On Additive Analogues of Certain Arithmetic Functions

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1. The Smarandache, Pseudo-Smarandache, resp. Smarandache-simple functions are defined as ([7], [6])

\[ S(n) = \min\{m \in \mathbb{N} : n|m!\}, \quad (1) \]

\[ Z(n) = \min\left\{ m \in \mathbb{N} : \frac{m(m+1)}{2} \right\}, \quad (2) \]

\[ S_p(n) = \min\{m \in \mathbb{N} : p|m!\} \text{ for fixed primes } p. \quad (3) \]

The duals of \( S \) and \( Z \) have been studied e.g. in [2], [5], [6]:

\[ S^*(n) = \max\{m \in \mathbb{N} : m|n\}, \quad (4) \]

\[ Z^*(n) = \max\left\{ m \in \mathbb{N} : \frac{m(m+1)}{2} | n \right\}. \quad (5) \]

We note here that the dual of the Smarandache simple function can be defined in a similar manner, namely by

\[ S^*_p(n) = \max\{m \in \mathbb{N} : m!|p^n\} \quad (6) \]

This dual will be studied in a separate paper (in preparation).

2. The additive analogues of the functions \( S \) and \( S^* \) are real variable functions, and have been defined and studied in paper [3]. (See also our book [6], pp. 171-174). These functions have been recently further extended, by the use of Euler's gamma function, in place of the factorial (see [1]). We note that in what follows, we could define also the additive analogues functions by the use of Euler's gamma function. However, we shall apply the more transparent notation of a factorial of a positive integer.

The additive analogues of \( S \) and \( S^* \) from (1) and (4) have been introduced in [3] as follows:

\[ S(x) = \min\{m \in \mathbb{N} : x \leq m!\}, \quad S : (1, \infty) \rightarrow \mathbb{R} \quad (7) \]

resp.

\[ S^*(x) = \max\{m \in \mathbb{N} : m! \leq x\}, \quad S^* : (1, \infty) \rightarrow \mathbb{R} \quad (8) \]
Besides of properties relating to continuity, differentiability, or Riemann integrability of these functions, we have proved the following results:

**Theorem 1.**
$$S_n(x) \sim \frac{\log x}{\log \log x} \quad (x \to \infty)$$  

(The same for $S(x)$.)

**Theorem 2.** The series
$$\sum_{n=1}^{\infty} \frac{1}{n(S(n))^\alpha}$$  
is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$ (the same for $S_n(n)$ replaced by $S(n)$).

3. The additive analogues of $Z$ and $Z_n$ from (2), resp. (4) will be defined as
$$Z(x) = \min \left\{ m \in \mathbb{N} : x \leq \frac{m(m+1)}{2} \right\},$$  

$$Z_n(x) = \max \left\{ m \in \mathbb{N} : \frac{m(m+1)}{2} \leq x \right\}$$  

In (11) we will assume $x \in (0, \infty)$, while in (12) $x \in [1, \infty)$. The two additive variants of $S_n(n)$ of (3) will be defined as
$$P(x) = S_n(x) = \min \{ m \in \mathbb{N} : p^x \leq m! \},$$  

(13)

where in this case $p > 1$ is an arbitrary fixed real number
$$P_n(x) = S_n(n) = \max \{ m \in \mathbb{N} : m! \leq p^x \}$$  

From the definitions follow at once that
$$Z(x) = k \Leftrightarrow x \in \left[ \frac{(k-1)k}{2}, \frac{k(k+1)}{2} \right]$$  

for $k \geq 1$

$$Z_n(x) = k \Leftrightarrow x \in \left[ \frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2} \right]$$  

For $x \geq 1$ it is immediate that
$$Z_n(x) + 1 \geq Z(x) \geq Z_n(x)$$  

Therefore, it is sufficient to study the function $Z_n(x)$.

The following theorems are easy consequences of the given definitions:

**Theorem 3.**
$$Z_n(x) \sim \frac{1}{2} \sqrt{8x + 1} \quad (x \to \infty)$$  

**Theorem 4.**
$$\sum_{n=1}^{\infty} \frac{1}{n Z_n(n)^\alpha}$$  
is convergent for $\alpha > 2$

and divergent for $\alpha \leq 2$. The series
$$\sum_{n=1}^{\infty} \frac{1}{n Z_n(n)^\alpha}$$  
is convergent for all $\alpha > 0$.  

