Approximation To The Smarandache Curves in the The Null Cone

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Abstract In this paper, we study the Smarandache curves according to the asymptotic orthonormal frame in Null Cone $Q^3$. By using cone frame formulas, we obtain some characterizations of the Smarandache curves and introduce cone frenet invariants of these curves.

1 Introduction

The idea of studying curves has been one of the impressive topic owing to having many application area from mathematics to the diverse branch of science. As a result of this case, many mathematicians have studied different type of curves by using Frenet frame in numerous spaces. Among these, Smarandache curves have attract major attention by investigators for a long while.

Smarandache geometry is a geometry which has at least one Smarandachely denied axiom [4]. An axiom is said to be Smarandachely denied, if it behaves in at least two different ways within the same space. Smarandache curve is defined as a regular curve whose position vector is composed by Frenet frame vectors of another regular curve. Smarandache curves in various ambient spaces have been classified in [1]-[8], [14]-[16].

In this study, we give special Smarandache curves such as $x\alpha, xy, x\beta, \alpha\beta, y\beta, \alpha y$-smarandache curves according to asymptotic orthonormal frame in the Null Cone $Q^3$ and we examine the curvature and the asymptotic orthonormal frame’s vectors of the Smarandache curves. We also present an example related to these curves.

2 Preliminaries

Some basics of the curves in the null cone are provided from, [9]- [10]. Let $E^4$ be the 4-dimensional pseudo-Euclidean space with the

$$\tilde{g}(X, Y) = \langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$$

for all $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4) \in E^4$. $E^4$ is a flat pseudo-Riemannian manifold of signature (3, 1).

Let $M$ be a submanifold of $E^4$. If the pseudo-Riemannian metric $\tilde{g}$ of $E^4$ induces a pseudo-Riemannian metric $g$ (respectively, a Riemannian metric, a degenerate quadratic form) on $M$, then $M$ is called a timelike( respectively, spacelike, degenerate) submanifold of $E^4$. Let $c$ be a fixed point in $E^4$. The pseudo-Riemannian lightlike cone (quadric cone ) is defined by

$$Q^3(c) = \{ x \in E^4 : g(x - c, x - c) = 0 \},$$

where the point $c$ is called the center of $Q^3(c)$. When $c = 0$, we simply denote $Q^3(0)$ by $Q^3$ and call it the null cone.

Let $E^4$ be 4-dimensional Minkowski space and $Q^3$ be the lightlike cone in $E^4$. A vector $V \neq 0$ in $E^4$ is called spacelike, timelike or lightlike, if $\langle V, V \rangle > 0, \langle V, V \rangle < 0$ or $\langle V, V \rangle = 0$, respectively. The norm of a vector $x \in E^4$ is given by $\|x\| = \sqrt{\langle x, x \rangle}$, [13].
We assume that curve \( x : I \rightarrow Q^3 \subset E^4_1 \) is a regular curve in \( Q^3 \) for \( t \in I \). In the following, we always assume that the curve is regular.

A frame field \( \{x, \alpha, \beta, y\} \) on \( E^4_1 \) is called an asymptotic orthonormal frame field, if

\[
\langle x, x \rangle = \langle y, y \rangle = \langle x, \alpha \rangle = \langle y, \alpha \rangle = \langle \beta, \alpha \rangle = \langle y, \beta \rangle = \langle x, \beta \rangle = 0, \\
\langle x, y \rangle = \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1.
\]

Using \( x'(s) = \alpha(s) \), we know that \( \{x(s), \alpha(s), \beta(s), y(s)\} \) from an asymptotic orthonormal frame along the curve \( x(s) \) and the cone frenet formulas of \( x(s) \) are given by

\[
x'(s) = \alpha(s) \\
\alpha'(s) = \kappa(s)x(s) - y(s) \tag{3.1}
\]

\[
\beta'(s) = \tau(s)x(s) \\
y'(s) = -\kappa(s)\alpha(s) - \tau(s)\beta(s),
\]

where the functions \( \kappa(s) \) and \( \tau(s) \) are called cone curvature functions of the curve \( x(s) \), [11].

Let \( x : I \rightarrow Q^3 \subset E^4_1 \) be a spacelike curve in \( Q^3 \) with an arc length parameter \( s \). Then \( x = x(s) = (x_1, x_2, x_3, x_4) \) can be written as

\[
x(s) = \frac{1}{2\sqrt{f^2 + g_2}} (2f, 2g, 1 - f^2 - g^2, 1 + f^2 + g^2), \tag{2.2}
\]

for some non constant function \( f(s) \) and \( g(s) \), [12].

## 3 The Smarandache Curves in The Null Cone \( Q^3 \)

In this section, we define binary Smarandache curves according to the asymptotic orthonormal frame in \( Q^3 \). Also, we obtain the asymptotic orthonormal frame and cone curvature functions of the Smarandache partners lying on \( Q^3 \) using cone frenet formulas.

Smarandache curve \( \gamma = \gamma(s^*(s)) \) of the curve \( x \) is a regular unit speed curve lying fully on \( Q^3 \). Let \( \{x, \alpha, \beta, y\} \) and \( \{\gamma, \alpha_\gamma, \beta_\gamma, y_\gamma\} \) be the moving asymptotic orthonormal frames of \( x \) and \( \gamma \), respectively.

