ON SOME CHARACTERIZATION OF SMARANDACHE - BOOLEAN NEAR - RING WITH SUB-DIRECT SUM STRUCTURE

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Abstract In this paper, we introduced Samarandache-2-algebraic structure of Boolean-near-ring namely Smarandache-Boolean-near-ring. A Samarandache-2-algebraic structure on a set N means a weak algebraic structure $S_1$ on N such that there exist a proper subset M of N, which is embedded with a stronger algebraic structure $S_2$, stronger algebraic structure means satisfying more axioms, that is $S_1 << S_2$, by proper subset one understands a subset different from the empty set, from the unit element if any, from the whole set [3]. We define Smarandache-Boolean-near-ring and obtain the some of its characterization through Boolean-ring with sub-direct sum structure. For basic concept of near-ring we refer to G.Pilz [11].

Keywords Boolean-ring, Boolean-near-ring, Smarandache-Boolean-near-ring, Compatibility, Maximal set, Idempotent and uni-element.

§1. Preliminaries

Definition 1.1. A (Left) near ring $A$ is a system with two Binary operations, addition and multiplication, such that

(i) the elements of $A$ form a group $(A, +)$ under addition,
(ii) the elements of $A$ form a multiplicative semi-group,
(iii) $x(y+z) = xy+xz$, for all $x, y$ and $z \in A$.

In particular, if $A$ contains a multiplicative semi-group $S$ whose elements generates $(A, +)$ and satisfy,
(iv) $(x+y)s = xs+ys$, for all $x,y \in A$ and $s \in S$, then we say that $A$ is a distributively generated near-ring.

Definition 1.2. A near-ring $(B, +, \cdot)$ is Boolean-near-ring if there exists a Boolean-ring $(A, +, \land, 1)$ with identity such that $\cdot$ is defined in terms of $+, \land$ and 1, and for any $b \in B$, $b\cdot b = b$.

Definition 1.3. A near-ring $(B, +, \cdot)$ is said to be idempotent if $x^2 = x$, for all $x \in B$.

Definition 1.4. Compatibility $a \in b$ (ie "a is compatibility to b") if $ab^2 = a^2b$.

Definition 1.5. Let $A = \{a, b, c, \ldots\}$ be a set of pairwise compatible elements of an associate ring $R$. Let $A$ be maximal in the sense that each element of $A$ is compatible with
every other element of A and no other such elements may be found in R. Then A is said to be a maximal compatible set or a maximal set.

**Definition 1.6.** If a sub-direct sum R of domains has an identity, and if R has the property that with each element a, it contains also the associated idempotent a$^0$ of a, then R is called an associate subdirect sum or an associate ring.

**Definition 1.7.** If the maximal set A contains an element u which has the property that a$< u$, for all a$\in A$, then u is called the uni-element of A.

**Definition 1.8.** Left zero divisors are right zero divisors, if ab=0 implies ba=0.

**Definition 1.9.** A Boolean- near- ring B is said to be Samarandache- Boolean- near-ring whose proper subset A is a Boolean-ring with respect to same induced operation of B.

**Theorem 1.1.** A Boolean-near-ring (B,$\lor$,$\land$) is having the proper subset A, is a maximal set with uni-element in an associate ring R, with identity under suitable definitions for (B,+,.)

\[ a \lor b = a + b - 2a^0 b \]
\[ a \land b = a \cap b = a^0 b = ab^0. \]

Then B is a Smarandache-Boolean-near-ring.

**Proof.** Given (B,$\lor$,$\land$) is a Boolean-near- ring whose proper subset (A,$\lor$,$\land$) is a maximal set with uni-element in an associate ring R, and if the maximal set A is also a subset of B.

Now to prove that B is Smarandache-Boolean-near-ring. It is enough to prove that the proper subset A of B is a Boolean-ring. Let a and b be two constants of A : If a is compatible to b, we define a$\land$b as follows :

If a$^i$ = b$^i$ in the i-component, let (a$\land$b)$^i$=0$^i$

If a$^i$ $\neq$ b$^i$, then since a$\sim$b precisely one of these is zero.

