On Algebraic Multi-Vector Spaces

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Abstract: A Smarandache multi-space is a union of \( n \) spaces \( A_1, A_2, \cdots, A_n \) with some additional conditions holding. Combining Smarandache multi-spaces with linear vector spaces in classical linear algebra, the conception of multi-vector spaces is introduced. Some characteristics of a multi-vector space are obtained in this paper.

Key words: vector, multi-space, multi-vector space, ideal subspace chain.

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1. Introduction

The notion of multi-spaces is introduced by Smarandache in [6] under his idea of hybrid mathematics: combining different fields into a unifying field([7]), which is defined as follows.

Definition 1.1 For any integer \( i, 1 \leq i \leq n \) let \( A_i \) be a set with ensemble of law \( L_i \), and the intersection of \( k \) sets \( A_{i_1}, A_{i_2}, \cdots, A_{i_k} \) of them constrains the law \( I(A_{i_1}, A_{i_2}, \cdots, A_{i_k}) \). Then the union of \( A_i, 1 \leq i \leq n \)

\[ \tilde{A} = \bigcup_{i=1}^{n} A_i \]

is called a multi-space.

As we known, a vector space or linear space consists of the following:

(i) a field \( F \) of scalars;

(ii) a set \( V \) of objects, called vectors;

(iii) an operation, called vector addition, which associates with each pair of vectors \( a, b \) in \( V \) a vector \( a + b \) in \( V \), called the sum of \( a \) and \( b \), in such a way that

(1) addition is commutative, \( a + b = b + a \);

(2) addition is associative, \( (a + b) + c = a + (b + c) \);

(3) there is a unique vector \( 0 \) in \( V \), called the zero vector, such that \( a + 0 = a \) for all \( a \) in \( V \);

(4) for each vector \( a \) in \( V \) there is a unique vector \( -a \) in \( V \) such that \( a + (-a) = 0 \);

(iv) an operation \( \cdot \), called scalar multiplication, which associates with each scalar \( k \) in \( F \) and a vector \( a \) in \( V \) a vector \( k \cdot a \) in \( V \), called the product of \( k \) with \( a \), in such a way that

(1) \( 1 \cdot a = a \) for every \( a \) in \( V \);
\[(2) \ (k_1 k_2) \cdot a = k_1 (k_2 \cdot a); \]
\[(3) \ k \cdot (a + b) = k \cdot a + k \cdot b; \]
\[(4) \ (k_1 + k_2) \cdot a = k_1 \cdot a + k_2 \cdot a. \]

We say that \(V\) is a vector space over the field \(F\), denoted by \((V; +, \cdot)\).

By combining Smarandache multi-spaces with linear spaces, a new kind of algebraic structure called multi-vector space is found, which is defined in the following.

**Definition 1.2** Let \(\tilde{V} = \bigcup_{i=1}^{k} V_i\) be a complete multi-space with binary operation set \(O(\tilde{V}) = \{(\hat{+}_i, \cdot_i) \mid 1 \leq i \leq m\}\) and \(\tilde{F} = \bigcup_{i=1}^{k} F_i\) a multi-filed space with double binary operation set \(O(\tilde{F}) = \{(+, \times_i) \mid 1 \leq i \leq k\}\). If for any integers \(i, j\), \(1 \leq i, j \leq k\) and \(\forall a, b, c \in \tilde{V}, k_1, k_2 \in \tilde{F}\),

(i) \((V_i; +_i, \cdot_i)\) is a vector space on \(F_i\) with vector additive \(+_i\) and scalar multiplication \(\cdot_i\);

(ii) \((a +_i b) +_j c = a +_i (b +_j c)\);

(iii) \((k_1 +_i k_2) \cdot_j a = k_1 +_i (k_2 \cdot_j a)\);

if all those operation results exist, then \(\tilde{V}\) is called a multi-vector space on the multi-filed space \(\tilde{F}\) with a binary operation set \(O(\tilde{V})\), denoted by \((\tilde{V}; \tilde{F})\).

