On the mean value of the Smarandache LCM function

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Abstract For any positive integer \( n \), the F.Smarandache LCM function \( SL(n) \) is defined as the smallest positive integer \( k \) such that \( n \mid [1, 2, \cdots, k] \), where \([1, 2, \cdots, k]\) denotes the least common multiple of \( 1, 2, \cdots, k \), and let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) be the factorization of \( n \) into prime powers, then \( \Omega(n) = \sum_{i=1}^{s} \alpha_i p_i \). The main purpose of this paper is using the elementary methods to study the mean value properties of \( \Omega(n)SL(n) \), and give a sharper asymptotic formula for it.

Keywords F.Smarandache LCM function, \( \Omega(n) \) function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer \( n \), the famous F.Smarandache LCM function \( SL(n) \) defined as the smallest positive integer \( k \) such that \( n \mid [1, 2, \cdots, k] \), where \([1, 2, \cdots, k]\) denotes the least common multiple of \( 1, 2, \cdots, k \). For example, the first few values of \( SL(n) \) are
\[
SL(1) = 1, \quad SL(2) = 2, \quad SL(3) = 3, \quad SL(4) = 4, \quad SL(5) = 5, \quad SL(6) = 3, \quad SL(7) = 7, \quad SL(8) = 4, \quad SL(9) = 6, \quad SL(10) = 5, \quad SL(11) = 11, \quad SL(12) = 4, \quad SL(13) = 13, \quad SL(14) = 7, \quad SL(15) = 5, \cdots.
\]

About the elementary properties of \( SL(n) \), some authors had studied it, and obtained some interesting results, see reference [2] and [3]. For example, Lv Zhongtian [4] studied the mean value properties of \( SL(n) \), and proved that for any fixed positive integer \( k \) and any real number \( x > 1 \), we have the asymptotic formula
\[
\sum_{n \leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \frac{k}{\ln^2 x} + O \left( \frac{x^2}{\ln^{k+1} x} \right),
\]
where \( b_i \) (\( i = 2, 3, \cdots, k \)) are computable constants.

On the other hand, Chen Jianbin [5] studied the value distribution properties of \( SL(n) \), and proved that for any real number \( x > 1 \), we have the asymptotic formula
\[
\sum_{n \leq x} (SL(n) - P(n))^2 = \frac{2}{5} \cdot \zeta \left( \frac{5}{2} \right) \cdot \frac{x^2}{\ln x} + O \left( \frac{x^2}{\ln^2 x} \right),
\]
where \( \zeta(s) \) is the Riemann zeta-function, and \( P(n) \) denotes the largest prime divisor of \( n \).

Now we define a new arithmetical function \( \Omega(n) \) as follows: \( \Omega(1) = 0 \); for \( n > 1 \), let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) be the factorization of \( n \) into prime powers, then \( \Omega(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s \).
Obviously, for any positive integers \( m \) and \( n \), we have \( \Omega(mn) = \Omega(m) + \Omega(n) \). That is, \( \Omega(n) \) is the additive function. The main purpose of this paper is using the elementary methods to study the mean value properties of \( \Omega(n)SL(n) \), and give a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

**Theorem.** For any real number \( x > 1 \), we have the asymptotic formula

\[
\sum_{n \leq x} \Omega(n)SL(n) = \sum_{i=1}^{k} \frac{d_i x^3}{\ln^{i} x} + O \left( \frac{x^3}{\ln^{k+1} x} \right),
\]

where \( d_i (i = 1, 2, \ldots, k) \) are computable constants.

### §2. Proof of the theorem

In this section, we shall use the elementary methods to complete the proof of the theorem.

In fact, for any positive integer \( n > 1 \), let \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) be the factorization of \( n \) into prime powers, then from [2] we know that

\[
SL(n) = \max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_s^{\alpha_s}\},
\]

and we easily to know that

\[
\Omega(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s.
\]

Now we consider the summation

\[
\sum_{n \leq x} \Omega(n)SL(n).
\]

We separate all integer \( n \) in the interval \([1, x]\) into four subsets \( A, B, C \) and \( D \) as follows:

- \( A: p \geq \sqrt{n} \) and \( n = m \cdot p; \)
- \( B: n^{1/2} < p_1 < p_2 \leq \sqrt{n} \) and \( n = m \cdot p_1 \cdot p_2 \), where \( p_i (i = 1, 2) \) are primes;
- \( C: n^{1/2} < p \leq \sqrt{n} \) and \( n = m \cdot p^2; \)
- \( D: \) otherwise.

