On DM- Compact Smarandache Topological Semigroups

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Abstract

In this present paper, we have introduced some new definitions On DM-compact Smarandache topological semigroup and DM-L. compact Smarandache topological semigroup for the compactness in topological spaces and groups. We obtained some results related to DM-compact Smarandache topological semigroup and DM-L. compact Smarandache topological semigroup, for example any infinite group can be a DM- compact Smarandache topological semigroup.

Keywords: semigroups, groups, Smarandache semigroup, DM-covering, topological semigroup, topological group, direct product, isomorphism.

Mathematics subject classification: 22-XX

1. Introduction

A topological group \((G, \tau, *)\) is said to be compact topological group, if a topological space \((G, \tau)\) is a compact space [1]. Also a group \((G, *)\) is said to be D-compact group if for every D-cover group of \((G, *)\), there exists a finite sub-D-cover group of \((G, *)\) [5].

Vasantha Kandasamy [6], introduced details on a Smarandache structure on a set \(G\) means a weak structure \(W\) on \(G\), where there exists a proper subset \(H\) of \(G\) embedded with a strong structure \(S\). Here, we investigated on DM-compact Smarandache topological semigroup and DM-L. compact Smarandache topological semigroup for the compaction in topological spaces and groups, we obtain some good results related to these concepts above. Al-Khafajy [7], introduced details on D-Compact Smarandache Groupoids. Al-Khafajy and Sadek [8], studied the D-Compact Topological Groups.

Motivated by this, we introduce and study the DM-compact Smarandache topological semigroup and DM-L. compact Smarandache topological semigroup.
2. Definitions

2.1 Definition

1- We say that the triple \((G, \tau, *)\) is a topological semigroup if \((G, \tau)\) is a topological space and \((G, *)\) is a semigroup, where \(* : G \times G \to G\) is a continuous, (the set \(G \times G\) has the product topology).

2- We say that the triple \((G, \tau, *)\) is a topological monoid if \((G, \tau)\) is a topological space and \((G, *)\) is a semigroup with a unit element (monoid).

3- The topological semigroup \((G, \tau, *)\) is called topological group and denoted by \((G, \tau, *)\) if \((G, \tau)\) is a group, such that, writing \(p(x) = x^{-1}\) the inversion map \(p : G \to G\) is continuous.

4- Let \((G, \tau, *)\) be a topological semigroup (monoid, group), the topological subsemigroup (submonoid, subgroup) \((H, \tau_H, *)\) is a subset \(H\) of \(G\) with the topological and semigroup (monoid, group) structures induced from \(G\) make \(H\) a topological semigroup (monoid, group), respectively, where \(\tau_H = H \cap \tau\)

2.2 Definition

Let \((G, \tau, *)\) be a topological semigroup, \(T\) is a non empty subset of \(G\), and \(I\) be an indexed (\(I\) is a finite or an infinite set), we say that:

1- The family \(\{A_i; A_i \in \tau, \forall i \in I\}\) is a DM-covering set of \(T\) if \(T \subseteq \bigcup_{i \in I} A_i\).

2- The set \(T\) is DM- compact Smarandache set if for any DM-covering set of \(T\), there is a finite DM-subcovering set of \(T\), \(\{A_j\}_{j \in J}\), \((J\) is a finite set), such that \(T = \bigcup_{j \in J} A_j\) and \((A_j, *)\) is a group \(\forall j \in J\) under the same operation \(*\) on \(G\).

3- The set \(T\) is DM-L. compact Smarandache set if for any DM-covering set of \(T\), there is a countable DM-subcovering set of \(T\), \(\{A_s\}_{s \in S}\), \((S\) is a countable set), such that \(T = \bigcup_{s \in S} A_s\) and \((A_s, *)\) is a group \(\forall s \in S\) under the same operation \(*\) on \(G\).