For subsets \(\tilde{V}_1 \subset \tilde{V}\) and \(\tilde{F}_1 \subset \tilde{F}\), if \((\tilde{V}_1; \tilde{F}_1)\) is also a multi-vector space, then call \((\tilde{V}_1; \tilde{F}_1)\) a multi-vector subspace of \((\tilde{V}; \tilde{F})\).

The subject of this paper is to find some characteristics of a multi-vector space. For terminology and notation not defined here can be seen in [1], [3] for linear algebraic terminologies and in [2], [4] – [11] for multi-spaces and logics.

2. Characteristics of a multi-vector space

First, we have the following result for multi-vector subspace of a multi-vector space.

**Theorem 2.1** For a multi-vector space \((\tilde{V}; \tilde{F})\), \(\tilde{V}_1 \subset \tilde{V}\) and \(\tilde{F}_1 \subset \tilde{F}\), \((\tilde{V}_1; \tilde{F}_1)\) is a multi-vector subspace of \((\tilde{V}; \tilde{F})\) if and only if for any vector additive \(+_i\), scalar multiplication \(\cdot_i\) in \((\tilde{V}_1; \tilde{F}_1)\) and \(\forall a, b \in \tilde{V}, \forall \alpha \in \tilde{F}\),

\[
\alpha \cdot a +_i b \in \tilde{V}_1
\]

if their operation result exist.

**Proof** Denote by \(\tilde{V} = \bigcup_{i=1}^{k} V_i\), \(\tilde{F} = \bigcup_{i=1}^{k} F_i\). Notice that \(\tilde{V}_1 = \bigcup_{i=1}^{k} (\tilde{V}_1 \cap V_i)\). By definition, we know that \((\tilde{V}_1; \tilde{F}_1)\) is a multi-vector subspace of \((\tilde{V}; \tilde{F})\) if and only if for any integer \(i, 1 \leq i \leq k\), \((\tilde{V}_1 \cap V_i; +_i, \cdot_i)\) is a vector subspace of \((V_i; +_i, \cdot_i)\) and \(\tilde{F}_1\) is a multi-filed subspace of \(\tilde{F}\) or \(\tilde{V}_1 \cap V_i = \emptyset\).
According to the criterion for linear subspaces of a linear space ([3]), we know that for any integer \( i, 1 \leq i \leq k \), \((\tilde{V}_i \cap V_i; \hat{+}_i; \cdot_i)\) is a vector subspace of \((V_i; \hat{+}_i; \cdot_i)\) if and only if for \( \forall a, b \in \tilde{V}_i \cap V_i, a \in F_i \),

\[
\alpha \cdot a \hat{+}_i b \in \tilde{V}_i \cap V_i.
\]

That is, for any vector additive \( \hat{+} \), scalar multiplication \( \cdot \) in \((\tilde{V}_1; \tilde{F}_1)\) and \( \forall a, b \in \tilde{V}, \forall \alpha \in \tilde{F}, if \alpha \cdot a \hat{+}_i b \) exists, then \( \alpha \cdot a \hat{+}_i b \in \tilde{V}_i \).

**Corollary 2.1** Let \((\tilde{U}; \tilde{F}_1), (\tilde{W}; \tilde{F}_2)\) be two multi-vector subspaces of a multi-vector space \((\tilde{V}; \tilde{F})\). Then \((\tilde{U} \cap \tilde{W}; \tilde{F}_1 \cap \tilde{F}_2)\) is a multi-vector space.

For a multi-vector space \((\tilde{V}; \tilde{F})\), vectors \(a_1, a_2, \cdots, a_n \in \tilde{V}\), if there are scalars \(\alpha_1, \alpha_2, \cdots, \alpha_n \in \tilde{F}\) such that

\[
\alpha_1 \cdot a_1 \hat{+}_i \alpha_2 \cdot a_2 \hat{+}_i \cdots \hat{+}_i \alpha_n \cdot a_n = 0,
\]

where \(0 \in \tilde{V}\) is an unit under an operation \(\hat{+}\) in \(\tilde{V}\) and \(1 \cdot a_i \hat{+}_i 0 \in O(\tilde{V})\), then the vectors \(a_1, a_2, \cdots, a_n\) are said to be linearly dependent. Otherwise, \(a_1, a_2, \cdots, a_n\) to be linearly independent.

