n-FOLD FILTERS IN SMARANDACHE RESIDUATED LATTICES, PART (II)

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In this paper we introduce the notions of n-fold BL-Smarandache n-fold BL-Smarandache fantastic filter and n-fold BL-Smarandache easy filter in Smarandache residuated lattices and study the relations among them. And we also introduce the notions of n-fold Smarandache n-fold Smarandache fantastic BL-residuated lattice and n-fold Smarandache easy BL-residuated lattice and investigate its properties.

Keywords: Smarandache residuated lattice, n-fold Smarandache easy filter, n-fold Smarandache fantastic filter, n-fold Smarandache easy residuated lattice, n-fold Smarandache fantastic residuated lattice.

MSC2010: 3B47, 03G25, 06D99.

1. Introduction and Preliminaries

BL-algebras (basic logic algebras) are the algebraic structures for Hájek basic logic [5], in order to investigate many valued logic by algebraic means. Residuated lattices play an important role in the study of fuzzy logic and filters are basic concepts in residuated lattices and other algebraic structures.

A Smarandache structure on a set L means a weak structure W on L such that there exists a proper subset B of L which is embedded with a strong structure S. In [12], Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids and strong Bol groupoids and obtained many interesting results about them. It will be very interesting to study the Smarandache structure in these algebraic structures. A BL-algebra is a weaker structure than residuated lattice, then we can consider in any residuated lattice a weaker structure as BL-algebra. Smarandache BL-algebra and Smarandache (implicative) ideals in BL-algebra are defined in [?] and the concepts of bi-Smarandache BL-algebra, bi-weak Smarandache BL-algebra, bi − Q-Smarandache ideal and bi − Q-Smarandache implicative filter are defined in [1]. The concept of Smarandache residuated lattice, Smarandache (positive) implicative filters and Smarandache fantastic filters defined in [2]. In [6, 10] the authors defined the notion of n-fold (positive) implicative filters, n-fold fantastic filters, n-fold obstinate filters in BL-algebras and studied the relation among many type of n-fold filters in BL-algebra. The aim of this paper is to extend this research to Smarandache residuated lattices.

[4] A residuated lattice is an algebra L = (L, ∧, ∨, ⊕, →, 0, 1) of type (2, 2, 2, 2, 0, 0) equipped with an order ≤ satisfying the following:

(LR₁) (L, ∧, ∨, 0, 1) is a bounded lattice,
(LR₂) (L, ⊕, 1) is a commutative ordered monoid,
(LR₃) ⊕ and → form an adjoint pair i.e, c ≤ a → b if and only if a ⊕ c ≤ b for all a, b, c ∈ L.

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A BL-algebra is a residuated lattice $L$ if satisfying the following identity, for all $a, b \in L$:

\[(BL_1) \quad (a \to b) \lor (b \to a) = 1,\]
\[(BL_2) \quad a \wedge b = a \circ (a \to b).\]

**Theorem 1.1.** [4, 7, 8, 9] Let $L$ be a residuated lattice. Then the following properties hold, for all $x, y, z \in L$:

1. $(Lr_1)$ $1 \to x = x, \; x \to x = 1$,
2. $(Lr_2)$ $x \to y \leq (z \to x) \to (z \to y) \leq z \to (x \to y),$
3. $(Lr_3)$ $x \to y \leq (y \to z) \to (x \to z)$ and $(x \to y) \circ (y \to z) \leq x \to z,$
4. $(Lr_4)$ $x \leq y \iff x \to y = 1, \; x \leq y \to x$,
5. $(Lr_5)$ $x \to (y \to z) = y \to (x \to z) = (x \circ y) \to z$,
6. $(Lr_6)$ $x \circ (x \to y) \leq y, x \leq y \to (x \circ y)$ and $y \leq (y \to x) \to x$,
7. $(Lr_7)$ If $x \leq y$, then $y \to z \leq x \to z, \; z \to x \leq z \to y$ and $y^* \leq x^*,$
8. $(Lr_8)$ $x \leq y$ and $z \leq w$ then $x \circ z \leq y \circ w,$
9. $(Lr_9)$ $x \leq x^{**}, \; x^{***} = x^*,$
10. $(Lr_{10}) \quad x^* \circ y^* \leq (x \circ y)^* \text{ (so, } (x^*)^n \leq (x^n)^* \text{ for every } n \geq 1),$\hspace{1cm}
11. $(Lr_{11}) \quad x^{**} \circ y^{**} \leq (x \circ y)^{**} \text{ (so, } (x^{**})^n \leq (x^n)^{**} \text{ for every } n \geq 1)$.

The following definitions and theorems are stated from [2].

A Smarandache $BL$-residuated lattice is a residuated lattice $L$ in which there exists a proper subset $B$ of $L$ such that $0, 1 \in B, \; |B| > 2$ and $B$ is a $BL$-algebra under the operations of $L$.

From now on $L_B = (L, \wedge, \vee, \circ, \to, 0, 1)$ is a Smarandache $BL$-residuated lattice and $B = (B, \wedge, \vee, \circ, \to, 0, 1)$ is a $BL$-algebra unless otherwise specified. A nonempty subset $F$ of $L_B$ is called a $BL$-Smarandache deductive system of $L_B$ if $1 \in F$ and if $x \in F, \; y \in B, \; x \to y \in F$ imply $y \in F$. A nonempty subset $F$ of $L_B$ is called a $BL$-Smarandache filter of $L_B$ if $x, y \in F$ imply $x \circ y \in F$ and if $x \in F, \; y \in B$ and $x \leq y$ imply $y \in F$.