2.3 Definition

Let \((G, \tau, *)\) be a topological semigroup and \(I\) be an indexed (\(I\) is a finite or an infinite set), we say that:

1- The family \(\{G_i; G_i \in \tau, \forall i \in I\}\) is DM-covering of \((G, \tau, *)\) if \(G = \bigcup_{i \in I} G_i\).

2- The topological semigroup \((G, \tau, *)\) is DM-weakly compact Smarandache topological semigroup, if there is a finite DM-covering of \((G, \tau, *)\), such that \((G_i, *)\) is a group \(\forall i \in I\) under the same operation \(*\) on \(G\).

3- The topological semigroup \((G, \tau, *)\) is DM-compact Smarandache topological semigroup if for every DM-covering of \((G, \tau, *)\) there is a finite sub-DM-covering \(\{G_j\}_{j \in J}\), \((J\) is a finite set), such that \(G = \bigcup_{j \in J} G_j\) and \((G_j, *)\) is a group \(\forall j \in J\) under the same operation \(*\) on \(G\).
4- The topological semigroup \((G,\tau,\ast)\) is DM-weakly L. compact Smarandache topological semigroup, if there is a countable DM-covering of \((G,\tau,\ast)\), such that \((G_j,\ast)\) is a group \(\forall j \in J\) under the same operation \(*\) on \(G\).

5- The topological semigroup \((G,\tau,\ast)\) is DM-L. compact Smarandache topological semigroup if for every DM-covering of \((G,\tau,\ast)\) there is a countable sub-DM-covering \(\{G_s\}_{s \in S}\). \((S\) is a countable set), such that \(G = \bigcup_{s \in S} G_s\) and \((G_s,\ast)\) is a group \(\forall s \in S\) under the same operation \(*\) on \(G\).

2.4 Definition

3Let \((G,\tau,\ast)\) and \((\bar{G},\bar{\tau},\bar{\ast})\) be two topological semigroups, we say that:
1- \(f : (G,\tau,\ast) \to (\bar{G},\bar{\tau},\bar{\ast})\) is a \textit{homomorphism} if \(f : (G,\tau) \to (\bar{G},\bar{\tau})\) is a continuous and \(f(x \ast y) = f(x) \ast f(y) \ \forall x, y \in G\).
2- \(f : (G,\tau,\ast) \to (\bar{G},\bar{\tau},\bar{\ast})\) is an \textit{isomorphism} if it is a topological homeomorphism and \(f(x \ast y) = f(x) \ast f(y) \ \forall x, y \in G\).

3 Main Results

The prove of the following lemma is direct, hence is omitted.

3.1 Lemma

Any DM- compact Smarandache topological semigroup is DM- weakly (DM-L.) compact Smarandache topological semigroup.

3.2 Theorem

Any infinite group can be a DM- compact Smarandache topological semigroup.

Proof

Let \((G,\ast)\) is any an infinite group, \(I\) is a set (finite or infinite), defined \(\tau = \{A_i \subseteq G : A_i^c\ \text{is a finite set}, (A_i,\ast) \ \text{group}\ \forall i \in I \ \& A_i \subseteq A_{i_2} \text{ for } i_1 \leq i_2 \} \cup \emptyset\).

It is clear that \(\tau \neq \emptyset\), since every finite group \(G\), \((\sigma(G) \geq 4)\), has nontrivial subgroups unless it is cyclic of prime order, but \(G\) is an infinite so \(G\) has nontrivial subgroups, [3].

It is easy to prove that \((G,\ast)\) is a topological space:
1- \(\emptyset \in \tau\) and \(G^c = \emptyset\) is a finite \(\implies G \in \tau\).
2- Let \(A_1, A_2 \in \tau\) so \(A_1^c, A_2^c\) are finite, but \((A_1 \cap A_2)^c = A_1^c \cup A_2^c \implies (A_1 \cap A_2)^c\) is finite and we know that \((A_1 \cap A_2,\ast)\) is a group \(\implies A_1 \cap A_2 \in \tau\).
3- Let \(A_s \in \tau, \forall s \in S \implies A_s^c\) is a finite \(\forall s \in S \implies \bigcap_{s \in S} A_s^c\) is a finite and \((\bigcup_{s \in S} A_s)^c = \bigcap_{s \in S} A_s^c\), and we know that \(U_{s \in S} A_s = A_t\) for some \(t\) where \(s \leq t\) \(\forall s \in S\) so \((U_{s \in S} A_s^c)\) is a group and hence \(U_{s \in S} A_s \in \tau\).

