ON 15-TH SMARANDACHE'S PROBLEM
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Introduction

The 15-th Smarandache's problem [10,11] is the following: "Smarandache's simple numbers:
2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, 27, 29, 31, ...
A number \( n \) is called "Smarandache's simple number" if the product of its proper divisors is less than or equal to \( n \). Generally speaking, \( n \) has the form \( n = p \), or \( n = p^2 \), or \( n = p^3 \), or \( n = pq \), where \( p \) and \( q \) are distinct primes".

Let us denote: by \( S \) - the sequence of all Smarandache's simple numbers and by \( s_n \) - the \( n \)-th term of \( S \); by \( P \) - the sequence of all primes and by \( p_n \) - the \( n \)-th term of \( P \); by \( P^2 \) - the sequence \( \{ p_n^2 \}_{n=1}^{\infty} \); by \( P^3 \) - the sequence \( \{ p_n^3 \}_{n=1}^{\infty} \); by \( PQ \) - the sequence \( \{ p_nq \}_{p,q \in P} \), where \( p < q \).

For an arbitrary increasing sequence of natural numbers \( C = \{ c_n \}_{n=1}^{\infty} \) we denote by \( \pi_C(n) \) the number of terms of \( C \), which are not greater than \( n \). When \( n < c_1 \) we must put \( \pi_C(n) = 0 \).

In the present paper we find \( \pi_S(n) \) in an explicit form and using this, we find the \( n \)-th term of \( S \) in explicit form, too.

1. \( \pi_S(n) \)-representation

First, we must note that instead of \( \pi_P(n) \) we shall use the well known denotation \( \pi(n) \). Hence

\[ \pi_P(n) = \pi(\sqrt{n}) \quad \pi_P(n) = \pi(\sqrt{n}). \]

Thus, using the definition of \( S \), we get

\[ \pi_S(n) = \pi(n) + \pi(\sqrt{n}) + \pi(\sqrt{n}) + \pi_PQ(n). \]

(1)

Our first aim is to express \( \pi_S(n) \) in an explicit form. For \( \pi(n) \) some explicit formulae are proposed in [2]. Other explicit formulae for \( \pi(n) \) are contained in [3]. One of them is known as Mináč's formula. It is given below

\[ \pi(n) = \sum_{k=2}^{n} \left[ \frac{(k-1)! + 1}{k} \right] - \left[ \frac{(k-1)!}{k} \right]. \]

(2)

where \([ \cdot ]\) denotes the function integer part. Therefore, the question about explicit formulae for functions \( \pi(n), \pi(\sqrt{n}), \pi(\sqrt{n}) \) is solved successfully. It remains only to express \( \pi_PQ(n) \) in an explicit form.

Let \( k \in \{ 1, 2, ..., \pi(\sqrt{n}) \} \) be fixed. We consider all numbers of the kind \( p_kq \), where \( q \in \mathbb{P}, q > p_k \) for which \( p_kq \leq n \). The number of these numbers is \( \pi(\frac{n}{p_k}) - \pi(p_k) \), or which is the same

\[ \pi(\frac{n}{p_k}) - k. \]

(3)

When \( k = 1, 2, ..., \pi(\sqrt{n}) \), numbers \( p_kq \), that were defined above, describe all numbers of the kind \( p_kq \), where \( p, q \in \mathbb{P}, p < q, p,q \leq n \). But the number of the last numbers is equal to \( \pi_PQ(n) \). Hence

\[ \pi_PQ(n) = \sum_{k=1}^{\pi(\sqrt{n})} \left( \pi(\frac{n}{p_k}) - k \right), \]

(4)

because of (3). The equality (4), after a simple computation yields the formula

\[ \pi_PQ(n) = \sum_{k=1}^{\pi(\sqrt{n})} \pi(\sqrt{n}) - \pi(\sqrt{n})(\pi(\sqrt{n}) + 1). \]

(5)