**Theorem 1.2.** (i) Let $F$ be a $BL$-Smarandache filter of $L_B$, then $F$ is a $BL$-Smarandache deductive system of $L_B$.
(ii) Let $F$ be a $BL$-Smarandache deductive system of $L_B$. If $F \subseteq B$, then $F$ is a $BL$-Smarandache filter of $L_B$.

A filter $F$ of a residuated lattice $L$ is called an easy filter if $x^{**} \to (y \to z) \in F$ and $x^{**} \to y \in F$ imply $x^{**} \to z \in F$, for all $x, y, z \in L$, [3].

**Definition 1.1.** [11] Let $F$ be a subset of Smarandache $BL$-residuated lattice $L_B$ and $1 \in F$.

1. $F$ is called an $n$-fold $BL$-Smarandache positive implicatve filter of $L_B$ if $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, then $x^n \to z \in F$, for all $x, y, z \in B$.
2. $F$ is called an $n$-fold $BL$-Smarandache positive implicatve filter of $L_B$ if $x \in F$ and $x \to (y^n \to z) \to y \in F$ then $y \in F$, for all $y, z \in B$.

**Definition 1.2.** [11] Let $L_B$ be a Smarandache $BL$-residuated lattice.

1. $L_B$ is called an $n$-fold Smarandache positive implicatve $BL$-residuated lattice if $x^{n+1} = x^n$ for all $x \in B$.
2. $L_B$ is called an $n$-fold Smarandache implicatve $BL$-residuated lattice if $(x^n)^* \to x = x$, for each $x \in B$.

**Theorem 1.3.** [11] Let $F$ be a $BL$-Smarandache deductive system of $L_B$. Then $F$ is an $n$-fold $BL$-Smarandache positive implicatve filter of $L_B$ if and only if $x^n \to x^{2n} \in F$, for all $x \in B$ and $n \in N$. 
Theorem 1.4. [11] Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. Then $L_B$ is an $n$-fold Smarandache positive implicative $B_L$-residuated lattice if and only if every $B_L$-Smarandache deductive system of $L_B$ is an $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$.

Theorem 1.5. [11] Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. Then $F$ is an $n$-fold $B_L$-Smarandache implicative filter if and only if $(x^n \to y) \to x \in F$ implies $x \in F$, for all $x, y \in B$.

Theorem 1.6. [11] $L_B$ is an $n$-fold Smarandache implicative $B_L$-residuated lattice if and only if every $B_L$-Smarandache deductive system $F$ of $L_B$ is an $n$-fold $B_L$-Smarandache implicative filter of $L_B$.

Theorem 1.7. [11] Let every $\text{Smarandache deductive system be a Smarandache filter. Then}$ every $\text{n-fold Smarandache implicative filter is n-fold Smarandache positive implicative filter.}$

Now, unless mentioned otherwise, $n \geq 1$ will be an integer.

2. $n$-Fold $B_L$-Smarandache Fantastic Filters

Definition 2.1. A subset $F$ of $L_B$ is called an $n$-fold Smarandache fantastic filter of $L_B$ related to $B$ (or briefly $n$-fold $B_L$-Smarandache fantastic filter of $L_B$) if $1 \in F$ and $z, z \to (y \to x) \in F$ imply $(x^n \to y) \to y \to x \in F$, for all $x, y \in B$.

Example 2.1. Let $L = \{0, a, b, c, d, 1\}$ be a residuated lattice such that $0 < a < c < d < 1$ and $0 < b < c < d < 1$. We define

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We can see that $L = (L, \wedge, \vee, \circ, \to, 0, 1)$ is a residuated lattice, in which $B = \{0, a, c, 1\}$ is a $B_L$-algebra which properly contained in $L$. Then $L$ is a $\text{Smarandache } B_L\text{-residuated lattice.}$

$F = \{d, 1\}$ is an $n$-fold $B_L$-Smarandache fantastic filter of $L_B$, for $n \geq 2$, while it is not $1$-fold $B_L$-Smarandache fantastic filter. Since $a, 0 \in B$ and $d, d \to (0 \to a) = 1 \in F$ but $(a \to 0) \to 0 \to a = c \notin F$.

Theorem 2.1. Every $n$-fold $B_L$-Smarandache fantastic filter of $L_B$ is a $B_L$-Smarandache deductive system of $L_B$.

Proof. Let $F$ be an $n$-fold $B_L$-Smarandache fantastic filter and $y, y \to x \in F$. We have $y \to x = y \to (1 \to x) \in F$. Then by Definition 2.1 we get $(x^n \to 1) \to x \in F$, i.e. $x \in F$. □

Theorem 2.2. Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. Then $F$ is an $n$-fold $B_L$-Smarandache fantastic filter if and only if $y \to x \in F$ implies $(x^n \to y) \to y \to x \in F$, for all $x, y \in B$.