Therefore \((G,\ast)\) is a topological space. And hence \((G,\tau,\ast)\) is a topological group, which is also a topological semigroup.

Let \(\{A_j : A_j \in \tau, A \in A\}\), indexed by \(A\), be any DM-covering of \((G,\tau,\ast)\), that is \(G = \bigcup_{A \in A} A_j\). Let \(A^c = \{A_j, A_{j_2}, \ldots, A_{j_n}\}\), where \(a_j \in G\ \forall j \in J\). For

\(\text{Proof}\)

Let \((G,\ast)\) is any an infinite group, \(I\) is a set (finite or infinite), defined
\(\tau = \{A_i \subseteq G : A_i^c\ \text{is a finite set}, (A_i,\ast) \ \text{group}\ \forall i \in I \ \& A_i \subseteq A_{i_2} \text{ for } i_1 \leq i_2 \} \cup \emptyset\).

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2- Let \(A_1, A_2 \in \tau\) so \(A_1^c, A_2^c\) are finite, but \((A_1 \cap A_2)^c = A_1^c \cup A_2^c \implies (A_1 \cap A_2)^c\) is finite and we know that \((A_1 \cap A_2,\ast)\) is a group \(\implies A_1 \cap A_2 \in \tau\).
3- Let \(A_s \in \tau, \forall s \in S \implies A_s^c\) is a finite \(\forall s \in S\) so \((U_{s \in S} A_s^c)\) is a group and hence \(U_{s \in S} A_s \in \tau\).

Therefore \((G,\ast)\) is a topological space. And hence \((G,\tau,\ast)\) is a topological group, which is also a topological semigroup.

Let \(\{A_j : A_j \in \tau, A \in A\}\), indexed by \(A\), be any DM-covering of \((G,\tau,\ast)\), that is \(G = \bigcup_{A \in A} A_j\). Let \(A^c = \{A_j, A_{j_2}, \ldots, A_{j_n}\}\), where \(a_j \in G\ \forall j \in J\). For
each \( j \in J \) there is \( A_{\lambda_j} \subseteq \{A_{\lambda}\}_{\lambda \in A} \) such that
\[
a_j \in A_{\lambda_j} \Rightarrow A^{c} = \bigcup_{j \in J} A_{\lambda_j}.
\]
But \( G = A^{c} \cup A \) \( \Rightarrow G = A^{c} \cup \bigcup_{j \in J} A_{\lambda_j} \), so that there is a finite
sub-DM-covering \( \{A^{c},A_{\lambda_1},A_{\lambda_2},\ldots,A_{\lambda_n}\} \) which is \( (A^{c},*) \) and \((A_{\lambda_j},*)\) are groups for each \( j \in J \),
and therefore \((G,\tau,*)\) is a DM-compact Smarandache topological semigroup.

By Lemma 3.1 and Theorem 3.2 we can prove the following, any infinite group can be a
DM-weak (DM-L.) compact Smarandache topological semigroup.

We can prove directly, by order the group and
Lemma 3.1, the following theorem,

3.3 Theorem

Let \((G,\tau,*)\) be a topological semigroup, such that \( G \) is a finite set. Then the following are
equivalents;
1- \((G,\tau,*)\) is a DM-compact Smarandache topological semigroup.
2- \((G,\tau,*)\) is a DM-L. compact Smarandache topological semigroup.

3.4 Theorem

Let \((G,\tau,*)\) be a topological semigroup and \( A, B \subseteq G \), if \( A, B \) are DM-compact
Smarandache set. Then \( A \cup B \) is a DM-compact Smarandache set.