If a$^i$=0, let (a$\land$b)$^i$ = b$^i$ $\neq$0;

If b$^i$=0, let (a$\land$b)$^i$ = a$^i$ $\neq$0

It is seen that if a$\land$b belongs to the associate ring R, then a$\land$b $< u$, where u is the uni-element of A, and therefore,a$\land$b$\in A$

Consider a$\land$b = a + b - 2ab$^0$

If in the i-component, 0$\neq a$-b, then since (a$^0$)$^i$=1$^i$, whence,

we have (a+b-2ab$^0)$) =0$^i$ and,

If 0$^i = a$ = b$^i$, then (a$^0$)$^i$ = 0 and (b$^0$)$^i$ =1, whence,

(a+b-2ab$^0$)$^i$ =b$^i$

If a$^i$ $\neq$ 0 and b$^i$=0 then (a+b-2ab$^0$) = 0$^i$

Therefore a$\land$b$\notin A$, the maximal set.

Similarly, the element a$\land$b = a$\lor$b = a$^0$b = ab$^0$ = glb(a,b) has defined and shown to belong to A as the glb (a,b) Now let us show that (A,$\lor$,$\land$) is a Boolean - ring: Firstly, a$\lor$a = 0, since a$^i=a$ in every i-component, whence (a$\lor$ a)$^i$ vanishes, by our definition of $\lor$. Secondly a$\land$a =
a\cap a = a^0a = a$, and so $a$ is idempotent under $\land$. We shown that $A$ is closed under $\land$ is $\lor$. And associativity is a direct verification, and each element is itself inverse under $\land$.

To prove associativity under $\land$:

For $a \land (b \land c) = a^0(b \land c)$

$= a^0(b^0c)$

$= a^0(bc^0)$

$= (a \land b)^0c = (a \land b) \land c$

$\Rightarrow a \land (b \land c) = (a \land b) \land c$, for all $a, b, c \in R$

For distributivity of $\land$ over $\lor$,

Let $c$ be an arbitrary in $A$

Now $c \land (a \lor b) = c^0(a \lor b)$

$= c^0(a \lor b) - c^0(a \land b)$

$= c^0a + c^0b - c^0a^0b$

$= (c \land a) \lor (c \land b)$

$\Rightarrow c \land (a \lor b) = (c \land a) \lor (c \land b)$

Hence $(A \lor, \land)$ is a Boolean-ring.

Theorem 1.2. A Boolean-near-ring $(B, \lor, \land)$ is having the proper subset $(A, +, \land, 1)$ is an associate ring in which the relation of compatibility is transitive for non-zero elements with identity under suitable definitions for $(B, +, \cdot)$ with corresponding lattices $(A, \leq)$ $(A, <)$ and

\[
\begin{align*}
a \lor b &= a + b - 2a^0b = (a \lor b) - (a \land b) \\
a \land b &= a \land b = a^0b = ab^0.
\end{align*}
\]

Then $B$ is a Smarandache-Boolean-near-ring.

**Proof.**

Assume that $(B, +, \cdot)$ be Boolean-near-ring having a proper subset $A$ is an associate ring in which the relation of compatibility is transitive for non-zero elements.

Now to prove that $B$ is a Smarandache-Boolean-near-ring.

(i.e) to prove that if the proper subset of $B$ is a Boolean-ring, then by definition $B$ is Smarandache-Boolean-near-ring. we have $0$ is compatible with all elements, whence all elements are compatible with $A$ and therefore, are idempotent.

Then assume that transitivity holds for compatibility of non-zero elements. It follows that non-zero elements from two maximal sets cannot be compatible (much less equal), and hence, except for the element $0$, the maximal sets are disjoint.

Let $a$ be an arbitrary, non-zero element of $R$. If $a$ is a zero-divisor of $R$, then the idempotent element $A-a^0 \neq 0$.

Further $A-a^0$ belongs to the maximal set generated by the non-zero divisor $a' = a + A-a^0$,
since it is \((A-a^0)a' = (A-a^0)(a+A-a^0)\)
\[= (A-a^0) = (A-a^0)^2\]
(ie) \(A-a^0 < a'\). Since also \(a < a'\) and \(a \sim A - a^0\) Therefore, \(a\) is idempotent.
(ie) All the zero-divisors of \(R\) are idempotent which is a maximal set then by theorem 1 and by definition \(A\) is a Boolean-ring. Then by definition, Therefore \(B\) is Smarandache-Boolean-
near-ring.

**Theorem 1.3.**

A Boolean-near-ring \((B, \lor, \land)\) is having the proper subset \(A\), the set \(A\) of idempotent elements of a ring \(R\), with suitable definitions for \(\lor\) and \(\land\),
\[a \lor b = a + b - 2a^0b = (a \lor b) - (a \land b)\]
\[a \land b = a \land b = a^0b = ab^0.\]

Then \(B\) is a Smarandache-Boolean-near-ring.