**Definition 3.1.** Let \( x \) be unit speed spacelike curve lying on \( Q^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \). Then, \( x\alpha \)-smarandache curve of \( x \) is defined by

\[
\gamma_{x\alpha}(s^*) = \frac{a}{b} x(s) + \alpha(s), \tag{3.1}
\]

where \( a, b \in \mathbb{R}^+_0 \).

**Theorem 3.2.** Let \( x \) be unit speed spacelike curve in \( Q^3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvatures \( \kappa(s), \tau(s) \) and let \( \gamma_{x\alpha} \) be \( x\alpha \)-smarandache curve with asymptotic orthonormal frame \( \{\gamma_{x\alpha}, \alpha_{x\alpha}, \beta_{x\alpha}, y_{x\alpha}\} \). Then the following relations hold:

1. The asymptotic orthonormal frame \( \{\gamma_{x\alpha}, \alpha_{x\alpha}, \beta_{x\alpha}, y_{x\alpha}\} \) of the \( x\alpha \)-smarandache curve \( \gamma_{x\alpha} \) is given as

\[
\begin{bmatrix}
\gamma_{x\alpha} \\
\alpha_{x\alpha} \\
\beta_{x\alpha} \\
y_{x\alpha}
\end{bmatrix} = 
\begin{bmatrix}
\frac{a}{b} & 0 & 0 & 0 \\
\frac{1}{\sqrt{a^2 - 2b^2}} & \frac{1}{\sqrt{a^2 - 2b^2}} & 0 & -\frac{a}{b} \\
B_1 & B_2 & B_3 & B_4 \\
Y_1 & Y_2 & Y_3 & Y_4
\end{bmatrix}
\begin{bmatrix}
x \\
\alpha \\
\beta \\
y
\end{bmatrix}, \tag{3.2}
\]

where

\[
\xi = \frac{1}{\sqrt{a^2 - 2b^2}}, w = \frac{1}{b} \sqrt{a^2 - 2b^2}, \\
B_1 = \frac{1}{w} (a\xi + b\xi' + b\xi^2), B_2 = \frac{1}{w} ((\alpha + b\kappa)\xi' + (\kappa' + \kappa)b\xi), \\
B_3 = \frac{1}{w} (b\xi \tau), B_4 = \frac{1}{w} (a\xi + b\xi') \tag{3.4}
\]

\[
\xi = \frac{1}{\sqrt{a^2 - 2b^2}}, w = \frac{1}{b} \sqrt{a^2 - 2b^2}, \\
B_1 = \frac{1}{w} (a\xi + b\xi' + b\xi^2), B_2 = \frac{1}{w} ((\alpha + b\kappa)\xi' + (\kappa' + \kappa)b\xi), \\
B_3 = \frac{1}{w} (b\xi \tau), B_4 = \frac{1}{w} (a\xi + b\xi') \tag{3.4}
\]
and
\[ Y_1 = -(B_1 + \frac{a}{2b} (2B_1 B_4 + B_2^2 + B_3^2)), \]
\[ Y_2 = -(B_2 + \frac{1}{2} (2B_1 B_4 + B_2^2 + B_3^2)), \]
\[ Y_3 = -B_3, Y_4 = -B_4. \] (3.5)

ii) The cone curvatures \( \kappa_{\gamma_{x\alpha}}(s^*) \) and \( \tau_{\gamma_{x\alpha}}(s^*) \) of the curve \( \gamma_{x\alpha} \) is given by
\[ \kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2} (2B_1 B_4 + B_2^2 + B_3^2) \]
\[ \tau_{\gamma_{x\alpha}}(s^*) = \sqrt{2(Y_1 - \kappa') Y_4 + (Y_2 - \kappa)^2 + Y_3^2 - \kappa_{\gamma_{x\alpha}}^2}, \] (3.6)
where
\[ s^* = \frac{1}{b} \int \sqrt{a^2 - 2b^2 \kappa(s)} ds. \]

**Proof.** i) We assume that the curve \( x \) is a unit speed spacelike curve with the asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvature \( \kappa, \tau \). Differentiating the equation (3.1) with respect to \( s \) and considering (2.1), we have
\[ \gamma_{x\alpha}'(s^*) = (a\xi)\alpha(s) + (b\xi)\alpha(s) + (-b\xi)\gamma(s), \] (3.7)
where
\[ \frac{ds^*}{ds} = \frac{1}{b} \sqrt{a^2 - 2b^2 \kappa(s)}, \] (3.8)
\[ \xi = \frac{1}{\sqrt{a^2 - 2b^2 \kappa(s)b^2}}. \] (3.9)

It can be easily seen that the tangent vector \( \gamma_{x\alpha}'(s^*) = \alpha_{x\alpha}(s^*) \) is a unit spacelike vector. Differentiating (3.7), we obtain equation as follows
\[ \gamma_{x\alpha}''(s^*) = B_1 x(s) + B_2 \alpha(s) + B_3 \beta(s) + B_4 y(s), \] (3.10)
where
\[ B_1 = \frac{1}{w} (a\xi \kappa + bn\kappa' \xi + b\kappa \xi'), \quad B_2 = \frac{1}{w} ((a + b\kappa)\xi' + (\kappa' + \kappa)b \xi), \]
\[ B_3 = \frac{1}{w} (b\xi \tau), \quad B_4 = -\frac{1}{w} (a\xi + b\xi'). \]

\[ y_{x\alpha}(s^*) = -\gamma_{x\alpha}''' - \frac{1}{2} (\gamma_{x\alpha}'', \gamma_{x\alpha}'') \gamma_{x\alpha}. \] (3.11)