Proof

Let \( \{U_{i}\}_{i \in I} \) be a DM-covering set of \( A \cup B \) where \( U_{i} \subseteq \tau \), \( \forall i \in I \), so that, \( A \subseteq \bigcup_{i \in I} U_{i} \) and
\( B \subseteq \bigcup_{i \in I} U_{i} \) but \( A \) and \( B \) are DM-compact Smarandache sets, then there are finite subsets
\( J_{1}, J_{2} \subseteq I \) such that \( A \subseteq \bigcup_{i \in J_{2}} U_{i} \) and
\( B \subseteq \bigcup_{i \in J_{2}} U_{i} \) where \( (U_{i},*) \) and \((U_{i},*)\) are groups for each \( s \in J_{1} \), \( t \in J_{2} \), hence \( A \cup B \)
\( \subseteq \bigcup_{s \in J_{1}} U_{s} \cup \bigcup_{t \in J_{2}} U_{t} = \bigcup_{i \in J_{1} \cup J_{2}} U_{i} \) where
\( J_{1} \cup J_{2} \) is a finite set and \((U_{s},*)\) is a group for each \( j \in J_{1} \cup J_{2} \). Therefore \( A \cup B \) is a DM-compact Smarandache set.

The prove of the following corollary is direct
from Theorem 3.4, hence is omitted.

3.5 Corollary

Let \((G,\tau,*)\) be a topological semigroup and
\( A, B \in \tau \) such that \( A \cup B \) is a DM-compact
Smarandache set, if \((B,*)\) is group. Then \( A \) is a
DM-compact Smarandache set.

3.6 Theorem

Let \((G,\tau,*)\) be a topological semigroup and \( A, B \subseteq G \), if
1- \( A \cup B \) is a DM-compact Smarandache set,
2- \( A \) and \( B \) are disjoint open sets,
3- \( (A,*) \) and \( (B,*) \) are groups,

Then \( A \) and \( B \) are DM-compact Smarandache sets.

Proof

Let \( \{U_{i}\}_{i \in I} \) be a DM-covering set of \( A \) where
\( U_{i} \subseteq \tau \), \( \forall i \in I \) \( \Rightarrow \bigcup_{i \in I} U_{i} \subseteq A \cup B \) but
\( A \cup B \) is a DM-Smarandache compact set, so that, there is a finite subset of \( I \) such that
\( A \cup B \subseteq \bigcup_{j \in J} U_{j} \cup B \) where \((U_{j},*)\) are group
\( \forall j \in J \) \( \Rightarrow (A \cup B) \cap A \subseteq \bigcup_{j \in J} U_{j} \) \( \cap A \Rightarrow \bigcup_{j \in J} U_{j} \) \( \Rightarrow A \subseteq \bigcup_{j \in J} U_{j} \),
hence \( A \) is a DM-compact Smarandache set.

By similarity can we prove that \( B \) is a DM-compact Smarandache set.
3.7 Theorem

Let \((G, \tau, *)\) be a topological semigroup and \(A \subseteq H \subseteq G\), if \((H, *)\) is a group and \(A\) is a DM-compact Smarandache set in \((G, \tau, *)\). Then \(A\) is a DM-compact Smarandache set in \((H, \tau_H, *_H)\).

Proof

Let \(\{H_i\}_{i \in I}\) be any DM-covering set of \(A\) in \((H, \tau_H, *_H)\), (where \(\tau_H = H \cap \tau\)), that is
\[
A \subseteq \bigcup_{i \in I} H_i \quad \text{and} \quad H_i = G_i \cap H, \quad G_i \in \tau, \quad \forall i \in I \implies A \subseteq \bigcup_{i \in I} (G_i \cap H) = (\bigcup_{i \in I} G_i) \cap H \implies A \subseteq \bigcup_{i \in I} G_i \quad \text{but} \quad A \text{ is a DM-compact Smarandache set in } (G, \tau, *), \text{ so there is a finite subset } J \subseteq I \text{ such that } A = \bigcup_{j \in J} G_j \quad \text{and} \quad (G_j, *) \text{ is a group } \forall j \in J \implies A = (\bigcup_{j \in J} G_j) \cap H = \bigcup_{j \in J} (G_j \cap H), \text{ where } (G_j \cap H, *) \text{ is a group } \forall j \in J.

