On Algebraic Multi-Ring Spaces

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Abstract A Smarandache multi-space is a union of $n$ spaces $A_1, A_2, \cdots, A_n$ with some additional conditions hold. Combining these Smarandache multi-spaces with rings in classical ring theory, the conception of multi-ring spaces is introduced and some characteristics of multi-ring spaces are obtained in this paper.

Keywords Ring, multi-space, multi-ring space, ideal subspace chain.

§1. Introduction

These multi-spaces is introduced by Smarandache in [6] under an idea of hybrid mathematics: combining different fields into a unifying field ([7]), which can be formally defined with mathematical words by the next definition.

Definition 1.1. For any integer $i, 1 \leq i \leq n$ let $A_i$ be a set with ensemble of law $L_i$, denoted by $(A_i; L_i)$. Then the union of $(A_i; L_i), 1 \leq i \leq n$

$\tilde{A} = \bigcup_{i=1}^{n}(A_i; L_i),$

is called a multi-space.

As we known, a set $R$ with two binary operation “$+$” and “$\circ$”, denoted by $(R; +, \circ)$, is said to be a ring if for $\forall x, y \in R, x + y \in R, x \circ y \in R$, the following conditions hold.

(i) $(R; +)$ is an abelian group;
(ii) $(R; \circ)$ is a semigroup;
(iii) For $\forall x, y, z \in R, x \circ (y + z) = x \circ y + x \circ z$ and $(x + y) \circ z = x \circ z + y \circ z$.

By combining these Smarandache multi-spaces with rings in classical mathematics, a new kind of algebraic structure called multi-ring spaces is found, which are defined in the next definition.

Definition 1.2. Let $\tilde{R} = \bigcup_{i=1}^{m} R_i$ be a complete multi-space with a double binary operation set $O(\tilde{R}) = \{(+, \times_i), 1 \leq i \leq m\}$. If for any integers $i, j, i \neq j, 1 \leq i, j \leq m, (R_i; +, \times_i)$ is a ring and for $\forall x, y, z \in \tilde{R},$

$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$

and

$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$
provided all these operation results exist, then $\tilde{R}$ is called a multi-ring space. If for any integer $1 \leq i \leq m$, $(R_i; +_i, \times_i)$ is a field, then $\tilde{R}$ is called a multi-field space.

For a multi-ring space $\tilde{R} = \bigcup_{i=1}^m R_i$, let $\tilde{S} \subset \tilde{R}$ and $O(\tilde{S}) \subset O(\tilde{R})$, if $\tilde{S}$ is also a multi-ring space with a double binary operation set $O(\tilde{S})$, then $\tilde{S}$ is said a multi-ring subspace of $\tilde{R}$.

The main object of this paper is to find some characteristics of multi-ring spaces. For terminology and notation not defined here can be seen in [1], [5], [12] for rings and [2], [6] – [11] for multi-spaces and logics.

§2. Characteristics of multi-ring spaces

First, we get a simple criterions for multi-ring subspaces of a multi-ring space.

Theorem 2.1. For a multi-ring space $\tilde{R} = \bigcup_{i=1}^m R_i$, a subset $\tilde{S} \subset \tilde{R}$ with a double binary operation set $O(\tilde{S}) \subset O(\tilde{R})$ is a multi-ring subspace of $\tilde{R}$ if and only if for any integer $k, 1 \leq k \leq m$, $(\tilde{S} \cap R_k; +_k, \times_k)$ is a subring of $(R_k; +_k, \times_k)$ or $\tilde{S} \cap R_k = \emptyset$.

Proof. For any integer $k, 1 \leq k \leq m$, if $(\tilde{S} \cap R_k; +_k, \times_k)$ is a subring of $(R_k; +_k, \times_k)$ or $\tilde{S} \cap R_k = \emptyset$, then since $\tilde{S} = \bigcup_{i=1}^m (\tilde{S} \cap R_i)$, we know that $\tilde{S}$ is a multi-ring subspace by definition of multi-ring spaces.