By the help of previous equation (3.11), we obtain
\[ y_{x\alpha}(s^*) = Y_1 x(s) + Y_2 \alpha(s) + Y_3 \beta(s) + Y_4 y(s), \] (3.12)
where \( Y_1 = -(B_1 + \frac{a}{2b} (2B_1 B_4 + B_2^2 + B_3^2)), Y_2 = -(B_2 + \frac{1}{2} (2B_1 B_4 + B_2^2 + B_3^2)), Y_3 = -B_3, Y_4 = -B_4. \)

ii) Using equations \( \kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2} (\gamma_{x\alpha}'', \gamma_{x\alpha}'') \) and \( \tau_{\gamma_{x\alpha}}^2(s^*) = (x'''' - \kappa' x, x'''' - \kappa x - \kappa' x) - \kappa_{\gamma_{x\alpha}}^2(s^*) \). The curvatures \( \kappa_{\gamma_{x\alpha}}(s^*) \) and \( \tau_{\gamma_{x\alpha}}(s^*) \) of the \( \gamma_{x\alpha} \) (s^*) are explicitly obtained by
\[ \kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2} (2B_1 B_4 + B_2^2 + B_3^2) \]
\[ \tau_{\gamma_{x\alpha}}^2(s^*) = 2(Y_1 - \kappa') Y_4 + (Y_2 - \kappa)^2 + Y_3^2 - \kappa_{\gamma_{x\alpha}}^2. \] (3.13)

Thus, the theorem is proved. \( \square \)
**Definition 3.3.** Let \( x \) be unit speed spacelike curve lying on \( \mathbb{Q}^3 \) with the moving asymptotic orthonormal frame \( \{ x, \alpha, \beta, y \} \). Then, \( xy \)-smarandache curve of \( x \) is defined by

\[
\gamma_{xy}(s^*) = \frac{1}{\sqrt{2ab}} \left( ax(s) + by(s) \right),
\]

(3.14)

where \( a, b \in \mathbb{R}^+_0 \).

**Theorem 3.4.** Let \( x \) be unit speed spacelike curve in \( \mathbb{Q}^3 \) with the moving asymptotic orthonormal frame \( \{ x, \alpha, \beta, y \} \) and cone curvature \( \kappa \) and let \( \gamma_{xy} \) be \( xy \)-smarandache curve with asymptotic orthonormal frame \( \{ \gamma_{xy}, \alpha_{xy}, \beta_{xy}, y_{xy} \} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{ \gamma_{xy}, \alpha_{xy}, \beta_{xy}, y_{xy} \} \) of the \( xy \)-smarandache curve \( \gamma_{xy} \) is given as

\[
\begin{bmatrix}
\gamma_{xy} \\
\alpha_{xy} \\
\beta_{xy} \\
y_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{a}{\sqrt{2ab}} & \eta_1 & \eta_2 & \frac{b}{\sqrt{2ab}} \\
0 & \eta_1 & \eta_2 & 0 \\
(w \eta_1 - b \tau) & \eta_1 & \eta_2 & \frac{-w \eta_1}{w} \\
(-aC & -\eta_1 & -\eta_2 & -aC)
\end{bmatrix} \begin{bmatrix}
x \\\n\alpha \\\n\beta \\\ny
\end{bmatrix},
\]

(3.15)

where

\[
\eta_1 = \frac{a - b \kappa}{w \sqrt{2ab}}, \quad \eta_2 = \frac{-b \tau}{w \sqrt{2ab}},
\]

\[
w = \frac{ds^*}{ds} = \sqrt{\frac{a}{2b} - \kappa + \frac{b}{2a} (\kappa^2 + \tau^2)},
\]

\[
C = \frac{1}{w^2} \left( -2 \eta_1 (\eta_1 \kappa + \eta_2 \tau) + (\eta_1')^2 + (\eta_2')^2 \right).
\]

ii) The cone curvature \( \kappa_{xy}(s^*) \) and \( \tau_{xy}(s^*) \) of the curve \( \gamma_{xy} \) is given by

\[
\kappa_{xy}(s^*) = -\frac{C}{2},
\]

\[
\tau_{xy}(s^*) = \frac{2 (\eta_1 \kappa + \eta_2 \tau + \frac{aC}{2 \sqrt{2ab}} - \kappa') (\frac{bC}{2 \sqrt{2ab}} - \eta_1) + (\eta_1')^2 + (\eta_2')^2 - \frac{C^2}{4}}{s^*},
\]

(3.16)

where

\[
s^* = \int \sqrt{\frac{a}{2b} - \kappa + \frac{b}{2a} (\kappa^2 + \tau^2)} ds.
\]

(3.17)