Therefore \(A\) is a DM-compact Smarandache set in \((H, \tau_H, *_H)\).

The prove of the following theorem is direct, hence is omitted.

3.8 Theorem

Let \((G, \tau, *)\) be a DM-compact Smarandache topological semigroup and \(H \subseteq G\), if \((H, *)\) is a subgroup of \((G, *)\). Then \((H, \tau_H, *_H)\) is a DM-compact Smarandache topological semigroup.

3.9 Corollary

Let \((G, \tau, *)\) is a DM-compact Smarandache topological semigroup and \(H_i \subseteq G\), indexed by \(I\), be any family of subset of \(G\) such that \((H_i, *)\) is a subgroup of \((G, *)\) for each \(i \in I\). Then \((\bigcap_{i \in I} H_i, \tau, *)\) is a DM-compact Smarandache topological semigroup, (where \(\tau = (\bigcap_{i \in I} H_i) \cap \tau\)).

3.10 Theorem

Let \((G, \tau, *)\) and \((\bar{G}, \bar{\tau}, \bar{*})\) are two topological semigroups, if \((G, *)\) is a group and \((\bar{G}, \bar{\tau}, \bar{*})\) is a DM-compact Smarandache topological semigroup. Then \((G \times \bar{G}, \tau \times \bar{\tau}, \emptyset)\) is a DM-compact Smarandache topological semigroup.

Proof

Let \(\{(G \times \bar{G}_i, \emptyset); \bar{G}_i \in \bar{\tau}, \forall i \in I\}\) be any DM-covering of \(G \times \bar{G} \implies G \times \bar{G} = \bigcup_{i \in I} (G \times \bar{G}_i) = G \times (\bigcup_{i \in I} \bar{G}_i) \implies \bar{G} = \bigcup_{i \in I} \bar{G}_i \text{ but } (\bar{G}, \bar{\tau}, \bar{*}) \text{ is a DM-compact Smarandache topological semigroup, so there is a finite subset } J \subseteq I \text{ such that } \bar{G} = \bigcup_{j \in J} \bar{G}_j \text{ and } (\bar{G}_j, \bar{*}) \text{ is a group } \forall j \in J.

\[
\implies G \times \bar{G} = G \times (\bigcup_{j \in J} \bar{G}_j) = \bigcup_{j \in J} (G \times \bar{G}_j)
\]

where \(G \times \bar{G}_j \in \tau \times \bar{\tau}\) and \((G \times \bar{G}_j, \emptyset)\) is a group for each \(j \in J\). Therefore \((G \times \bar{G}, \tau \times \bar{\tau}, \emptyset)\) is a DM-compact Smarandache topological semigroup.

3.11 Theorem

Let \((G, \tau, *)\) and \((\bar{G}, \bar{\tau}, \bar{*})\) be two DM-compact Smarandache topological semigroups. Then \((G \times \bar{G}, \tau \times \bar{\tau}, \emptyset)\) is a DM-compact Smarandache topological semigroup.

Proof

Let \((G, \tau, *)\) and \((\bar{G}, \bar{\tau}, \bar{*})\) are two DM-compact Smarandache topological semigroup \(\Rightarrow \) there exists a DM-covering \(\{G_a\}_{a \in A}\) and \(\{\bar{G}_b\}_{b \in B}\) of \(G\) and \(\bar{G}\), respectively, \(\Rightarrow G \times \bar{G} = (\bigcup_{a \in A} G_a) \times (\bigcup_{b \in B} \bar{G}_b) = \bigcup_{a \in A, b \in B} (G_a \times \bar{G}_b) \Rightarrow (G_a \times \bar{G}_b)_{a \in A, b \in B}\) is a DM-covering of \((G \times \bar{G}, \tau \times \bar{\tau}, \emptyset)\).