Proof. The proof by the Definition 2.1, is clear. □

Proposition 2.1. Any $n$-fold $B_L$-Smarandache fantastic filter is an $(n+1)$-fold $B_L$-Smarandache fantastic filter.
Proposition 2.2. $n$-fold Smarandache implicative filters are $n$-fold Smarandache fantastic filters.

Proof. Assume that $F$ is an $n$-fold $B_L$-Smarandache implicative filter. Let $y \to x \in F$, where $x, y \in B$. By Theorem 1.1, we have $x \leq (((x^n \to y) \to y) \to y)$. Then $x^n \leq (((x^n \to y) \to y) \to y)$, hence $(x^n \to y) \geq (((x^n \to y) \to y) \to y)$. By (I) we get $y \to x \leq (((x^n \to y) \to y) \to y)$. Then by (I) we get $((x^n \to y) \to y) \leq y \to x$. Therefore $((x^n \to y) \to y) \leq y \to x$. Hence $(x^n \to y) \to y \leq y \to x$. By Theorem 1.1, we get $(x^n \to y) \to y \leq y \to x$. Therefore $(x^n \to y) \to y$ is an $n$-fold $B_L$-Smarandache fantastic filter.

Example 2.2. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $\odot$ and $\to$ as follows:

\[
\begin{array}{c|cccc}
\odot & 0 & a & b & 1 \\
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0 & 0 & 0 & 0 & 0 \\
a & 0 & a & a & a \\
b & 0 & a & b & b \\
1 & 0 & a & b & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\to & 0 & a & b & 1 \\
\hline
0 & 1 & 1 & 1 & 1 \\
a & a & 1 & 1 & 1 \\
b & 0 & a & 1 & 1 \\
1 & 0 & a & b & 1 \\
\end{array}
\]

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice and $B = \{0, a, 1\}$ is a BL-algebra, which properly contained in $L$. Then $L$ is a Smarandache $B_L$-residuated lattice. $F = \{1\}$ is an $n$-fold $B_L$-Smarandache fantastic filter of $L_B$, while it is not 1-fold $B_L$-Smarandache implicative filter. Since $(a \to 0) \to a \in F$ but $a \not\in F$.

Theorem 2.3. Let every $B_L$-Smarandache deductive system of $L_B$ be a $B_L$-Smarandache filter of $L_B$ and $F$ be a $B_L$-Smarandache deductive system of $L_B$. Then the following statements are equivalent:

(i) $F$ is an $n$-fold $B_L$-Smarandache fantastic filter and $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$.

(ii) $F$ is an $n$-fold $B_L$-Smarandache implicative filter of $L_B$.

Proof. (i) $\Rightarrow$ (ii) Let $x, y \in B$ such that $(x^n \to y) \to x \in F$. Since $F$ is an $n$-fold $B_L$-Smarandache fantastic filter, by the fact that $(x^n \to y) \to x \in F$, we have $(x^n \to (x^n \to y)) \to (x^n \to y) \to x \in F$. By Theorem 1.1 we get

\[
x^n \to x^{2n} \leq (x^n \to y) \to (x^n \to y) = ((x^n \odot x^n) \to y) \to (x^n \to y) = (x^n \to (x^n \to y)) \to (x^n \to y).
\]

Hence $(x^n \to x^{2n}) \to x \geq ((x^n \to (x^n \to y)) \to (x^n \to y)) \to x$. Since $F$ is a $B_L$-Smarandache filter, by the fact that $(x^n \to (x^n \to y)) \to (x^n \to y) \to x \in F$, we have $(x^n \to x^{2n}) \to x \in F$. Since $F$ is an $n$-fold $B_L$-Smarandache positive implicative filter,
by Theorem 1.3, we get \( x^n \to x^{2n} \in F \) and so \( x \in F \). By Theorem 1.5 \( F \) is an \( n \)-fold \( B_L \)-Smarandache implicative filter.

(ii) \( \Rightarrow \) (i) By Theorem 1.7 and Proposition 2.2, the proof is clear.

\[ \square \]

**Definition 2.2.** \( L_B \) is said to be \( n \)-fold Smarandache fantastic \( B_L \)-residuated lattice if for all \( x, y \in B \), \( y \to x = ((x^n \to y) \to y) \to x \).

**Example 2.3.** (i) Let \( L = \{0, a, b, c, d \} \) be a residuated lattice such that \( 0 < b < a < 1 \) and \( 0 < d < a, c < 1 \). We define

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Then \( (L, \land, V, \odot, \to, 0, 1) \) is a residuated lattice, in which \( B = \{0, b, c, 1\} \) is a \( BL \)-algebra which properly contained in \( L \). Then \( L \) is a Smarandache \( B_L \)-residuated lattice. \( L_B \) is an \( n \)-fold Smarandache fantastic \( B_L \)-residuated lattice.

(ii) In Example 2.1, \( L_B \) is not a \( 1 \)-fold Smarandache fantastic \( B_L \)-residuated lattice. Since \((a \to 0) \to a \neq 0 \to a\).

**Proposition 2.3.** \( L_B \) is an \( n \)-fold Smarandache fantastic \( B_L \)-residuated lattice if and only if the inequality \( (x^n \to y) \to y \leq (y \to x) \to x \) holds for all \( x, y \in B \).