Now if $\tilde{S} = \bigcup_{j=1}^s S_{ij}$ is a multi-ring subspace of $\tilde{R}$ with a double binary operation set $O(\tilde{S}) = \{(+_j, \times_j), 1 \leq j \leq s\}$, then $(S_{ij}; +_l, \times_l)$ is a subring of $(R_l; +_l, \times_l)$. Therefore, for any integer $j, 1 \leq j \leq s, S_{ij} = R_i \cap \tilde{S}$. But for any integer $l \in \{i; 1 \leq i \leq m\} \setminus \{ij; 1 \leq j \leq s\}$, $\tilde{S} \cap S_l = \emptyset$.

Applying a criterion for subrings of a ring, we get the following result.

Theorem 2.2. For a multi-ring space $\tilde{R} = \bigcup_{i=1}^m R_i$, a subset $\tilde{S} \subset \tilde{R}$ with a double binary operation set $O(\tilde{S}) \subset O(\tilde{R})$ is a multi-ring subspace of $\tilde{R}$ if and only if for any double binary operations $(+_j, \times_j) \in O(\tilde{S}), (\tilde{S} \cap R_j; +_j) \subset (R_j; +_j)$ and $(\tilde{S}; \times_j)$ is complete.

Proof. According to Theorem 2.1, we know that $\tilde{S}$ is a multi-ring subspace if and only if for any integer $i, 1 \leq i \leq m$, $(\tilde{S} \cap R_i; +_i, \times_i)$ is a subring of $(R_i; +_i, \times_i)$ or $\tilde{S} \cap R_i = \emptyset$. By a well known criterion for subrings of a ring (see also [5]), we know that $(\tilde{S} \cap R_i; +_i, \times_i)$ is a subring of $(R_i; +_i, \times_i)$ if and only if for any double binary operations $(+_j, \times_j) \in O(\tilde{S}), (\tilde{S} \cap R_j; +_j) \subset (R_j; +_j)$ and $(\tilde{S}; \times_j)$ is a complete set. This completes the proof.

We use these ideal subspace chains of a multi-ring space to characteristic its structure properties. An ideal subspace $\tilde{I}$ of a multi-ring space $\tilde{R} = \bigcup_{i=1}^m R_i$ with a double binary operation set $O(\tilde{R})$ is a multi-ring subspace of $\tilde{R}$ satisfying the following conditions:

(i) $\tilde{I}$ is a multi-group subspace with an operation set $\{+, \times\} \subset O(\tilde{I})$;

(ii) for any $r \in \tilde{R}, a \in \tilde{I}$ and $(+, \times) \in O(\tilde{I}), r \times a \in \tilde{I}$ and $a \times r \in \tilde{I}$ provided these operation results exist.

Theorem 2.3. A subset $\tilde{I}$ with $O(\tilde{I}), O(\tilde{I}) \subset O(\tilde{R})$ of a multi-ring space $\tilde{R} = \bigcup_{i=1}^m R_i$ with a double binary operation set $O(\tilde{R}) = \{(+_i, \times_i) | 1 \leq i \leq m\}$ is a multi-ideal subspace if and only
if for any integer $i, 1 \leq i \leq m$, $(\bar{I} \cap R_i, +, \times_i)$ is an ideal of the ring $(R_i, +, \times_i)$ or $\bar{I} \cap R_i = \emptyset$.

**Proof.** By definition of an ideal subspace, the necessity of conditions is obvious.

For the sufficiency, denote by $\bar{R}(+, \times)$ the set of elements in $\bar{R}$ with binary operations “+” and “×”. If there exists an integer $i$ such that $\bar{I} \cap R_i \neq \emptyset$ and $(\bar{I} \cap R_i, +, \times_i)$ is an ideal of $(R_i, +, \times_i)$, then for all $a \in \bar{I} \cap R_i$, $\forall r_i \in R_i$, we know that

$$r_i \times_i a \in \bar{I} \cap R_i; \quad a \times_i r_i \in \bar{I} \cap R_i.$$  

Notice that $\bar{R}(+, \times_i) = R_i$. Therefore, we get that for all $r \in \bar{R}$,

$$r \times_i a \in \bar{I} \cap R_i; \quad \text{and } a \times_i r \in \bar{I} \cap R_i$$

provided these operation results exist. Whence, $\bar{I}$ is an ideal subspace of $\bar{R}$.