**Proof.** i) We assume that the curve \( x \) is a unit speed spacelike curve with the asymptotic orthonormal frame \( \{ x, \alpha, \beta, y \} \) and cone curvature \( \kappa, \tau \). Differentiating the equation (3.14) with respect to \( s \) and considering (2.1), we have

\[
\gamma_{xy}'(s^*) = \frac{(a - b \kappa(s))}{\sqrt{2ab}} \alpha(s) - \frac{b \tau(s)}{\sqrt{2ab}} \beta(s)
\]

(3.18)

or

\[
\gamma_{xy}'(s^*) = \eta_1 \alpha + \eta_2 \beta.
\]

(3.19)

By considering (3.17), we get

\[
\gamma_{xy}'(s^*) = \alpha(s) = \alpha_{xy}.
\]

(3.20)

Here, it can be easily seen that the tangent vector \( \overrightarrow{\alpha_{xy}} \) is a unit spacelike vector. Differentiating (3.19) and using (3.17), we obtain

\[
\gamma_{xy}''(s^*) = \left( \frac{\eta_1 \kappa + \eta_2 \tau}{w} \right) x(s) + \frac{\eta_1'}{w} \alpha + \frac{\eta_2'}{w} \beta - \frac{\eta_1}{w} y(s).
\]
By the help of equation $y_{xy}(s^*) = -\gamma''_{xy} - \frac{1}{2} \langle \gamma''_{xy}, \gamma''_{xy} \rangle \gamma_{xy}$, we write
\[ y_{xy}(s^*) = \left( -\frac{\eta_1 \kappa - \eta_2 \tau}{w} - \frac{aC}{2 \sqrt{2} ab} \right) x(s) - \frac{\eta_1}{w} \alpha + \frac{\eta_2}{w} \beta + \frac{aC}{2 \sqrt{2} ab} y(s). \] (3.21)

ii) \[
\kappa_{xy}(s^*) = -\frac{1}{2} \langle \gamma''_{xy}, \gamma''_{xy} \rangle.
\]
\[
\tau_{xy}^2(s^*) = (\beta - \kappa \alpha - \kappa' x, \beta - \kappa \alpha - \kappa' x) - \kappa_{xy}^2.
\] (3.22)

By using (3.22), the curvatures $\kappa_{xy}(s^*)$ and $\tau_{xy}(s^*)$ of the $\gamma_{xy}(s^*)$ are explicitly obtained
\[
\kappa_{xy}(s^*) = -\frac{1}{2} \langle \gamma''_{xy}, \gamma''_{xy} \rangle = \frac{M}{2},
\]
\[
\tau_{xy}^2(s^*) = 2(\eta_1 \kappa + \eta_2 \tau + \frac{aC}{2 \sqrt{2} ab} - \kappa') \left( \frac{bC}{2 \sqrt{2} ab} - \eta_1 \right) + (\eta_1^2 + \kappa^2) + \frac{C^2}{4}.
\]

**Definition 3.5.** Let $x$ be unit speed spacelike curve lying on $Q^3$ with the moving asymptotic orthonormal frame \{x, $\alpha, \beta, y$\}. Then, $\alpha y$-smarandache curve of $x$ is defined by
\[
\gamma_{\alpha y}(s^*) = \alpha(s) + \frac{b}{a} y(s),
\] (3.23)
where $a, b \in \mathbb{R}^+_U$.

**Theorem 3.6.** Let $x$ be unit speed spacelike curve in $Q^3$ with the moving asymptotic orthonormal frame \{x, $\alpha, \beta, y$\} and cone curvature $\kappa$ and let $\gamma_{\alpha y}$ be $\alpha y$-smarandache curve with asymptotic orthonormal frame \{\gamma_{\alpha y}, \alpha_{\alpha y}, \beta_{\alpha y}, y_{\alpha y}\}. Then the following relations hold:

i) The asymptotic orthonormal frame \{\gamma_{\alpha y}, \alpha_{\alpha y}, \beta_{\alpha y}, y_{\alpha y}\} of the $\alpha y$-smarandache curve $\gamma_{\alpha y}$ is given as
\[
\begin{bmatrix}
\gamma_{\alpha y} \\
\alpha_{\alpha y} \\
\beta_{\alpha y} \\
y_{\alpha y}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & \frac{b}{a} \\
\frac{\rho_1}{M} & \frac{\rho_2}{M} & \frac{\rho_3}{M} & \frac{\rho_4}{M} \\
-\frac{\rho_1 + \kappa \rho_2 + \rho_3 \tau}{M} & -\frac{\rho_2 + \kappa \rho_3 - \kappa \rho_4}{M} & -\frac{\rho_3 + \tau \rho_4}{M} & -\frac{-\rho_3 + \rho_4}{M} \\
-\frac{\rho_1 + \kappa \rho_2 + \rho_3 \tau}{M} & -\frac{\rho_2 + \kappa \rho_3 - \kappa \rho_4}{M} & -\frac{\rho_3 + \tau \rho_4}{M} & -\frac{-\rho_3 + \rho_4}{M}
\end{bmatrix} \begin{bmatrix}
x \\
\alpha \\
\beta \\
y
\end{bmatrix},
\] (3.24)
where
\[
\rho_1 = \frac{\kappa}{M}, \rho_2 = -\frac{b}{a} \left( \frac{\kappa}{M} \right), \rho_3 = -\frac{b}{a} \left( \frac{\tau}{M} \right), \rho_4 = \frac{1}{M},
\]
\[
M = \sqrt{\frac{b}{a^2} (\kappa^2 + \tau^2) - 2 \kappa}.
\] (3.25)
\[
D = \frac{2}{M^2} (\rho_1 + \kappa \rho_2 + \rho_3 \tau) (-\rho_3 + \rho_4) + \frac{1}{M^2} ((\rho_2 + \rho_3 - \kappa \rho_4)^2 + (\rho_2 - \tau \rho_3)^2).
\]