**Proof.** Assume that \( L_B \) is an \( n \)-fold Smarandache fantastic \( B_L \)-residuated lattice. Let \( x, y \in B \). We have

\( ((x^n \to y) \to y) \to ((y \to x) \to x) = (y \to x) \to (((x^n \to y) \to y) \to x) \).

By hypothesis \( y \to x = ((x^n \to y) \to y) \to x \). Hence \( (y \to x) \to (((x^n \to y) \to y) \to x) = 1 \).

Then we get \( ((x^n \to y) \to y) \to ((y \to x) \to x) = 1 \) or equivalently \( ((x^n \to y) \to y) \leq ((y \to x) \to x) \).

Conversely, suppose that the inequality \( (x^n \to y) \to y \leq (y \to x) \to x \) holds. Then \( (y \to x) \to (((x^n \to y) \to y) \to x) = (y \to x) \to (((x^n \to y) \to y) \to x) \), (I). Since \( (x^n \to y) \to y \leq (y \to x) \to x \), we get

\( ((x^n \to y) \to y) \to ((x^n \to y) \to y) \leq ((x^n \to y) \to y) \to ((y \to x) \to x) \), that is

\( ((x^n \to y) \to y) \to ((y \to x) \to x) = 1 \). Then \( (y \to x) \to ((x^n \to y) \to y) \to x) = 1 \).

It follows that \( (y \to x) \leq ((x^n \to y) \to y) \to x \). Since \( y \leq (x^n \to y) \to y \), we also get \( (y \to x) \geq ((x^n \to y) \to y) \to x \). Therefore we obtain \( (y \to x) = ((x^n \to y) \to y) \to x \). Hence \( L_B \) is an \( n \)-fold Smarandache fantastic \( B_L \)-residuated lattice. \( \square \)

**Proposition 2.4.** The following conditions are equivalent:

(i) \( L_B \) is an \( n \)-fold Smarandache fantastic \( B_L \)-residuated lattice.

(ii) Every \( B_L \)-Smarandache deductive system of \( L_B \) is an \( n \)-fold \( B_L \)-Smarandache fantastic filter of \( L_B \).

(iii) \( (x^n \to y) \to y) \to ((y \to x) \to x) \in F \), for every \( B_L \)-Smarandache deductive system \( F \) of \( L_B \).

(iv) \( \{1\} \) is an \( n \)-fold \( B_L \)-Smarandache fantastic filter of \( L_B \).

**Proof.** (i) \( \Rightarrow \) (ii) Assume that \( \{1\} \) is an \( n \)-fold \( B_L \)-Smarandache fantastic filter. Let \( x, y \in B \) and \( t = (y \to x) \to x \). By Theorem 1.1, \( y \leq t \). So \( y \to t = 1 \) and by the hypothesis, we
have \(((t^n \rightarrow y) \rightarrow y) \rightarrow t = 1\), that is \((t^n \rightarrow y) \rightarrow y \leq t\), (I). On the other hand, \(x \leq t\) implies \(x^n \leq t^n\), hence \((x^n \rightarrow y) \rightarrow y \leq (t^n \rightarrow y) \rightarrow y\). Then by (I) it follows that \((x^n \rightarrow y) \rightarrow y \leq t = (y \rightarrow x) \rightarrow x\). Hence by Proposition 2.3, \(L_B\) is an \(n\)-fold Smarandache fantastic \(B_L\)-residuated lattice.

(i) \(\Rightarrow\) (iii) By Proposition 2.3, \((x^n \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = 1 \in F\), for every \(B_L\)-Smarandache deductive system \(F\) of \(L_B\).

(iii) \(\Rightarrow\) (iv) By (iii), \((x^n \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) \in \{1\}\). Hence \((x^n \rightarrow y) \rightarrow y \leq ((y \rightarrow x) \rightarrow x)\). So by Proposition 2.3, \(L_B\) is an \(n\)-fold Smarandache fantastic \(B_L\)-residuated lattice. Let \(y \rightarrow x \in \{1\}\). By the fact that \(L_B\) is an \(n\)-fold Smarandache fantastic \(B_L\)-residuated lattice we get \((x^n \rightarrow y) \rightarrow y \rightarrow x \in \{1\}\), i.e. \(\{1\}\) is an \(n\)-fold \(B_L\)-Smarandache fantastic filter of \(L_B\).

\[\square\]

**Corollary 2.1.** Let \(L_B\) be an \(n\)-fold Smarandache fantastic \(B_L\)-residuated lattice and \(F\) be a \(B_L\)-Smarandache deductive system of \(L_B\). Then \((x^n)^{**} \rightarrow x \in F\), for all \(x \in B\).

**Proof.** In Proposition 2.4, take \(y = 0\). Thus the proof is clear. \[\square\]

**Corollary 2.2.** Let every \(B_L\)-Smarandache deductive system of \(L_B\) be a \(B_L\)-Smarandache filter of \(L_B\) and \(F\) be a \(B_L\)-Smarandache deductive system of \(L_B\). Then the following conditions are equivalent:

(i) \(L_B\) is an \(n\)-fold Smarandache implicative \(B_L\)-residuated lattice.

(ii) \(L_B\) is an \(n\)-fold Smarandache fantastic \(B_L\)-residuated lattice and \(n\)-fold Smarandache positive implicative \(B_L\)-residuated lattice.