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4 See [4]
Let \( \{ \mathcal{W}_i \}_{i \in I} \) be any DM-covering of \( (G \times G, \tau \times \bar{\tau}, \emptyset) \Rightarrow G \times G = \bigcup_{i \in I} \mathcal{W}_i \), such that \( \mathcal{W}_i = \mathcal{U}_i \times \mathcal{V}_i \), where \( \mathcal{U}_i \in \tau \), \( \mathcal{V}_i \in \bar{\tau} \) for each \( i \in I \). But \((G, \tau, *)\) is a DM-compact Smarandache topological semigroup, so there is a finite subset of \( I \) such that \( G = \bigcup_{j \in J} \mathcal{U}_j \) and \((\mathcal{U}_j, *)\) is a group for each \( j \in J \).

Let \( \mathcal{U}_j \in \{ \mathcal{U}_j \}_{j \in J} \Rightarrow \{ \mathcal{U}_j \times \mathcal{V}_j \}_{i \in I} \) is a DM-covering of \( \mathcal{U}_j \times \bar{\mathcal{G}} = \bigcup_{i \in I} (\mathcal{U}_j \times \mathcal{V}_i) \), but \( \mathcal{U}_j \times \bar{\mathcal{G}} \) is a DM-compact Smarandache topological semigroup from Theorem 3.11 since \((\mathcal{U}_j, *)\) is a group and \((\bar{\mathcal{G}}, \bar{\tau})\) is a DM-compact Smarandache topological semigroup, so there is a finite set \( S \subseteq I \) such that \( \{ \mathcal{U}_j \times \mathcal{V}_j \}_{s \in S} \) is a group \( \forall s \in S \) and \( \mathcal{U}_j \times \bar{\mathcal{G}} = \bigcup_{s \in S} (\mathcal{U}_j \times \mathcal{V}_s) \)

\[ = \mathcal{U}_j \times (\bigcup_{s \in S} \mathcal{V}_s) \Rightarrow \mathcal{U}_j \times (\bigcup_{s \in S} \mathcal{V}_s) \]

\[ = (\bigcup_{j \in J} \mathcal{U}_j) \times (\bigcup_{s \in S} \mathcal{V}_s) = G \times \bar{\mathcal{G}} \Rightarrow G \times \bar{\mathcal{G}} \]

\[ (\bigcup_{j \in J} \mathcal{U}_j) \times (\bigcup_{s \in S} \mathcal{V}_s) = \bigcup_{s \in S} (\mathcal{U}_j \times \mathcal{V}_s), \]

where \( \bigcup_{j \in J} \mathcal{U}_j \times \mathcal{V}_s \) are groups for each \( j \in J \), \( s \in S \). Therefore \((G \times G, \tau \times \bar{\tau}, \emptyset)\) is a DM-compact Smarandache topological semigroup.

The prove of the following corollary is direct, hence is omitted.

3.12 Corollary

Let \((G, \tau, *)\) is a DM-compact Smarandache topological semigroup and \( H, S \) are two subsets of \( G \). Then \( H \times S \) is a DM-compact Smarandache set in \((G \times G, \tau \times \bar{\tau}, \emptyset)\).

3.13 Theorem

Let \( \{ G_i : i \in I \} \) be a family of topological groups. Then the direct \( G = \prod_{i \in I} G_i \), equipped with the product topology is a topological group.

From Theorem 3.11 and Theorem 3.13, respectively, and by induction we can prove the following theorem;

3.14 Theorem

If \((G, \tau, *)\) is a DM-compact Smarandache topological semigroup, then \((G^n, \tau^n, \emptyset)\) is a DM-compact Smarandache topological semigroup, where \((x \otimes y) = (x_1 * y_1, ..., x_n * y_n)\) for each \( x_i, y_i \in G \), \( i = 1, 2, \ldots, n \).