**Proof.**

Assume that the set \(A\) of idempotent elements of a ring \(R\), which is also a subset of \(B\). Now to prove that \(B\) is a Smarandache-Boolean-near-ring. It is sufficient to prove that the set \(A\) of idempotent elements of a ring \(R\) with identity forms a maximal set in \(R\) with uni-element.

By the definition of compatible, then we have every element of \(R\) is compatible with every other idempotent element.

If \(a \in R\) is not idempotent then,
\[a^2.1 \neq a.1^2,\] since the definition of compatible. Hence no non-idempotent can belong to this maximal set. Thus the set \(A\) is idempotent element of \(R\) with identity forms a maximal set in \(R\) whose uni-element is the identity of \(R\), by theorem 1 and by definition. \(A\), a maximal set of \(B\) forms a Boolean ring

Then by definition

It conclude that \(B\) is Smarandache-Boolean-near-ring.

**Theorem 1.4.**

A Boolean-near-ring \((B, \lor, \land)\) is having the proper subset having a non-zero divisor of \(A\), as an associate ring. with suitable definitions for \(\lor\) and \(\land\),
\[a \lor b = a + b - 2a^0b = (a \lor b) - (a \land b)\]
\[a \land b = a \land b = a^0b = ab^0.\]

Then \(B\) is a Smarandache-Boolean-near-ring.

**Proof.**

Let \(B\) is Boolean-near-ring whose proper subset having a non-zero divisor of associate ring \(A\).

Now to prove that \(B\) Smarandache-Boolean-near-ring.
It is enough to prove that every non-divisor of \(A\) determines uniquely a maximal set of \(A\) with uni-element.

Let \(a\) be the uni-element of a maximal set \(A\) then we have \(b < a\), for \(b \in A\)
Consider all the elements of \(A\) in whose sub-direct display one or more component \(ai\) duplicate the corresponding component \(ui\) of \(u\), the other components of \(a\) being zeros.
(ie) all the element \(a\) such that \(a < u\), becomes \(u\) is uni-element.

Clearly, these elements are compatible with each other and together with \(u\) form a maximal set
in A, for which u is the uni-element.
Hence A is a maximal set with uni-element and by theorem 1 and definition A, a maximal set of B forms a Boolean ring.

Then by definition Therefore B is Smarandache-Boolean-near-ring.

**Theorem 1.5.**

A Boolean-near-ring \((B, \lor, \land)\) is having the proper subset A, associate ring is of the form 
\[A = uJ,\]
where u is a non-zero of A and J is the set of idempotent elements of A, with suitable definitions for \(\lor\) and \(\land\),

\[a \lor b = a + b - 2a^0b = (a \cup b) - (a \cap b)\]

\[a \land b = a \cap b = a^0b = ab^0.\]

Then B is a Smarandache-Boolean-near-ring.

**Proof.**

Assume that the proper subset A of a Boolean-near-ring B is of the form \(A = uJ\), where u is non-zero divisor of A and J is the set of idempotent elements of A.

Now to prove B is Smarandache-Boolean-near-ring.
It is enough to prove that A is a maximal set with uni-element.

(i) It is sufficient to show that the set \(uJ\) is a maximal set having u as its uni-element and

(ii) If b belongs to the maximal set determined by u, then b has the required form \(b = eu\), for some \(e \in J\)

**Proof of (i)** It is seen that \(ue \sim uf\) (ie) \(ue\) is compatible to \(uf\) with uni-element u, for all \(e, f \in J\), since idempotent belongs to the center of A. Also, \(ue < u\) and \(ue.u = u^2e = (ue)^2\)

**Proof of (ii)** We know that A is an associate ring, the associated idempotent \(a^0\) of a has the property:

If \(a \sim b\) then \(a0b = ab^0 = b^0a = ba^0\)
If \(a \in A_u\) then since \(a < u\) and \(u^0 = 1\),

we have \(A = u^0a = au^0 = a^0u\), for all \(a^0 \in J\)

Hence A is a maximal set with uni-element of of B by suitable definition and by theorem 1 then we have A is a Boolean-ring.

Then by definition, 
Hence B is Smarandache-Boolean-near-ring.

**References**