**Proof.** By Theorem 1.2(i), Propositions 2.4, 1.6, 1.4 and Theorem 2.3, the proof is easy. \[\square\]

**Corollary 2.3.** Let \(F\) be a \(B_L\)-Smarandache deductive system of \(L_B\). \(F\) is an \(n\)-fold \(B_L\)-Smarandache fantastic filter of \(L_B\) if and only if \(L_B/F\) is an \(n\)-fold Smarandache fantastic \(B_{L/F}\)-residuated lattice.

**Proof.** Assume that \(F\) is an \(n\)-fold \(B_L\)-Smarandache fantastic filter of \(L_B\). Let \(x, y \in B\) be such that \(y/F \rightarrow x/F \in \{1/F\}\), then \((y \rightarrow x)/F = 1/F\) or equivalently \(y \rightarrow x \in F\). Since \(F\) is an \(n\)-fold \(B_L\)-Smarandache fantastic filter, we get \((x^n \rightarrow y) \rightarrow x \in F\) or equivalently \(((x^n \rightarrow y) \rightarrow y) \rightarrow x) = 1/F\), so \((((x/F)^n \rightarrow y/F) \rightarrow y/F) \rightarrow x/F) \in \{1/F\}\), for all \(x/F \in B/F\). Hence \(1/F\) is an \(n\)-fold \(B_{L/F}\)-Smarandache fantastic filter of \(L_{B/F}\), therefore by Proposition 2.4, \(L_{B/F}\) is an \(n\)-fold Smarandache fantastic \(B_{L/F}\)-residuated lattice.

Conversely, assume that \(L_{B/F}\) is an \(n\)-fold Smarandache fantastic \(B_{L/F}\)-residuated lattice. Let \(x, y \in B\) be such that \(y \rightarrow x \in F\) then \((y \rightarrow x)/F = 1/F\) or equivalently \(y \rightarrow x \in F\). Since \(L_{B/F}\) is an \(n\)-fold Smarandache fantastic \(B_{L/F}\)-residuated lattice, by Proposition 2.4, \(1/F\) is an \(n\)-fold \(B_{L/F}\)-Smarandache fantastic filter of \(L_{B/F}\). From this and the fact that \(y/F \rightarrow x/F \in \{1/F\}\), we have \((((x/F)^n \rightarrow y/F) \rightarrow y/F) \rightarrow x/F) \in \{1/F\}\) or equivalently \(((x^n \rightarrow y) \rightarrow y) \rightarrow x) = 1/F\), so \((x^n \rightarrow y) \rightarrow y \rightarrow x \in F\). Hence \(F\) is an \(n\)-fold \(B_L\)-Smarandache fantastic filter of \(L_B\).

\[\square\]

**Theorem 2.4.** Let \(F\) and \(G\) be two \(B_L\)-Smarandache deductive system of \(L_B\) such that \(F \subseteq G\). If \(F\) is an \(n\)-fold \(B_L\)-Smarandache fantastic filter, then so is \(G\).

**Proof.** Let \(x, y \in B\) be such that \(y \rightarrow x \in G\). Since \(F\) is an \(n\)-fold \(B_L\)-Smarandache fantastic filter, by Corollary 2.3, \(L_{B/F}\) is an \(n\)-fold Smarandache fantastic \(B_{L/F}\)-residuated lattice.

So \(((x/F)^n \rightarrow y/F) \rightarrow y/F) \rightarrow x/F = y/F \rightarrow x/F\), hence \((y \rightarrow x) \rightarrow (((x^n \rightarrow y) \rightarrow y) \rightarrow x) \in F\), so \((y \rightarrow x) \rightarrow (((x^n \rightarrow y) \rightarrow y) \rightarrow x) \in G\). By the fact that \(y \rightarrow x \in G\) we we
get \(((x^n \rightarrow y) \rightarrow y) \rightarrow x) \in G\). Hence \(G\) is an \(n\)-fold \(BL\)-Smarandache fantastic filter of \(L_B\).

### 3. \(n\)-Fold \(BL\)-Smarandache Easy Filters

**Definition 3.1.** A subset \(F\) of \(L_B\) is called an \(n\)-fold Smarandache easy filter of \(L_B\) related to \(B\) (for briefly \(n\)-fold \(BL\)-Smarandache easy filter of \(L_B\)) if \((x^{**})^n \rightarrow (y \rightarrow z) \in F\) and \((x^{**})^n \rightarrow y \in F\) imply \((x^{**})^n \rightarrow z \in F\), for all \(x, y, z \in B\).

**Example 3.1.** In Example 2.1, \(F = \{d, 1\}\) is an \(n\)-fold \(BL\)-Smarandache easy filter, for \(n \geq 2\). \(F\) is not a 1-fold \(BL\)-Smarandache easy filter. Since \(a^{**} \rightarrow (c \rightarrow a) = 1 \in F\) and \(a^{**} \rightarrow c = 1 \in F\) but \(a^{**} \rightarrow a = c \notin F\).

**Theorem 3.1.** Every \(n\)-fold \(BL\)-Smarandache easy filter of \(L_B\) is a \(BL\)-Smarandache deductive system of \(L_B\).