3.15 Theorem

The product of any finite collection of DM-compact Smarandache topological semigroups is a DM-compact Smarandache topological semigroup.

The following corollary is direct from Corollary 3.12 and Theorem 3.15;

3.16 Corollary

Suppose \( I \) is non-empty set and \((G_i, \tau_i, *)\) is a DM-compact Smarandache topological semigroups for each \( i \in I \), if \( H_i \) is a subset of \( G_i \), \( \forall i \in I \). Then \( \prod_{i \in I} H_i \) is a DM-compact Smarandache set in \( \prod_{i \in I} G_i, S, (\emptyset) \), where \( S = \tau \prod_{i \in I} G_i \) the usual product topology.

3.17 Theorem

Let \((G, \tau, *)\) and \((\bar{G}, \bar{\tau}, \bar{*})\) be two topological semigroups and \( f : (G, \tau, *) \rightarrow (\bar{G}, \bar{\tau}, \bar{*}) \) is an isomorphism. Then

\[ \text{See [4]} \]

\[ \text{proposition 3.3.4., p.18, [1].} \]
1- If \( A \) is a DM- compact Smarandache set in \((G, \tau, *)\) \( \Rightarrow f(A) \) is a DM- compact Smarandache set in \((\bar{G}, \bar{\tau}, \bar{*})\).

2- If \( B \) is a DM- compact Smarandache set in \((\bar{G}, \bar{\tau}, \bar{*})\) and \( f \) is an open map \( \Rightarrow f^{-1}(B) \) is a DM- compact Smarandache set in \((G, \tau, *)\).

**Proof**

1- Let \( \{G_i\}_{i \in I} \) be any DM-covering set of \( f(A) \) in \((\bar{G}, \bar{\tau}, \bar{*})\) that is \( f(A) \subseteq \bigcup_{i \in I} G_i \) \( \Rightarrow A \subseteq f^{-1}(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} f^{-1}(G_i) \), it is clear that \( f^{-1}(G_i) \) is a group so \( f^{-1}(G_i) \in \tau \) \( \forall i \in I \) since \( G_i \in \bar{\tau} \) for each \( i \in I \) and \( f \) is continuous , but \( A \) is a DM- compact Smarandache set in \((G, \tau, *)\), so there is a finite subset of \( I \) such that

\[
A = \bigcup_{i \in J} f^{-1}(G_i) \text{ and } (f^{-1}(G_i), *) \text{ is a group}
\]

\( \forall j \in J \) \( \Rightarrow A = f^{-1}(\bigcup_{j \in J} G_j) \) \( \Rightarrow f(A) = f\left(f^{-1}(\bigcup_{j \in J} G_j)\right) = \bigcup_{j \in J} G_j \text{ where } (G_j, \bar{*}) \text{ is a group} \)

\( \forall j \in J \) since \( f \) is an isomorphism \( \Rightarrow f(A) \) is a DM- compact Smarandache set in \((\bar{G}, \bar{\tau}, \bar{*})\).

2- Let \( \{G_i\}_{i \in I} \) be any DM-covering set of \( f^{-1}(B) \) in \((G, \tau, *)\) \( \Rightarrow f^{-1}(B) \subseteq \bigcup_{i \in I} G_i \) \( \forall i \in I \) \( \Rightarrow B \subseteq f(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} f(G_i) \), it is clear that \( f(G_i) \) is an open map , but \( B \) is a DM- compact Smarandache set in \((\bar{G}, \bar{\tau}, \bar{*})\), so there is a finite subset \( J \subseteq I \) such that \( B = \bigcup_{j \in J} f(G_j) \) where \( (f(G_j), \bar{*}) \) is a group

\( \forall j \in J \) \( \Rightarrow B = f(\bigcup_{j \in J} G_j) \Rightarrow f^{-1}(B) = \bigcup_{j \in J} G_j \),

where \( (G_j, \bar{*}) \) is a group \( \forall j \in J \) since \( f \) is an isomorphism \( \Rightarrow f^{-1}(B) \) is a DM- compact Smarandache set in \((G, \tau, *)\).