ii) The cone curvatures $\kappa_{\alpha y}(s^*)$ and $\tau_{\alpha y}(s^*)$ of the curve $\gamma_{\alpha y}$ is given by
\[
\kappa_{\alpha y}(s^*) = -\frac{D}{2},
\] (3.26)
\[
\tau_{\alpha y}^2(s^*) = 2\left( \frac{\rho_1^2 + \kappa \rho_2^2 + \rho_3^2 \tau}{M} - \kappa' \right) \left( -\frac{\rho_3}{M} + \frac{bD}{a} \right) + \left( \frac{\rho_2^2 + \rho_3^2 - \kappa \rho_4}{M} + \frac{D}{2} + \kappa \right)^2 + \left( -\frac{\rho_3^2 + \tau \rho_4}{M} - \frac{D^2}{4} \right).
\] (3.27)

where
\[
s^* = \int \sqrt{\frac{b}{a^2} (\kappa^2 + \tau^2) - 2 \kappa} ds.
\] (3.28)
Proof. i) Let the curve $x$ be a unit speed spacelike curve with the asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvature $\kappa, \tau$. Differentiating the equation (3.23) with respect to $s$ and considering (2.1), we find

\[
\gamma'_{\alpha y}(s^*) \frac{ds^*}{ds} = \kappa x(s) - \frac{b}{c} \kappa \alpha(s) - \frac{b}{a} \tau \beta(s) - y(s).
\]

This can be written as following

\[
\alpha_{\alpha y}(s^*) \frac{ds^*}{ds} = \frac{\kappa}{M} x(s) - \frac{b}{c} \kappa \alpha(s) - \frac{b}{a} \tau \beta(s) - \frac{1}{M} y(s),
\]

where

\[
M = \frac{ds^*}{ds} = \sqrt{\frac{b}{a^2} (\kappa^2 + \tau^2) - 2 \kappa}.
\]

Differentiating (3.29) and using (3.30), we get

\[
\gamma''_{\alpha y} = \left( \frac{\rho_1' + \kappa \rho_2 + \rho_2 \tau}{M} \right) x + \left( \frac{\rho_1' + \rho_1 - \kappa \rho_2}{M} \right) \alpha(s) + \left( \frac{\rho_1' - \tau \rho_2}{M} \right) \beta(s) + \left( \frac{- \rho_2 + \rho_2}{M} \right) y(s),
\]

where $\rho_1 = \frac{\kappa}{M}, \rho_2 = \frac{-b}{a} \left( \frac{\kappa}{M} \right), \rho_3 = \frac{-b}{a} \left( \frac{\tau}{M} \right), \rho_4 = \frac{1}{M}$.

By the help of equation (3.32), we obtain

\[
y_{\alpha y}(s^*) = -\gamma''_{\alpha y} - \frac{1}{2} \langle \gamma''_{\alpha y}, \gamma''_{\alpha y} \rangle \gamma_{\alpha y} \text{ and } \langle \gamma''_{\alpha y}, \gamma''_{\alpha y} \rangle = D.
\]

By the help of equation (3.32), we obtain

\[
y_{\alpha y}(s^*) = \left( \frac{- \rho_1' + \kappa \rho_2 + \rho_2 \tau}{M} \right) x(s) + \left( \frac{- \rho_1' + \rho_1 - \kappa \rho_2}{M} \right) \alpha(s) + \left( \frac{- \rho_2 - \rho_2}{M} \right) y(s),
\]

ii) Using (3.22), we have (3.26) and (3.27).

Definition 3.7. Let $x$ be unit speed spacelike curve lying on $Q^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$. Then, $x\beta$-smarandache curve of $x$ is defined by

\[
\gamma_{x\beta}(s^*) = \frac{a}{b} x(s) + \beta(s),
\]

where $a, b \in \mathbb{R}_0^+$.  

Theorem 3.8. Let $x$ be unit speed spacelike curve in $Q^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature $\kappa$ and let $\gamma_{x\beta}$ be $x\beta$-smarandache curve with asymptotic orthonormal frame $\{\gamma_{x\beta}, \alpha_{x\beta}, \beta_{x\beta}, y_{x\beta}\}$. Then the following relations hold:

i) The asymptotic orthonormal frame $\{\gamma_{x\beta}, \alpha_{x\beta}, \beta_{x\beta}, y_{x\beta}\}$ of the $x\beta$-smarandache curve $\gamma_{x\beta}$ is given as

\[
\begin{bmatrix}
\gamma_{x\beta} \\
\alpha_{x\beta} \\
\beta_{x\beta} \\
y_{x\beta}
\end{bmatrix} = \begin{bmatrix}
a & b & 0 & 1 & 0 \\
b & a & \tau & 0 & 0 \\
\frac{b}{a} \kappa & \left( \frac{b}{a} \right)^2 \tau' & \tau & 0 & -\frac{b}{a} \\
- \left( \frac{b}{a} \right)^3 \tau^2 & - \left( \frac{b}{a} \right)^3 \tau & M & \frac{b}{a} \\
\end{bmatrix} x,
\]