Notice that in a multi-vector space, there are two cases for linearly independent vectors \(a_1, a_2, \cdots, a_n\):

(i) for any scalars \(\alpha_1, \alpha_2, \cdots, \alpha_n \in \tilde{F}\), if

\[
\alpha_1 \cdot a_1 \hat{+}_i \alpha_2 \cdot a_2 \hat{+}_i \cdots \hat{+}_i \alpha_n \cdot a_n = 0,
\]

where \(0\) is a unit of \(\tilde{V}\) under an operation \(\hat{+}\) in \(O(\tilde{V})\), then \(\alpha_1 = 0, \alpha_2 = 0, \alpha_n = 0\), where \(0 \cdot a_i \hat{+}_i 1 \leq i \leq n\) are the units under the operation \(\hat{+}\) in \(\tilde{F}\).

(ii) the operation result of \(\alpha_1 \cdot a_1 \hat{+}_i \alpha_2 \cdot a_2 \hat{+}_i \cdots \hat{+}_i \alpha_n \cdot a_n\) does not exist.

Now for a subset \(\hat{S} \subset \tilde{V}\), define its linearly spanning set \(\langle \hat{S} \rangle\) to be

\[
\langle \hat{S} \rangle = \{ a | a = a_1 \cdot a_1 \hat{+}_i a_2 \cdot a_2 \hat{+}_i \cdots \hat{+}_i a_i \cdot a_i \in \tilde{V}, a_i \in \hat{S}, a_i \in \tilde{F}, i \geq 1 \}.
\]

For a multi-vector space \((\tilde{V}; \tilde{F})\), if there exists a subset \(\hat{S} \subset \tilde{V}\) such that \(\tilde{V} = \langle \hat{S} \rangle\), then we say \(\hat{S}\) is a linearly spanning set of the multi-vector space \(\tilde{V}\). If the vectors in a linearly spanning set \(\hat{S}\) of the multi-vector space \(\tilde{V}\) are linearly independent, then \(\hat{S}\) is said to be a basis of \(\tilde{V}\).

**Theorem 2.2** Any multi-vector space \((\tilde{V}; \tilde{F})\) has a basis.

*Proof* Assume \(\tilde{V} = \bigcup_{i=1}^{k} V_i, \tilde{F} = \bigcup_{i=1}^{k} F_i\) and the basis of the vector space \((V_i; \hat{+}_i; \cdot_i)\) is \(\Delta_i = \{a_{i1}, a_{i2}, \cdots, a_{in_i}\}, 1 \leq i \leq k\). Define
\[ \hat{\Delta} = \bigcup_{i=1}^{k} \Delta_i. \]

Then \( \hat{\Delta} \) is a linearly spanning set for \( \tilde{V} \) by definition.

If vectors in \( \hat{\Delta} \) are linearly independent, then \( \hat{\Delta} \) is a basis of \( \tilde{V} \). Otherwise, choose a vector \( b_1 \in \hat{\Delta} \) and define \( \hat{\Delta}_1 = \hat{\Delta} \setminus \{ b_1 \} \).

If we have obtained the set \( \hat{\Delta}_s, s \geq 1 \) and it is not a basis, choose a vector \( b_{s+1} \in \hat{\Delta}_s \) and define \( \hat{\Delta}_{s+1} = \hat{\Delta}_s \setminus \{ b_{s+1} \} \).

If the vectors in \( \hat{\Delta}_{s+1} \) are linearly independent, then \( \hat{\Delta}_{s+1} \) is a basis of \( \tilde{V} \). Otherwise, we can define the set \( \hat{\Delta}_{s+1} \). Continue this process. Notice that for any integer \( i, 1 \leq i \leq k \), the vectors in \( \Delta_i \) are linearly independent. Therefore, we can finally get a basis of \( \tilde{V} \).