**Proof.** Let \(F\) be an \(n\)-fold \(BL\)-Smarandache easy filter of \(L_B\). Suppose \(z \in B\), such that \(y, y \rightarrow z \in F\). We have \((1^{**})^n \rightarrow y, (1^{**})^n \rightarrow (y \rightarrow z) \in F\), these imply \(z = (1^{**})^n \rightarrow z \in F\). Hence \(F\) is a \(BL\)-Smarandache deductive system of \(L_B\).

**Theorem 3.2.** Every \(n\)-fold \(BL\)-Smarandache positive implicative filter of \(L_B\) is an \(n\)-fold \(BL\)-Smarandache easy filter of \(L_B\).

**Proof.** Let \(F\) be an \(n\)-fold \(BL\)-Smarandache positive implicative filter of \(L_B\) and \(x, y, z \in B\) such that \((x^{**})^n \rightarrow (y \rightarrow z) \in F\) and \((x^{**})^n \rightarrow y \in F\). So by Definition 1.1(i), \((x^{**})^n \rightarrow z \in F\). So \(F\) is an \(n\)-fold \(BL\)-Smarandache easy filter of \(L_B\).

**Example 3.2.** Let \(L = \{0, a, b, c, 1\}\), where \(0 < a < b < c < 1\). Define \(\circ\) and \(\rightarrow\) as follows:

<table>
<thead>
<tr>
<th>(\circ)</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

We can see that \(L = (L, \wedge, \vee, \circ, \rightarrow, 0, 1)\) is a residuated lattice, in which \(B = \{0, a, 1\}\) is a \(BL\)-algebra which properly contained in \(L\). Then \(L\) is a Smarandache \(BL\)-residuated lattice. \(F = \{1\}\) is an \(n\)-fold \(BL\)-Smarandache easy filter of \(L_B\), while is not 1-fold and 2-fold \(BL\)-Smarandache positive implicative filter. Since \(c^1 \rightarrow (c \rightarrow c^1) = 1 \in F\), but \(c^1 \rightarrow c^2 = c \notin F\). Also \(c^2 \rightarrow (c \rightarrow c^1) = 1 \in F\) and \(c^2 \rightarrow c^1 = 1 \in F\), but \(c^2 \rightarrow c^3 = c \notin F\).

**Theorem 3.3.** Let \(F\) be a \(BL\)-Smarandache filter of \(L_B\). The following conditions are equivalent:

(i) \(F\) is an \(n\)-fold \(BL\)-Smarandache easy filter of \(L_B\);

(ii) \((x^{**})^n \rightarrow (y \rightarrow z) \in F\) implies \(((x^{**})^n \rightarrow y) \rightarrow ((x^{**})^n \rightarrow z) \in F\), for all \(x, y, z \in B\);

(iii) \((x^{**})^n \rightarrow ((x^{**})^n \rightarrow z) \in F\) implies \((x^{**})^n \rightarrow z \in F\), for all \(x, z \in B\);

(iv) \((x^{**})^n \rightarrow (x^{**})^n \in F\), for all \(x \in B\).

**Proof.** (i) \(\Rightarrow\) (ii) Let \(F\) be an \(n\)-fold \(BL\)-Smarandache easy filter and \(x, y, z \in B\) such that \((x^{**})^n \rightarrow (y \rightarrow z) \in F\). By Theorem 1.1 we have \((x^{**})^n \rightarrow ((x^{**})^n \rightarrow ((x^{**})^n \rightarrow y) \rightarrow z)\) \(= (x^{**})^n \rightarrow (((x^{**})^n \rightarrow y) \rightarrow ((x^{**})^n \rightarrow z)) \geq (x^{**})^n \rightarrow (y \rightarrow z)\). Hence \((x^{**})^n \rightarrow ((x^{**})^n \rightarrow (y \rightarrow z)) \in F\). From \((x^{**})^n \rightarrow (x^{**})^n = 1 \in F\) and the hypothesis we get \((x^{**})^n \rightarrow ((x^{**})^n \rightarrow y) \rightarrow z) \in F\). By Theorem 1.1, we obtain \(((x^{**})^n \rightarrow y) \rightarrow (x^{**})^n \rightarrow z) \in F\).
(ii) ⇒ (iii) Take \( y = (x^*)^n \), hence the proof is clear.

(iii) ⇒ (i) Let \((x^*)^n \rightarrow (y \rightarrow z) \in F\) and \((x^*)^n \rightarrow y \in F\). By Theorem 1.1 we have \((x^*)^n \rightarrow (y \rightarrow z) = y \rightarrow ((x^*)^n \rightarrow z)\). So \(((x^*)^n \rightarrow y) \rightarrow ((x^*)^n \rightarrow ((x^*)^n \rightarrow z)) \in F\).

Since \((x^*)^n \rightarrow y \in F\) and \(F\) is an \(n\)-fold \(B_L\)-Smarandache filter, \((x^*)^n \rightarrow ((x^*)^n \rightarrow z) \in F\). By Theorem 1.1 we have \((x^*)^n \rightarrow ((x^*)^n \rightarrow z) = (x^*)^{2n} \rightarrow z \in F\). By (ii), \((x^*)^n \rightarrow (x^*)^{2n} \in F\). So \(((x^*)^n \rightarrow (x^*)^{2n}) \circ ((x^*)^{2n} \rightarrow z) \in F\). By Theorem 1.1, \(((x^*)^n \rightarrow (x^*)^{2n}) \circ ((x^*)^{2n} \rightarrow z) \leq (x^*)^n \rightarrow z\). So \((x^*)^n \rightarrow z \in F\). Therefore \(F\) is an \(n\)-fold \(B_L\)-Smarandache easy filter.