where

\[
M = -\frac{b^3}{2a} \kappa + \frac{b^2}{a^2} \tau' + \frac{b^3}{a^3} \kappa + \frac{b^3}{a^3} \tau'.
\]

ii) The cone curvatures $\kappa_{\gamma_{x\beta}}(s^*)$ and $\tau_{\gamma_{x\beta}}(s^*)$ of the curve $\gamma_{x\beta}$ is given by

\[
\kappa_{\gamma_{x\beta}}(s^*) = -\frac{b^2}{a^2} \left( \frac{b^2}{a^2} \tau^2 - 2 \kappa - 2 \frac{b}{a} \tau' \right)
\]

\[
\tau_{\gamma_{x\beta}}(s^*) = M^2 - 2 \frac{b}{a} \kappa' + \kappa^2 - \frac{b^2}{a^2} \kappa - \frac{b^4}{a^3} \kappa - 2 \frac{b^3}{a^2} \tau',
\]

where

\[
s^* = \frac{a}{b} s + A; a, b, A \in \mathbb{R}_0^+.
\]
Proof. i) Differentiating the equation (3.34) with respect to \( s \) and considering (2.1), we find

\[
\gamma'_{x\beta}(s^*) \frac{ds^*}{ds} = \frac{a}{b} \alpha(s) + \tau x(s).
\]  
(3.40)

This can be written as follows

\[
\alpha_{x\beta}(s^*) = \frac{b\tau}{a} \frac{\gamma'_{x\beta}}{x(s)} + \alpha(s),
\]  
(3.41)

where

\[
\frac{ds^*}{ds} = \frac{a}{b}.
\]  
(3.42)

Differentiating (3.41) and using (3.42), we get

\[
\gamma''_{x\beta}(s^*) = \left( \frac{b}{a} \kappa + \left( \frac{b}{a} \right)^2 \tau^2 \right) x(s) + \left( \tau \left( \frac{b}{a} \right)^2 \right) \alpha(s) - \frac{b}{a} y(s)
\]  
(3.43)

By the help of equation (3.43), we obtain

\[
y_{x\beta}(s^*) = -\gamma''_{x\beta} - \frac{1}{2} \langle \gamma''_{x\beta} \frac{s}{x_{\beta}} \rangle 
\]  
(3.44)

where \( M = -\frac{b^2}{a^2} \tau^2 + \frac{b^4}{a^4} \kappa + \frac{b^3}{a^3} \tau^2 \).

ii) Using (3.22), we have (3.36) and (3.37). \( \square \)

**Definition 3.9.** Let \( x \) be a unit speed spacelike curve lying on \( \mathbb{Q}^3 \) with the moving asymptotic orthonormal frame \( \{ x, \alpha, \beta, y \} \). Then, \( \alpha\beta \)-smarandache curve of \( x \) is defined by

\[
\gamma_{\alpha\beta}(s^*) = \frac{1}{\sqrt{a^2 + b^2}} \left( a\alpha(s) + b\beta(s) \right),
\]  
(3.45)

where \( a, b \in \mathbb{R}_0^+ \).

**Theorem 3.10.** Let \( x \) be a unit speed spacelike curve in \( \mathbb{Q}^3 \) with the moving asymptotic orthonormal frame \( \{ x, \alpha, \beta, y \} \) and cone curvature \( \kappa \) and \( \tau \) let \( \gamma_{\alpha\beta} \) be \( \alpha\beta \)-smarandache curve with asymptotic orthonormal frame \( \{ \gamma_{\alpha\beta}, \alpha_{\alpha\beta}, \beta_{\alpha\beta}, y_{\alpha\beta} \} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{ \gamma_{\alpha\beta}, \alpha_{\alpha\beta}, \beta_{\alpha\beta}, y_{\alpha\beta} \} \) of the \( \alpha\beta \)-smarandache curve \( \gamma_{\alpha\beta} \) is given as

\[
\begin{bmatrix}
\gamma_{\alpha\beta} \\
\alpha_{\alpha\beta} \\
\beta_{\alpha\beta} \\
y_{\alpha\beta}
\end{bmatrix}
= \begin{bmatrix}
0 & a & b & 0 \\
\frac{a}{\sqrt{a^2 + b^2}} & 0 & \frac{b}{\sqrt{a^2 + b^2}} & 0 \\
Y_1 & Y_1 - \kappa Y_2 & -\gamma Y_2 & Y_2 \\
Y_1 E & -Y_1 - \kappa Y_2 & \gamma Y_2 & -Y_2 E
\end{bmatrix}
\begin{bmatrix}
x \\
\alpha \\
\beta \\
y
\end{bmatrix},
\]  
(3.46)

where

\[
E = \frac{ds^*}{ds} = \sqrt{\frac{2}{a^2 + b^2} - \frac{a}{\kappa + b\tau}}.
\]  
(3.47)