An ideal subspace $\bar{I}$ of a multi-ring space $\bar{R}$ is maximal if for any ideal subspace $\bar{I}'$, if $\bar{R} \supseteq \bar{I}' \supseteq \bar{I}$, then $\bar{I}' = \bar{R}$ or $\bar{I}' = \bar{I}$. For any order of these double binary operations in $O(\bar{R})$ of a multi-ring space $\tilde{R} = \bigcup_{i=1}^{m} R_i$, not loss of generality, assume it being $(+, x_1) \succ (+, x_2) \succ \cdots \succ (+, x_m)$, we can construct an *ideal subspace chain* of $\bar{R}$ by the following programming.

(i) Construct an ideal subspace chain

$$\bar{R} \supset \bar{R}_{11} \supset \bar{R}_{12} \supset \cdots \supset \bar{R}_{1s_1}$$

under the double binary operation $(+, x_1)$, where $\bar{R}_{11}$ is a maximal ideal subspace of $\bar{R}$ and in general, for any integer $i, 1 \leq i \leq m - 1$, $\bar{R}_{1(i+1)}$ is a maximal ideal subspace of $\bar{R}_{1i}$.

(ii) If the ideal subspace

$$\bar{R} \supset \bar{R}_{11} \supset \bar{R}_{12} \supset \cdots \supset \bar{R}_{1s_1} \supset \cdots \supset \bar{R}_{i1} \supset \cdots \supset \bar{R}_{is_i}$$

has been constructed for $(+, x_1) \succ (+, x_2) \succ \cdots \succ (+, x_i), 1 \leq i \leq m - 1$, then construct an ideal subspace chain of $\bar{R}_{is_i}$,

$$\bar{R}_{is_i} \supset \bar{R}_{(i+1)1} \supset \bar{R}_{(i+1)2} \supset \cdots \supset \bar{R}_{(i+1)s_{i}}$$

under the operations $(+, x_{i+1})$, where $\bar{R}_{(i+1)1}$ is a maximal ideal subspace of $\bar{R}_{is_i}$ and in general, $\bar{R}_{(i+1)(j+1)}$ is a maximal ideal subspace of $\bar{R}_{(i+1)j}$ for any integer $j, 1 \leq j \leq s_i - 1$. Define an ideal subspace chain of $\bar{R}$ under $(+, x_1) \succ (+, x_2) \succ \cdots \succ (+, x_{i+1}, x_{i+1})$ being

$$\bar{R} \supset \bar{R}_{11} \supset \cdots \supset \bar{R}_{1s_1} \supset \cdots \supset \bar{R}_{i1} \supset \cdots \supset \bar{R}_{is_i} \supset \bar{R}_{(i+1)1} \supset \cdots \supset \bar{R}_{(i+1)s_{i+1}}.$$

Similar to a multi-group space ([3]), we get the following result for ideal subspace chains of multi-ring spaces.

**Theorem 2.4.** For a multi-ring space $\bar{R} = \bigcup_{i=1}^{m} R_i$, its ideal subspace chain only has finite terms if and only if for any integer $i, 1 \leq i \leq m$, the ideal chain of the ring $(R_i; +, \times_i)$ has finite terms, i.e., each ring $(R_i; +, \times_i)$ is an Artin ring.