(iii) ⇒ (iv) For \(x, y \in B\), we have \((x^*)^n \rightarrow ((x^*)^n \rightarrow (x^*)^{2n}) = (x^*)^{2n} \rightarrow (x^*)^{2n} = 1 \in F\). So by (iii), \((x^*)^n \rightarrow (x^*)^{2n} \in F\).

(iv) ⇒ (iii) Let \(x, y \in B\) such that \((x^*)^n \rightarrow ((x^*)^n \rightarrow y) \in F\), so \((x^*)^{2n} \rightarrow y \in F\). By (iv), \((x^*)^n \rightarrow (x^*)^{2n} \in F\). By Theorem 1.1, \((x^*)^n \rightarrow (x^*)^{2n} \leq ((x^*)^{2n} \rightarrow y) \rightarrow ((x^*)^n \rightarrow y)\). Since \(F\) is a \(B_L\)-Smarandache filter, \(((x^*)^{2n} \rightarrow y) \rightarrow ((x^*)^n \rightarrow y) \in F\), so \((x^*)^n \rightarrow y \in F\). Hence \(F\) is an \(n\)-fold \(B_L\)-Smarandache easy filter.

**Theorem 3.4.** Let \(F\) and \(G\) be \(B_L\)-Smarandache filters of \(L_B\) and \(F \subseteq G\). If \(F\) is an \(n\)-fold \(B_L\)-Smarandache easy filter of \(L_B\), then \(G\) is an \(n\)-fold \(B_L\)-Smarandache easy filter of \(L_B\).

**Proof.** Follows from Theorem 3.3 (iv) ⇔ (i). □

**Proposition 3.1.** For a Smarandache \(B_L\)-residuated lattice \(L_B\), the following conditions are equivalent:

(i) \(\{1\}\) is an \(n\)-fold \(B_L\)-Smarandache easy filter of \(L_B\);

(ii) Every \(B_L\)-Smarandache easy filter of \(L_B\) is an \(n\)-fold \(B_L\)-Smarandache easy filter of \(L_B\);

(iii) \((x^*)^n = (x^*)^{2n}\), for all \(x \in B\).

**Proof.** (i) ⇒ (ii) The proof is clear, by Theorem 3.4.

(i) ⇔ (iii) By Theorem 3.3, we have \((x^*)^n \rightarrow (x^*)^{2n} \in \{1\} ⇔ (x^*)^n \rightarrow (x^*)^{2n} = 1 ⇔ (x^*)^n \leq (x^*)^{2n} ⇔ (x^*)^n = (x^*)^{2n},\) for all \(x \in B\). □

**Corollary 3.1.** If \(\{1\}\) is an \(n\)-fold \(B_L\)-Smarandache easy filter of \(L_B\), then \((x^2)^* = x^*\), for all \(x \in B\).

**Proof.** By Proposition 3.1, \((x^*)^n = (x^*)^{2n}\), for all \(x \in B\) and \(n \in N\). Let \(n = 1\). Then \(x^* = (x^*)^2\) (I). We know \((x^*)^n \leq (x^n)^*\), for all \(n \in N\). Let \(n = 2\). Then \((x^*)^2 \leq (x^2)^*\). Then by (I), we get \(x^* \leq (x^2)^*\). By Theorem 1.1, we get \((x^2)^* \leq x^*\). (II). By Theorem 1.1, we know \(x^2 \leq x\), so \(x^* \leq (x^2)^*\). Therefore by (II), \((x^2)^* = x^*\). □

**Definition 3.2.** \(L_B\) is said to be an \(n\)-fold Smarandache easy \(B_L\)-residuated lattice if for all \(x \in B\), \((x^*)^n = (x^*)^{2n}\).

**Example 3.3.** (i) Consider the Smarandache \(B_L\)-residuated lattice \(L\) in Example 2.3(i). Clearly \(L_B\) is an \(n\)-fold Smarandache easy \(B_L\)-residuated lattice.

(ii) Consider the Smarandache \(B_L\)-residuated lattice \(L\) in Example 2.1. Clearly \(L_B\) is not a 1-fold Smarandache easy \(B_L\)-residuated lattice, since \(c = (a^*)^1 \neq (a^*)^2 = 0\).

**Corollary 3.2.** For a Smarandache \(B_L\)-residuated lattice \(L_B\), the following conditions are equivalent:

(i) \(\{1\}\) is an \(n\)-fold \(B_L\)-Smarandache easy filter of \(L_B\);

(ii) Every \(B_L\)-Smarandache filter of \(L_B\) is an \(n\)-fold \(B_L\)-Smarandache easy filter of \(L_B\);

(iii) \(L_B\) is an \(n\)-fold Smarandache easy \(B_L\)-residuated lattice.