3.18 Theorem

Let \((G, \tau, *)\) and \((\bar{G}, \bar{\tau}, \bar{*})\) be two topological semigroups and \( f : (G, \tau, *) \rightarrow (\bar{G}, \bar{\tau}, \bar{*}) \) is an isomorphism. Then the following are equivalents;

1-\((G, \tau, *)\) is a DM-compact Smarandache topological semigroup.

2-\((\bar{G}, \bar{\tau}, \bar{*})\) is a DM- compact Smarandache topological semigroup.

**Proof**

(\(\Rightarrow\)) Suppose that \((G, \tau, *)\) is a DM- compact Smarandache topological semigroup , let \( \{G_i ; G_i \in \bar{\tau}, \forall i \in I\} \) be any DM- covering of \((\bar{G}, \bar{\tau}, \bar{*})\) \( \Rightarrow \bar{G} = \bigcup_{i \in I} \bar{G}_i \Rightarrow \bar{G} = f^1(\bar{G}) = f^{-1}(\bigcup_{i \in I} \bar{G}_i) \Rightarrow \bar{G} = \bigcup_{i \in I} f^{-1}(\bar{G}_i) \) be any DM- compact Smarandache topological semigroup, so there is a finite subset of \( I \) such that \( G = \bigcup_{i \in J} f^{-1}(G_i) \) \( \Rightarrow f^{-1}(\bigcup_{i \in J} \bar{G}_i) \Rightarrow \bar{G} = f(G) = f(f^{-1}(\bigcup_{j \in J} \bar{G}_j)) = \bigcup_{j \in J} \bar{G}_j \),

where \( (G_j, \bar{*}) \) is a group \( \forall j \in J \). Therefore \((\bar{G}, \bar{\tau}, \bar{*})\) is a DM- compact Smarandache topological semigroup.

(\(\Leftarrow\)) Suppose that \((\bar{G}, \bar{\tau}, \bar{*})\) is a DM- compact Smarandache topological semigroup, let \( \{G_i ; G_i \in \tau, \forall i \in I\} \) be any DM-covering of \((G, \tau, *)\) \( \Rightarrow \bar{G} = \bigcup_{i \in I} \bar{G}_i \Rightarrow \bar{G} = f(G) = f(\bigcup_{i \in I} G_i) \Rightarrow \bar{G} = \bigcup_{i \in I} f(G_i) \), but \((\bar{G}, \bar{\tau}, \bar{*})\) is a DM- compact Smarandache topological semigroup, so there is a finite subset of \( I \) such that

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7 See [4]
that $\mathcal{G} = \bigcup_{j \in J} f(G_j)\,$ and $(f(G_j), \bar{x})$ is a group

\[ \forall j \in J \Rightarrow \mathcal{G} = f(\bigcup_{j \in J} G_j) = G = f^{-1}(\mathcal{G}) = f^{-1}\left(f(\bigcup_{j \in J} G_j)\right) = \bigcup_{j \in J} G_j \],

where $(G_j, \ast)$ is a group $\forall j \in J$.

Therefore $(G, \cdot, \ast)$ is a DM-compact Smarandache topological semigroup.

References

حوالي تراص سمارانداش لشيء الزمزم التوبولوجية من نوع - DM

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قدمنا في هذا البحث بعض التعريف الجديد حول تراص سمارانداش لشيء الزمزم التوبولوجية من نوع - DM و تراص سمارانداش لشيء الزمزم التوبولوجية من نوع - DM-L. وهي ترتبط الفضاءات التوبولوجية ونظرية الزمزم. حصلنا على بعض النتائج تتعلق بتراص سمارانداش لشيء الزمزم التوبولوجية من نوع - DM و تراص سمارانداش لشيء الزمزم التوبولوجية من نوع - DM-L. ومنها كل زمزم غير منتهية ممكن تكون تراص سمارانداش لشيء زمزم توبولوجية من نوع - DM.