\[
Y_1 = \frac{\kappa + b\tau}{E\sqrt{a^2 + b^2}}, Y_1 = \frac{-a}{E\sqrt{a^2 + b^2}}.
\]  
(3.48)

\[
L = \frac{1}{E^2} \left( 2Y_1 Y_2' + (Y_1 - \kappa Y_2)^2 + \tau^2 Y_2^2 \right).
\]  
(3.49)

ii) The cone curvatures \( \kappa_{\alpha\beta}(s^*) \) and \( \tau_{\alpha\beta}(s^*) \) of the curve \( \gamma_{\alpha\beta} \) is given by

\[
\kappa_{\alpha\beta}(s^*) = \frac{L}{2},
\]  
(3.49)
\[ \tau_{\gamma_{\alpha\beta}}^2(s^*) = 2\left(\frac{Y'_{\alpha}}{E} + \kappa'\right)(-\frac{Y'_{\beta}}{E}) + (\frac{\kappa Y_{\gamma} - Y_{\gamma}}{E} - \frac{a}{2\sqrt{a^2 + b^2}}L - \kappa)^2 \]
\[ + \left(\frac{\tau Y_{\gamma}}{E} - \frac{b}{2\sqrt{a^2 + b^2}}L \right)^2 - \frac{L^2}{4}, \]
(3.51)

where

\[ s^* = \int \sqrt{\frac{2}{a^2 + b^2}} \left| -a(ak + br) \right| ds. \]
(3.52)

Proof. i) Let the curve \( x \) be a unit speed spacelike curve with the asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvature \( \kappa, \tau \). Differentiating the equation (3.45) with respect to \( s \) and considering (2.1), we find

\[ \gamma_{\alpha\beta}(s^*) \frac{ds^*}{ds} = \frac{ak + br}{\sqrt{a^2 + b^2}} x(s) - \frac{a}{\sqrt{a^2 + b^2}} y(s), \]
(3.53)

where

\[ E = \frac{ds^*}{ds} = \sqrt{\frac{2}{a^2 + b^2}} \left| -a(ak + br) \right|. \]

Differentiating (3.53) and using (3.47), we get

\[ \gamma_{\alpha\beta}''(s^*) = \left(\frac{Y'_1}{E}\right) x(s) + (\frac{Y_1 - \kappa Y_{\gamma}}{E}) \alpha(s) + (\frac{-\tau Y_{\gamma}}{E}) \beta(s) + (\frac{Y'_{\gamma}}{E}) y(s), \]
(3.54)

where \( Y_1 = \frac{ak + br}{E\sqrt{a^2 + b^2}}, Y_{\gamma} = \frac{-a}{E\sqrt{a^2 + b^2}} \).

\[ y_{\alpha\beta}(s^*) = -\gamma_{\alpha\beta}'' - \frac{1}{2} \left( \gamma_{\alpha\beta}, \gamma_{\alpha\beta}'' \right) \gamma_{\alpha\beta}, \quad \text{and} \quad \left( \gamma_{\alpha\beta}, \gamma_{\alpha\beta}'' \right) = L. \]
(3.55)

By the help of equation (3.55), we obtain

\[ y_{\alpha\beta}(s^*) = \left(\frac{-Y'_1}{E}\right) x + (\frac{\kappa Y_{\gamma} - Y_1}{E}) \alpha - \frac{aL}{2\sqrt{a^2 + b^2}} \alpha, \]

\[ + \left(\frac{\tau Y_{\gamma}}{E} - \frac{bL}{2\sqrt{a^2 + b^2}}\right) \beta + (\frac{Y'_{\gamma}}{E}) y, \]
(3.56)

ii) Using (3.22), we have (3.50) and (3.51).
\[ \square \]

Definition 3.11. Let \( x \) be unit speed spacelike curve lying on \( Q_1 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \). Then, \( \beta y \)-smarandache curve of \( x \) is defined by

\[ \gamma_{\beta y}(s^*) = \beta(s) + \frac{b}{a} y(s), \]
(3.57)

where \( a, b \in \mathbb{R}_0^+ \).

Theorem 3.12. Let \( x \) be unit speed spacelike curve in \( Q_3 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, \beta, y\} \) and cone curvature \( \kappa \) and let \( \gamma_{\beta y} \) be \( \beta y \)-smarandache curve with asymptotic orthonormal frame \( \{\gamma_{\beta y}, \alpha_{\beta y}, \beta_{\beta y}, y_{\beta y}\} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{\gamma_{\beta y}, \alpha_{\beta y}, \beta_{\beta y}, y_{\beta y}\} \) of the \( \beta y \)-smarandache curve \( \gamma_{\beta y} \) is given as

\[ \begin{bmatrix} \gamma_{\beta y} \\ \alpha_{\beta y} \\ \beta_{\beta y} \\ y_{\beta y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{b}{a} \\ \frac{b\sqrt{\kappa^2 + \tau^2}}{\omega_1 + \omega_2 + \omega_3} & 0 & 0 & \frac{b}{a} \\ \frac{\sqrt{\kappa^2 + \tau^2}}{\omega_1 + \omega_2 + \omega_3} & -\frac{\sqrt{\kappa^2 + \tau^2}}{\omega_1 + \omega_2 + \omega_3} & 0 & \frac{b}{a} \\ \frac{\omega_1^2 + \omega_2^2 + \omega_3^2}{Z} & \frac{\omega_3^2}{Z} & -\frac{\omega_3}{Z} & 0 & \frac{bL}{aZ} \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ \beta \\ y \end{bmatrix}, \]
(3.58)

where

\[ Z = \frac{ds^*}{ds} = \frac{b}{a} \sqrt{\kappa^2 + \tau^2}, \]
(3.59)
\[ \omega_1 = \frac{a \tau}{b \sqrt{\kappa^2 + \tau^2}}, \omega_2 = -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}, \omega_3 = -\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \]  