**Proof.** By Proposition 3.1, the proof is clear. □
Corollary 3.3. Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. Then $F$ is an n-fold $B_L$-Smarandache easy filter of $L_B$ (in short n-fold $B_L$-SEF) if and only if $L_B/F$ is an n-fold Smarandache easy $B_L/F$-residuated lattice (in short n-fold $B_L$-SERL).

Proof. By Theorem 3.3 we get:

$$F \text{ is an n-fold } B_L\text{-SEF} \iff (x^{*})^n \rightarrow (x^{*})^{2n} \in F, \forall x \in B,$$

$$\iff (x^{*})^n \rightarrow (x^{*})^{2n} = 1/F, \forall x/F \in B/F,$$

$$\iff (x^{*}/F)^n \rightarrow (x^{*}/F)^{2n} = 1/F, \forall x/F \in B/F,$$

$$\iff ((x/F)^{*})^n \leq ((x/F)^{*})^{2n}, \forall x/F \in B/F,$$

$$\iff ((x/F)^{*})^n = ((x/F)^{*})^{2n}, \forall x/F \in B/F,$$

$$\iff L_B/F \text{ is an n-fold } B_L/F\text{-SERL}.$$ 

By Corollary 3.2 and 3.1, we have the following corollary.

Corollary 3.4. Let $L_B$ be an n-fold Smarandache easy $B_L$-residuated lattice. Then $(x^{2})^* = x^*$, for all $x \in B$.

Proposition 3.2. Every n-fold Smarandache positive implicative $B_L$-residuated lattice is an n-fold Smarandache easy $B_L$-residuated lattice.

Proof. Let $L_B$ be an n-fold Smarandache positive implicative $B_L$-residuated lattice. By Theorem 1.4, every $B_L$-Smarandache deductive system of $L_B$ is an n-fold $B_L$-Smarandache positive implicative filter of $L_B$. Hence by Theorem 3.2, every $B_L$-Smarandache deductive system of $L_B$ is an n-fold $B_L$-Smarandache easy filter of $L_B$. Since every $B_L$-Smarandache filter is a $B_L$-Smarandache deductive system, then every $B_L$-Smarandache filter is an n-fold $B_L$-Smarandache easy filter of $L_B$. Therefore by Corollary 3.2, $L_B$ is an n-fold Smarandache easy $B_L$-residuated lattice.

Theorem 3.5. For a Smarandache $B_L$-residuated lattice $L_B$, the following conditions are equivalent:

(i) $L_B$ is an n-fold Smarandache easy $B_L$-residuated lattice;

(ii) $(x^{*})^n \leq y \rightarrow z$ implies $(x^{*})^n \rightarrow y \leq (x^{*})^n \rightarrow z$, for all $x,y,z \in B$;

(iii) $(x^{*})^n \leq (x^{*})^n \rightarrow z$ implies $(x^{*})^n \leq z$, for all $x,z \in B$.

Proof. (i) $\Rightarrow$ (ii) Let $L_B$ be an n-fold Smarandache easy $B_L$-residuated lattice. By Corollary 3.2, $\{1\}$ is an n-fold $B_L$-Smarandache easy filter of $L_B$. Let $(x^{*})^n \leq y \rightarrow z$, for all $x,y,z \in B$. So $(x^{*})^n \rightarrow (y \rightarrow z) \in \{1\}$. Then by Theorem 3.3, we get $((x^{*})^n \rightarrow y) \rightarrow ((x^{*})^n \rightarrow z) \in \{1\}$. And so $(x^{*})^n \rightarrow y \leq (x^{*})^n \rightarrow z$.

(iii) $\Rightarrow$ (i) Take $y = (x^{*})^0$, hence the proof is clear.

(iii) $\Rightarrow$ (ii) Let $(x^{*})^n \rightarrow ((x^{*})^n \rightarrow z) \in \{1\}$. So $(x^{*})^n \leq (x^{*})^n \rightarrow z$. Hence by (iii), $(x^{*})^n \leq z$, for all $x,z \in B$. Then $(x^{*})^n \rightarrow z \in \{1\}$, for all $x,z \in B$. By Theorem 3.3, $\{1\}$ is an n-fold $B_L$-Smarandache easy filter of $L_B$. Therefore by Corollary 3.2, $L_B$ is an n-fold Smarandache easy $B_L$-residuated lattice.

4. Conclusion

In this paper we introduced the notion of n-fold $B_L$-Smarandache (easy) fantastic filter in Smarandache residuated lattices and we have established extension property for them. Also we defined the notions of n-fold Smarandache fantastic $B_L$-residuated lattice and n-fold Smarandache easy $B_L$-residuated lattice and we presented a characterization and many important properties of them. We proved that if $L_B$ is an n-fold Smarandache
fantastic $B_L$-residuated lattice, then $F$ is a $B_L$-Smarandache deductive system iff $F$ is an $n$-fold $B_L$-Smarandache fantastic filter. Also we proved that $L_B$ is an $n$-fold Smarandache easy $B_L$-residuated lattice iff every $B_L$-Smarandache filter of $L_B$ is an $n$-fold $B_L$-Smarandache easy filter of $L_B$. We show every $n$-fold Smarandache positive implicative $B_L$-residuated lattice is an $n$-fold Smarandache easy $B_L$-residuated lattice. We hope this work would serve as a foundation for further studies on the structure of Smarandache residuated lattices.

REFERENCES