Proof. By (16) one can write $\frac{k(k+1)}{2} \leq x < \frac{(k+1)(k+2)}{2}$, so $k^2 + k - 2x \leq 0$ and $k^2 + 3k + 2 - 2x > 0$. Since the solutions of these quadratic equations are $k_{1,2} = \frac{-1 \pm \sqrt{8x + 1}}{2}$, resp. $k_{3,4} = \frac{-3 \pm \sqrt{8x + 1}}{2}$, and remarking that $\sqrt{8x + 1} - 1 \geq 1 \Leftrightarrow x \geq 3$, we obtain that the solution of the above system of inequalities is:
$$\left\{ \begin{array}{ll}
k & \in \left[ 1, \frac{\sqrt{1+8x-1}}{2} \right] \quad \text{if } x \in [1,3); \\
k & \in \left[ \frac{\sqrt{1+8x-3}}{2}, \frac{\sqrt{1+8x-1}}{2} \right] \quad \text{if } x \in [3, \infty) \end{array} \right.$$  

So, for $x \geq 3$
$$\frac{\sqrt{1+8x-3}}{2} < Z_n(x) \leq \frac{\sqrt{1+8x-1}}{2}$$  

implying relation (18).

Theorem 4 now follows by (18) and the known fact that the generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^\theta}$ is convergent only for $\theta > 1$.

The things are slightly more complicated in the case of functions $P$ and $P_n$. Here it is sufficient to consider $P_n$, too.

First remark that
$$P_n(x) = m \Leftrightarrow x \in \left[ \frac{\log m!}{\log p}, \frac{\log (m+1)!}{\log p} \right].$$  

(22)

The following asymptotic results have been proved in [31 (Lemma 2) (see also [61, p. 172)]
$$\log m! \sim m \log m, \quad \frac{m \log m!}{\log m!} \sim 1, \quad \frac{\log m!}{\log (m+1)!} \sim 1 \quad (m \to \infty)$$  

(23)

By (22) one can write
$$\frac{m \log m!}{\log m!} - \frac{m \log m!}{\log (m+1)!} \leq \frac{m \log x}{\log m!} \leq \frac{m \log m!}{\log m!} - \frac{m \log x}{\log (m+1)!} - (\log p) \frac{m}{\log m!},$$

giving $\frac{m \log x}{\log m!} \to 1 \quad (m \to \infty)$, and by (23) one gets $\log x \sim \log m$. This means that:

**Theorem 5.**
$$\log P_n(x) \sim \log x \quad (x \to \infty)$$  

(24)

The following theorem is a consequence of (24), and a convergence theorem established in [3]:

**Theorem 6.** The series
$$\sum_{n=1}^{\infty} \frac{1}{n \left( \log P_n(n) \right)^\alpha}$$  
is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.

Indeed, by (24) it is sufficient to study the series
$$\sum_{n=1}^{\infty} \frac{1}{n \left( \log P_n(n) \right)^\alpha}$$  
(\text{where } n_0 \in \mathbb{N} \text{ is a fixed positive integer). This series has been proved to be convergent for } \alpha > 1 \text{ and divergent for } \alpha \leq 1 \text{ (see [6], p. 174).}
References


Let $I'$ be a number.

2. As a generalization of the integer part of a number one defines the Inferior Smarandache Prime Part as:

$$ISPP(I') = p$$

is the largest prime less than or equal to $I'$.

For example:

$$ISPP(9) = 7$$

Similarly the Superior Smarandache Prime Part is defined as:

$$SSPP(I') = p$$

is smallest prime greater than or equal to $I'$.

For example:

$$SSPP(13) = 13$$

Questions:

1) Show that a number $I'$ is prime if and only if

$$ISPP(I') = SSPP(I')$$

Solution by Hans Gunter, Koln (Germany)

The Inferior Smarandache Prime Part, $ISPP(I')$, does not exist for $I' < 2$.

1) The first question is obvious (Carlos Rivera).

2) The second question:

a) If $I' = 2p$, where $p$ is a prime (i.e., $I'$ is the double of a prime), then the Smarandache diophantine equation

$$ISPP(I') + SSPP(I') = 2p$$

has one solution: $I' = p$ (Carlos Rivera).

b) If $I'$ is equal to the sum of two consecutive primes, $I' = p_n + p_{n+1}$, then the above Smarandache diophantine equation has many solutions: $I' = p_n + p_{n+1}$ (Teresinha DaCosta).

Let's consider an example:

$$ISPP(I') + SSPP(I') = 24$$

has the only solution $I' = 12$ because $11 < 12 < 13$ and $24 = 11 + 13$.