Now we consider the finite-dimensional multi-vector space. A multi-vector space \( \tilde{V} \) is finite-dimensional if it has a finite basis. By Theorem 2.2, if for any integer \( i, 1 \leq i \leq k \), the vector space \( (V_i; +, \cdot) \) is finite-dimensional, then \( (\tilde{V}; \tilde{F}) \) is finite-dimensional. On the other hand, if there is an integer \( i_0, 1 \leq i_0 \leq k \), such that the vector space \( (V_{i_0}; +_{i_0}, \cdot_{i_0}) \) is infinite-dimensional, then \( (\tilde{V}; \tilde{F}) \) is infinite-dimensional. This enables us to get the following corollary.

**Corollary 2.2** Let \( (\tilde{V}; \tilde{F}) \) be a multi-vector space with \( \tilde{V} = \bigcup_{i=1}^{k} V_i, \tilde{F} = \bigcup_{i=1}^{l} F_i. \) Then \( (\tilde{V}; \tilde{F}) \) is finite-dimensional if and only if for any integer \( i, 1 \leq i \leq k \), \( (V_i; +, \cdot) \) is finite-dimensional.

**Theorem 2.3** For a finite-dimensional multi-vector space \( (\tilde{V}; \tilde{F}) \), any two bases have the same number of vectors.

**Proof** Let \( \tilde{V} = \bigcup_{i=1}^{k} V_i \) and \( \tilde{F} = \bigcup_{i=1}^{l} F_i. \) The proof is by the induction on \( k \). For \( k = 1 \), the assertion is true by Theorem 4 of Chapter 2 in [3].

For the case of \( k = 2 \), notice that by a result in linearly vector space theory (see also [3]), for two subspaces \( W_1, W_2 \) of a finite-dimensional vector space, if the basis of \( W_1 \cap W_2 \) is \( \{ a_1, a_2, \cdots, a_t \} \), then the basis of \( W_1 \cup W_2 \) is

\[ \{ a_1, a_2, \cdots, a_t, b_{t+1}, b_{t+2}, \cdots, b_{\dim W_1}, c_{t+1}, c_{t+2}, \cdots, c_{\dim W_2} \}, \]

where, \( \{ a_1, a_2, \cdots, a_t, b_{t+1}, b_{t+2}, \cdots, b_{\dim W_1} \} \) is a basis of \( W_1 \) and \( \{ a_1, a_2, \cdots, a_t, c_{t+1}, c_{t+2}, \cdots, c_{\dim W_2} \} \) a basis of \( W_2 \).

Whence, if \( \tilde{V} = W_1 \cup W_2 \) and \( \tilde{F} = F_1 \cup F_2 \), then the basis of \( \tilde{V} \) is also

\[ \{ a_1, a_2, \cdots, a_t, b_{t+1}, b_{t+2}, \cdots, b_{\dim W_1}, c_{t+1}, c_{t+2}, \cdots, c_{\dim W_2} \}. \]

Assume the assertion is true for \( k = l, l \geq 2 \). Now we consider the case of \( k = l + 1 \). In this case, since
\[ \tilde{V} = \bigcup_{i=1}^{l} V_i \cup V_{l+1}, \quad \tilde{F} = \bigcup_{i=1}^{l} F_i \cup F_{l+1}, \]

by the induction assumption, we know that any two bases of the multi-vector space \((\bigcup_{i=1}^{l} V_i; \bigcup_{i=1}^{l} F_i)\) have the same number \(p\) of vectors. If the basis of \((\bigcup_{i=1}^{l} V_i) \cap V_{l+1}\) is \(\{e_1, e_2, \ldots, e_n\}\), then the basis of \(\tilde{V}\) is

\[ \{e_1, e_2, \ldots, e_n, f_{n+1}, f_{n+2}, \ldots, f_p, g_{n+1}, g_{n+2}, \ldots, g_{\dim V_{l+1}}\}, \]

where \(\{e_1, e_2, \ldots, e_n, f_{n+1}, f_{n+2}, \ldots, f_p\}\) is a basis of \((\bigcup_{i=1}^{l} V_i; \bigcup_{i=1}^{l} F_i)\) and \(\{e_1, e_2, \ldots, e_n, g_{n+1}, g_{n+2}, \ldots, g_{\dim V_{l+1}}\}\) a basis of \(V_{l+1}\). Whence, the number of vectors in a basis of \(\tilde{V}\) is \(p + \dim V_{l+1} - n\) for the case \(n = l + 1\).