**Proof.** Let the order of double operations in $O(\bar{R})$ be
$(+1, \times_1) \succ (+2, \times_2) \succ \cdots \succ (+m, \times_m)$

and a maximal ideal chain in the ring $(R_1; +_1, \times_1)$ is

$$R_1 \succ R_{11} \succ \cdots \succ R_{1t_1}.$$ 

Calculation shows that

$$\tilde{R}_{11} = \tilde{R} \setminus \{ R_1 \setminus R_{11} \} = R_{11} \bigcup_{i=2}^{m} R_i,$$

$$\tilde{R}_{12} = \tilde{R}_{11} \setminus \{ R_{11} \setminus R_{12} \} = R_{12} \bigcup_{i=2}^{m} R_i,$$

$$\dotsc$$

$$\tilde{R}_{1t_1} = \tilde{R}_{1t_1} \setminus \{ R_{t_1; t_1-1} \setminus R_{1t_1} \} = R_{1t_1} \bigcup_{i=2}^{m} R_i.$$

According to Theorem 3.10, we know that

$$\tilde{R} \supset \tilde{R}_{11} \supset \tilde{R}_{12} \supset \cdots \supset \tilde{R}_{1t_1}$$

is a maximal ideal subspace chain of $\tilde{R}$ under the double binary operation $(+_1, \times_1)$. In general, for any integer $i$, $1 \leq i \leq m - 1$, assume

$$R_i \succ R_{1i} \succ \cdots \succ R_{iti}$$

is a maximal ideal chain in the ring $(R_{i; t_{i-1}}, +_i, \times_i)$. Calculate

$$\tilde{R}_{ik} = R_{ik} \bigcup_{j=i+1}^{m} \tilde{R}_{ik} \cap R_i$$

Then we know that

$$\tilde{R}_{(i-1)t_{i-1}} \supset \tilde{R}_{1i} \supset \tilde{R}_{2i} \supset \cdots \supset \tilde{R}_{iti}$$

is a maximal ideal subspace chain of $\tilde{R}_{(i-1)t_{i-1}}$ under the double operation $(+_i, \times_i)$ by Theorem 2.3. Whence, if for any integer $i$, $1 \leq i \leq m$, the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms, then the ideal subspace chain of the multi-ring space $\tilde{R}$ only has finite terms. On the other hand, if there exists one integer $i_0$ such that the ideal chain of the ring $(R_{i_0}; +_{i_0}, \times_{i_0})$ has infinite terms, then there must be infinite terms in the ideal subspace chain of the multi-ring space $\tilde{R}$.

A multi-ring space is called an Artin multi-ring space if each ideal subspace chain only has finite terms. We have consequence by Theorem 3.11.
Corollary 2.1. A multi-ring space $\tilde{R} = \bigcup_{i=1}^{m} R_i$ with a double binary operation set $O(\tilde{R}) = \{(+, \times_i) | 1 \leq i \leq m\}$ is an Artin multi-ring space if and only if for any integer $i, 1 \leq i \leq m$, the ring $(R_i; +, \times_i)$ is an Artin ring.

For a multi-ring space $\tilde{R} = \bigcup_{i=1}^{m} R_i$ with a double binary operation set $O(\tilde{R}) = \{(+, \times) | 1 \leq i \leq m\}$, an element $e$ is an idempotent element if $e^2 = e \times e = e$ for a double binary operation $(+, \times) \in O(\tilde{R})$. We define the directed sum $\tilde{I}$ of two ideal subspaces $\tilde{I}_1$ and $\tilde{I}_2$ as follows:

(i) $\tilde{I} = \tilde{I}_1 \cup \tilde{I}_2$;
(ii) $\tilde{I}_1 \cap \tilde{I}_2 = \{0_+\}$, or $\tilde{I}_1 \cap \tilde{I}_2 = \emptyset$, where $0_+$ denotes an unit element under the operation $+$.  

Denote the directed sum of $\tilde{I}_1$ and $\tilde{I}_2$ by

$$\tilde{I} = \tilde{I}_1 \bigoplus \tilde{I}_2.$$ 

If for any $\tilde{I}_1, \tilde{I}_2$, $\tilde{I} = \tilde{I}_1 \bigoplus \tilde{I}_2$ implies that $\tilde{I}_1 = \tilde{I}$ or $\tilde{I}_2 = \tilde{I}$, then $\tilde{I}$ is said to be non-reducible. We get the following result for these Artin multi-ring spaces, which is similar to a well-known result for these Artin rings (see [12]).