(3.60)

**ii) The cone curvatures** \( \kappa_{\gamma_\beta y}(s^*) \) and \( \tau_{\gamma_\beta y}(s^*) \) of the curve \( \gamma_{\beta y} \) is given by

\[ \kappa_{\gamma_{\beta y}}(s^*) = -\frac{F}{2} \]  

(3.61)

\[ \tau_{\gamma_{\beta y}}(s^*) = 2 \left( \frac{\omega'_1 + \kappa \omega'_2 + \omega_1 \tau}{Z} \right) \left( \frac{bF}{2a} \frac{\omega_3}{Z} \right) + \left( \frac{\omega_1 + \omega'_3}{Z} + \kappa \right)^2 + \frac{F^2}{4}, \]  

(3.62)

where

\[ s^* = \frac{a}{b} \sqrt{\kappa^2 + \tau^2}; \quad a, b, \in \mathbb{R}_0^+. \]  

(3.63)

**Proof.** i) Differentiating the equation (3.57) with respect to \( s \) and considering (2.1), we find

\[ \gamma'_{\beta y}(s^*) \frac{ds^*}{ds} = \tau x(s) - \frac{b}{a} \alpha(s) - \frac{b}{a} \beta(s). \]  

(3.64)

This can be written as follows

\[ \alpha_{\beta y}(s^*) = \frac{a \tau}{b \sqrt{\kappa^2 + \tau^2}} \tau(s) - \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \alpha(s) - \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \beta(s), \]  

(3.65)

where

\[ \frac{ds^*}{ds} = \frac{b}{a} \sqrt{\kappa^2 + \tau^2}. \]  

(3.66)

Differentiating (3.65) and using (3.66), we get

\[ \gamma''_{\beta y}(s^*) = \left( \frac{\omega'_1 + \kappa \omega'_2 + \omega_1 \tau}{Z} \right) x(s) + \left( \frac{\omega_1 + \omega'_3}{Z} \right) \alpha(s) + \left( \frac{\omega_3}{Z} \right) \beta(s) + \left( \frac{-\omega_2}{Z} \right) y(s), \]  

(3.67)

By the help of equation (3.67), we obtain

\[ y_{\beta y}(s^*) = -\gamma''_{\beta y} \frac{1}{2} \gamma^\prime\prime_{\beta y} \gamma_{\beta y}. \]  

(3.68)

where \( Z = \frac{ds^*}{ds} = \frac{b}{a} \sqrt{\kappa^2 + \tau^2}. \)

ii) Using (3.22), we have (3.61) and (3.62).

\[ \square \]

**Example 3.13.** Let \( x \) be a spacelike curve in \( \mathbb{Q}^1 \) with arc length parameter \( s \) given by

\[ x(s) = (\sin s, \cos s, 0, 1). \]

Then we can write the smarandache curves of the \( x \)-curve as follows:

1) \( x\alpha \)-smarandache curve \( \gamma_{\alpha x} \) is given by \( \gamma_{\alpha x}(s) = \left( \frac{a}{\sqrt{ab}} \sin s + \cos s, \frac{a}{\sqrt{ab}} \cos s + \sin s, 0, \frac{a}{\sqrt{ab}} \right) \)

2) \( x\beta \)-smarandache curve \( \gamma_{x\beta} \) is given by \( \gamma_{x\beta}(s) = \left( \frac{a}{b} - 1 \right) \sin s, \left( \frac{a}{b} - 1 \right) \cos s, 0, \frac{a}{b} \)

3) \( x\gamma \)-smarandache curve \( \gamma_{x\gamma} \) is given by \( \gamma_{x\gamma}(s) = \left( \frac{a}{\sqrt{ab}} \sin s - \cos s, \frac{a}{\sqrt{ab}} \cos s + \sin s, 0, \frac{a}{\sqrt{ab}} \right) \)

4) \( \alpha y \)-smarandache curve \( \gamma_{\alpha y} \) is given by \( \gamma_{\alpha y}(s) = \left( (1 - \frac{a}{b}) \cos s, (1 - \frac{a}{b}) \sin s, 0, 0 \right) \)

5) \( \alpha \beta \)-smarandache curve \( \gamma_{\alpha \beta} \) is given by \( \gamma_{\alpha \beta}(s) = \frac{1}{\sqrt{ab}} (b \sin s - a \cos s, a \sin s - b \cos s, 0, 0) \)

6) \( \beta y \)-smarandache curve \( \gamma_{\beta y} \) is given by \( \gamma_{\beta y}(s) = \left( -\sin s - \frac{a}{b} \cos s, -\cos s + \frac{a}{b} \sin s, 0, 0 \right), \)

where \( a, b, \in \mathbb{R}_0^+. \)
References


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