In [4] the identity

\[ \sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) = \pi(\frac{n}{b}) \pi(b) + \sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) \]

(6)

is proved, under the condition \( b \geq 2 \) (\( b \) is a real number). When \( \pi(\frac{b}{2}) = \pi(\frac{b}{2}) \), the right hand-side of (6) reduces to \( \pi(\frac{b}{2}) \pi(b) \). In the case \( b = \sqrt{n} \) and \( n \geq 4 \) equality (6) yields

\[ \sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) = \pi(\sqrt{n})^{\pi(\sqrt{n})} + \sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) \]

(7)

If we compare (5) with (7) we obtain for \( n \geq 4 \)

\[ \pi_PQ(n) = \pi(\sqrt{n})(\pi(\sqrt{n}) - 1) + \sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) \]

(8)

Thus, we have two different explicit representations for \( \pi_PQ(n) \). These are formulae (5) and (8). We must note that the right hand-side of (8) reduces to \( \pi(\sqrt{n})(\pi(\sqrt{n}) - 1) \), when \( \pi(\frac{b}{2}) = \pi(\sqrt{n}) \).

Finally, we observe that (1) gives an explicit representation for \( \pi_S(n) \), since we may use formula (2) for \( \pi(n) \) (or other explicit formulae for \( \pi(n) \)) and (5), or (8) for \( \pi_PQ(n) \).
2. Explicit formulae for $s_n$

The following assertion decides the question about explicit representation of $s_n$.

**Theorem:** The $n$-th term $s_n$ of $S$ admits the following three different explicit representations:

$$s_n = \sum_{k=0}^{\pi(n)} \frac{1}{1 + \left\lfloor \frac{\pi(n)}{n} \right\rfloor}; \quad (9)$$

$$s_n = -2 \sum_{k=0}^{\pi(n)} \theta(-2\left\lfloor \frac{\pi(n)}{n} \right\rfloor); \quad (10)$$

$$s_n = \sum_{k=0}^{\pi(n)} \frac{1}{\Gamma(1 - \left\lfloor \frac{\pi(n)}{n} \right\rfloor)}; \quad (11)$$

where

$$\theta(n) = \left\lfloor \frac{n^2 + 3n + 4}{4} \right\rfloor, \quad n = 1, 2, \ldots, \quad (12)$$

$\zeta$ is the Riemann function zeta and $\Gamma$ is Euler's function gamma.

**Remark.** We must note that in (9)-(11) $\pi_S(k)$ is given by (1), $\pi(k)$ is given by (2) (or by others formulae like (2)) and $\pi_{CQ}(n)$ is given by (5), or by (8). Therefore, formulae (9)-(11) are explicit.

**Proof of the Theorem.** In [2] the following three universal formulae are proposed, using $\pi_{C\{k\}}(k = 0, 1, \ldots)$, which one could apply to represent $c_n$. They are the following

$$c_n = \sum_{k=0}^{\infty} \frac{1}{1 + \left\lfloor \pi_{C\{k\}} \right\rfloor}; \quad (13)$$

$$c_n = -2 \sum_{k=0}^{\infty} \zeta(-2\left\lfloor \frac{\pi_{C\{k\}}}{n} \right\rfloor); \quad (14)$$

$$c_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1 - \left\lfloor \frac{\pi_{C\{k\}}}{n} \right\rfloor)}; \quad (15)$$

In [5] is shown that the inequality

$$p_n \leq \theta(n), \quad n = 1, 2, \ldots, \quad (16)$$

holds. Hence

$$s_n = \theta(n), \quad n = 1, 2, \ldots, \quad (17)$$

since we have obviously

$$s_n \leq p_n, \quad n = 1, 2, \ldots. \quad (18)$$

Then to prove the Theorem it remains only to apply (13)-(15) in the case $C = S$, i.e., for $c_n = s_n$, putting there $\pi_S(k)$ instead of $\pi_{C\{k\}}$ and $\theta(n)$ instead of $\infty$.

**REFERENCES:**