Theorem 2.5. Any Artin multi-ring space $\tilde{R} = \bigcup_{i=1}^{m} R_i$ with a double binary operation set $O(\tilde{R}) = \{(+, \times_i) | 1 \leq i \leq m\}$ is a directed sum of finite non-reducible ideal subspaces, and if for any integer $i, 1 \leq i \leq m$, $(R_i; +, \times_i)$ has unit $1_{x_i}$, then

$$\tilde{R} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i),$$

where $e_{ij}, 1 \leq j \leq s_i$, are orthogonal idempotent elements of the ring $R_i$.

Proof. Denote by $\tilde{M}$ the set of ideal subspaces which can not be represented by a directed sum of finite ideal subspaces in $\tilde{R}$. According to Theorem 2.4, there is a minimal ideal subspace $\tilde{I}_0$ in $\tilde{M}$. It is obvious that $\tilde{I}_0$ is reducible.

Assume that $\tilde{I}_0 = \tilde{I}_1 + \tilde{I}_2$. Then $\tilde{I}_1 \notin \tilde{M}$ and $\tilde{I}_2 \notin \tilde{M}$. Therefore, $\tilde{I}_1$ and $\tilde{I}_2$ can be represented by directed sums of finite ideal subspaces. Whence, $\tilde{I}_0$ can be also represented by a directed sum of finite ideal subspaces. Contradicts that $\tilde{I}_0 \in \tilde{M}$.

Now let

$$\tilde{R} = \bigoplus_{i=1}^{s} \tilde{I}_i,$$

where each $\tilde{I}_i, 1 \leq i \leq s$, is non-reducible. Notice that for a double operation $(+, \times)$, each non-reducible ideal subspace of $\tilde{R}$ has the form

$$(e \times R(\times)) \bigcup (R(\times) \times e), \ e \in R(\times).$$

Whence, we know that there is a set $T \subset \tilde{R}$ such that

$$\tilde{R} = \bigoplus_{e \in T, x \in O(\tilde{R})} (e \times R(\times)) \bigcup (R(\times) \times e).$$
For any operation $\times \in O(\tilde{R})$ and a unit $1_\times$, assume that

$$1_\times = e_1 \oplus e_2 \oplus \cdots \oplus e_i, \ e_i \in T, \ 1 \leq i \leq s.$$ 

Then

$$e_i \times 1_\times = (e_i \times e_1) \oplus (e_i \times e_2) \oplus \cdots \oplus (e_i \times e_i).$$

Therefore, we get that

$$e_i = e_i \times e_i = e_i^2 \quad \text{and} \quad e_i \times e_j = 0 \quad \text{for} \ i \neq j.$$ 

That is, $e_i, 1 \leq i \leq l$, are orthogonal idempotent elements of $\tilde{R}(\times)$. Notice that $\tilde{R}(\times) = R_h$ for some integer $h$. We know that $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of the ring $(R_h, +, \times)$. Denote by $e_{hj}$ for $e_j, 1 \leq j \leq l$. Consider all units in $\tilde{R}$, we get that

$$\tilde{R} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i).$$

This completes the proof.

**Corollary 2.2** ([12]) Any Artin ring $(R; +, \times)$ is a directed sum of finite ideals, and if $(R; +, \times)$ has unit $1_\times$, then

$$R = \bigoplus_{i=1}^{s} R_i e_i,$$

where $e_i, 1 \leq i \leq s$ are orthogonal idempotent elements of the ring $(R; +, \times)$.

§3. Open problems for a multi-ring space

Similar to Artin multi-ring spaces, we can also define Noether multi-ring spaces, simple multi-ring spaces, half-simple multi-ring spaces, etc.. Open problems for these new algebraic structures are as follows.

**Problem 3.1.** Call a ring $R$ a Noether ring if its every ideal chain only has finite terms. Similarly, for a multi-ring space $\tilde{R}$, if its every ideal multi-ring subspace chain only has finite terms, it is called a Noether multi-ring space. Whether can we find its structures similar to Corollary 2.2 and Theorem 2.5?

**Problem 3.2.** Similar to ring theory, define a Jacobson or Brown-McCoy radical for multi-ring spaces and determine their contribution to multi-ring spaces.

References