 2n\]
We will obtain that:

\[
\frac{\pi(C_n)}{n} < 2 \Rightarrow E\left[\frac{\pi(C_n)}{n}\right] \leq 1 \quad ; C_n \geq p_n \Rightarrow E\left[\frac{\pi(C_n)}{n}\right] = 1
\]

We can experimentaly check this last inequality saying that it checks for a lot of values and the difference tends to increase, wich makes to think that it is true for all \( n \).

Therefore if we prove that the (1) and (2) inequalities are true for all \( n \) which seems to be very probable; we will have that the general term of the prime numbers sequence is:

\[
p_n = 1 + \sum_{k=1}^{2(E(n \log n)+1)} \left( 1 - E\left[ \sum_{j=2}^{k} \left[ 1 + E\left[ \frac{\sum_{s=1}^{j}(E(j/s) - E((j-1)/s)) - 2}{j} \right] \right] \right] \right) - \sum_{k=1}^{E(n \log n)+1} \left( 1 - E\left[ \sum_{j=2}^{k} \left[ 1 + E\left[ \frac{\sum_{s=1}^{j}(E(j/s) - E((j-1)/s)) - 2}{j} \right] \right] \right] \right)
\]

Reference:
Http://www.gallup.unm.edu/~Smarandache/primfnct.txt
Chapter 6: Expressions of the Smarandache Coprime Function

Smarandache Coprime function is defined this way:

\[
C_k(n_1, n_2, \ldots, n_k) = \begin{cases} 
0 & \text{if } n_1, n_2, \ldots, n_k \text{ are coprime numbers} \\
1 & \text{otherwise}
\end{cases}
\]

We see two expressions of the Smarandache Coprime Function for \( k = 2 \).

**EXPRESSION 1:**

\[
C_2(n_1, n_2) = \left\lfloor -\frac{n_1 n_2 - \text{lcm}(n_1, n_2)}{n_1 n_2} \right\rfloor
\]

\( \lfloor x \rfloor \) is the biggest integer number smaller or equal than \( x \).

If \( n_1, n_2 \) are coprime numbers:

\[
lcm(n_1, n_2) = n_1 n_2 \quad \text{therefore:} \quad C_2(n_1, n_2) = -\left\lfloor \frac{0}{n_1 n_2} \right\rfloor = 0
\]

If \( n_1, n_2 \) aren’t coprime numbers:

\[
lcm(n_1, n_2) < n_1 n_2 \Rightarrow 0 < \frac{n_1 n_2 - lcm(n_1, n_2)}{n_1 n_2} < 1 \Rightarrow C_2(n_1, n_2) = 1
\]

**EXPRESSION 2:**

\[
C_2(n_1, n_2) = 1 + \left\lfloor \prod_{d \mid n_1} \prod_{d' \mid n_2} \frac{|d - d'|}{\prod_{d > 1, d' > 1} \prod_{d \mid n_1, d' \mid n_2} (d + d')} \right\rfloor
\]
If \( n_1, n_2 \) are coprime numbers then \( d \neq d' \quad \forall d, d' \neq 1 \)

\[
\prod_{d \mid n_1} \prod_{d' \mid n_2} |d - d'| \\
\Rightarrow 0 < \prod_{d \mid n_1} \prod_{d' \mid n_2} (d + d') \quad < 1 \Rightarrow C_2(n_1, n_2) = 0
\]

If \( n_1, n_2 \) aren’t coprime numbers \( \exists d = d' \quad d > 1, d' > 1 \Rightarrow C_2(n_1, n_2) = 1 \)

**EXPRESSION 3:**

Smarandache Coprime Function for \( k \geq 2 \):

\[
C_k(n_1, n_2, \ldots, n_k) = \left[ \frac{1}{GCD(n_1, n_2, \ldots, n_k)} - 1 \right]
\]

If \( n_1, n_2, \ldots, n_k \) are coprime numbers:

\[
GCD(n_1, n_2, \ldots, n_k) = 1 \Rightarrow C_k(n_1, n_2, \ldots, n_k) = 0
\]

If \( n_1, n_2, \ldots, n_k \) aren’t coprime numbers: \( GCD(n_1, n_2, \ldots, n_k) > 1 \)

\[
0 < \frac{1}{GCD} < 1 \Rightarrow -\left[ \frac{1}{GCD} - 1 \right] = 1 = C_k(n_1, n_2, \ldots, n_k)
\]

References:

1. E. Burton, “Smarandache Prime and Coprime Function”
Chapter 7: New Prime Numbers

I have found some new prime numbers using the PROTH program of Yves Gallot.
This program is based on the following theorem:

**Proth Theorem (1878):**
Let \( N = k \cdot 2^n + 1 \) where \( k < 2^n \). If there is an integer number \( a \) so that
\[
\frac{N-1}{2} \equiv -1 \pmod{N}
\]
therefore \( N \) is prime.

The Proth program is a test for primality of greater numbers defined as
\( k \cdot b^n + 1 \) or \( k \cdot b^n - 1 \). The program is made to look for numbers of less
than 5,000,000 digits and it is optimized for numbers of more than 1000
digits.

Using this Program, I have found the following prime numbers:

- \( 3239 \cdot 2^{12345} + 1 \) with 3720 digits \( a = 3, \ a = 7 \)
- \( 7551 \cdot 2^{12345} + 1 \) with 3721 digits \( a = 5, \ a = 7 \)
- \( 7595 \cdot 2^{12345} + 1 \) with 3721 digits \( a = 3, \ a = 11 \)
- \( 9363 \cdot 2^{12321} + 1 \) with 3713 digits \( a = 5, \ a = 7 \)

Since the exponents of the first three numbers are Smarandache number
\( Sm(5) = 12345 \) we can call this type of prime numbers, prime numbers
of Smarandache.

Helped by the MATHEMATICA program, I have also found new prime
numbers which are a variant of prime numbers of Fermat. They are the
following:

\[
2^n \cdot 3^{2^n} - 2^{2^n} - 3^{2^n} \quad \text{for } n=1, 4, 5, 7.
\]

It is important to mention that for \( n=7 \) the number which is obtained has
100 digits.
Chris Nash has verified the values \( n=8 \) to \( n=20 \), this last one being a number of 815,951 digits, obtaining that they are all composite. All of them have a tiny factor except \( n=13 \).

**References:**

2. Chris Caldwell, The Prime Pages, [www.utm.edu/research/primes](http://www.utm.edu/research/primes)
A book for people who love numbers:
Smarandache Function applied to perfect numbers, congruences.
Also, the Smarandache Prime and Coprime functions in connection with the expressions of the prime numbers.

$5.95