Therefore, by the induction principle, we know the assertion is true for any integer \(k\).

The number of a finite-dimensional multi-vector space \(\tilde{V}\) is called its dimension, denoted by \(\dim \tilde{V}\).

**Theorem 2.4 (dimensional formula)** For a multi-vector space \((\tilde{V}; \tilde{F})\) with \(\tilde{V} = \bigcup_{i=1}^{k} V_i\) and \(\tilde{F} = \bigcup_{i=1}^{k} F_i\), the dimension \(\dim \tilde{V}\) of \(\tilde{V}\) is

\[ \dim \tilde{V} = \sum_{i=1}^{k} (-1)^{i-1} \sum_{\{i_1,i_2,\ldots,i_l\} \subset \{1,2,\ldots,k\}} \dim (V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_l}). \]

**Proof** The proof is by induction on \(k\). For \(k = 1\), the formula is the trivial case of \(\dim \tilde{V} = \dim V_1\). For \(k = 2\), the formula is

\[ \dim \tilde{V} = \dim V_1 + \dim V_2 - \dim (V_1 \cap \dim V_2), \]

which is true by Theorem 6 of Chapter 2 in [3].

Now assume the formula is true for \(k = n\). Consider the case of \(k = n + 1\). According to the proof of Theorem 2.15, we know that

\[ \dim \tilde{V} = \dim \bigcup_{i=1}^{n} V_i + \dim V_{n+1} - \dim \bigcup_{i=1}^{n} (V_i \cap V_{n+1}) \]

\[ = \dim \bigcup_{i=1}^{n} V_i + \dim V_{n+1} - \dim \bigcup_{i=1}^{n} (V_i \cap V_{n+1}) \]

\[ = \dim V_{n+1} + \sum_{i=1}^{n} (-1)^{i-1} \sum_{\{i_1,i_2,\ldots,i_l\} \subset \{1,2,\ldots,n\}} \dim (V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_l}) \]

\[ = \dim V_{n+1} + \sum_{i=1}^{n} (-1)^{i-1} \end{equation} \]
\[ + \sum_{i=1}^{n} (-1)^{i-1} \sum_{\{i_1,i_2,\ldots,i_l\} \subset \{1,2,\ldots,n\}} \dim(V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_l}) \]
\[ = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\{i_1,i_2,\ldots,i_k\} \subset \{1,2,\ldots,k\}} \dim(V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_k}). \]

By the induction principle, we know this formula is true for any integer \(k\).

From Theorem 2.4, we get the following additive formula for any two multi-vector spaces.

**Corollary 2.3** (additive formula) For any two multi-vector spaces \(\tilde{V}_1, \tilde{V}_2\),
\[ \dim(\tilde{V}_1 \cup \tilde{V}_2) = \dim\tilde{V}_1 + \dim\tilde{V}_2 - \dim(\tilde{V}_1 \cap \tilde{V}_2). \]

3. Open problems for a multi-ring space

Notice that Theorem 2.3 has told us there is a similar linear theory for multi-vector spaces, but the situation is more complex. Here, we present some open problems for further research.

**Problem 3.1** Similar to linear spaces, define linear transformations on multi-vector spaces. Can we establish a new matrix theory for linear transformations?

**Problem 3.2** Whether a multi-vector space must be a linear space?

**Conjecture A** There are non-linear multi-vector spaces in multi-vector spaces.

Based on Conjecture A, there is a fundamental problem for multi-vector spaces.

**Problem 3.3** Can we apply multi-vector spaces to non-linear spaces?

References