It is clear that if \( n \in A \), then from (1) we know that \( SL(n) = p \), and from (2) we know that \( \Omega(n) = \Omega(m) + p \). Therefore, by the Abel’s summation formula (See Theorem 4.2 of [6]) and the Prime Theorem (See Theorem 3.2 of [7]):

\[
\pi(x) = \sum_{i=1}^{k} \frac{a_i x}{\ln^{i} x} + O \left( \frac{x}{\ln^{k+1} x} \right),
\]

where \( a_i (i = 1, 2, \ldots, k) \) are computable constants and \( a_1 = 1 \).
We have

\[
\sum_{n \in A} \Omega(n) SL(n) = \sum_{m \leq \sqrt{x}} \sum_{m < p} \Omega(m) + p)p = \sum_{m \leq \sqrt{x}} \sum_{m < p} p^2 + \sum_{p \leq x} (\Omega(m)p)
\]

\[
= \sum_{m \leq \sqrt{x}} \sum_{m < p} p^2 + O(x^2)
\]

\[
= \sum_{m \leq \sqrt{x}} \left[ \pi \left( \frac{x}{m} \right) \frac{x^2}{m^2} - \pi(m)m^2 - 2 \int_{m}^{\sqrt{x}} \pi(t)dt \right] + O(x^2)
\]

\[
= \frac{1}{3} \zeta(3) \frac{x^3}{\ln x} + \frac{k}{\ln^k x} + O \left( \frac{x^3 \ln^{k+1} x}{\ln^{k+1} x} \right),
\]

(4)

where \( \zeta(s) \) is the Riemann zeta-function, and \( b_i \) (i = 2, 3, \cdots, k) are computable constants.

Similarly, if \( n \in B \), then we have \( SL(n) = p_2 \) and \( \Omega(n) = \Omega(m) + p_1 + p_2 \). So

\[
\sum_{n \in B} \Omega(n) SL(n) = \sum_{m_1 \leq x} \sum_{m_1 < p_1} \sum_{m_2 < p_2} \Omega(m_1) + (p_1 + p_2)p_2 = \sum_{m_1 \leq x} \sum_{m_1 < p_1} p_2^2 + O \left( \sum_{m_1 \leq x} \sum_{m_1 < p_1} p_1p_2 \right)
\]

\[
= \sum_{m_2 \leq \sqrt{x}} \sum_{m_2 < p_2} \left[ \pi \left( \frac{x}{p_1m_2} \right) \frac{x^2}{p_2^2 m_2^2} - \pi(p_1m_2)p_2^2 - 2 \int_{p_1}^{\sqrt{x}} \pi(p_1m_2) \pi(t)dt \right]
\]

\[
= \sum_{m_1 \leq x^{\frac{1}{2}}} \sum_{m_1 < p_1 \leq \sqrt{x}} \frac{c_1 x^3}{\ln^k x} + O \left( \frac{x^3 \ln^{k+1} x}{\ln^{k+1} x} \right),
\]

(5)

where \( c_i \) (i = 1, 2, \cdots, k) are computable constants.

Now we estimate the error terms in set C. Using the same method of proving (4), we have \( SL(n) = p^2 \) and \( \Omega(n) = \Omega(m) + 2p \), so

\[
\sum_{n \in C} \Omega(n) SL(n) = \sum_{m \leq x} \sum_{m < p} (\Omega(m) + 2p)p^2 = 2 \sum_{m \leq x} p^3 + \sum_{m \leq x} (\Omega(m)p^2)
\]

\[
= 2 \sum_{m \leq x} \sum_{m < p} p^3 + O(x^{\frac{3}{2}})
\]

\[
= O(x^3).
\]

(6)

Finally, we estimate the error terms in set D. For any integer \( n \in D \), if \( SL(n) = p \) then
$p \leq \sqrt{n}$; if $SL(n) = p^2$, then $p \leq n^{\frac{1}{3}}$; or $SL(n) = p^\alpha, \alpha \geq 3$. So we have

$$\sum_{n \in D} \Omega(n)SL(n) \ll \sum_{mp \leq x, p \leq m} (\Omega(m) + p)p + \sum_{mp^2 \leq x, p \leq m} (\Omega(m) + 2p)p$$

$$+ \sum_{mp^\alpha \leq x, p \leq \sqrt{x}, \alpha \geq 3} (\Omega(m)ap)p^\alpha \ll \frac{x^2}{\ln x}.$$  \hspace{1cm} (7)

Combining (4), (5), (6) and (7) we may immediately obtain the asymptotic formula

$$\sum_{n \leq x} \Omega(n)SL(n) = \sum_{n \in A} \Omega(n)SL(n) + \sum_{n \in B} \Omega(n)SL(n)$$

$$+ \sum_{n \in C} \Omega(n)SL(n) + \sum_{n \in D} \Omega(n)SL(n)$$

$$= \sum_{i=1}^{k} d_i x^3 \ln^i x + O \left( \frac{x^3}{\ln^{k+1} x} \right),$$

where $d_i (i = 1, 2, \cdots, k)$ are computable constants.

This completes the proof of Theorem.

References