ON THE SMARANDACHE RECIPROCAL PARTITION SETS

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ABSTRACT

If the sum of the reciprocals of some distinct integer numbers is unity, then the set of those integers is called the Smarandache distinct reciprocal partition of unity. A prior research introduced this concept and denoted the Smarandache distinct reciprocal partition of unity in \( n \) partitions by SDRPS(\( n \)). This study has developed some methods to find out the Smarandache distinct reciprocal partition of unity, especially SDRPS(3) and SDRPS(4).

Keywords: Diophantine equation, Smarandache reciprocal partition of unity

INTRODUCTION

Murthy (2000) introduced the idea of the sets of Smarandache reciprocal partition of unity, which was studied in a systematic manner by Murthy and Ashbacher (2005). We start with the definition below, due to Murthy (2000).

**Definition 1:** The Smarandache distinct reciprocal partition of unity in \( n \) partitions, is denoted by SDRPS(\( n \)), and is defined by

\[
\text{SDRPS}(n) = \{(a_1, a_2, \ldots, a_n); 0 < a_1 < a_2 < \ldots < a_n; \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1\}.
\]

The order of the set SDRPS(\( n \)) is denoted by \( f_{DR}(n) \).

Note that, the \( n \) (distinct) integers \( a_1, a_2, \ldots, a_n \) can always be rearranged, if necessary, to satisfy the condition that \( a_1 < a_2 < \ldots < a_n \). Thus,

\[
\text{SDRPS}(2) = \{(a, b); \frac{1}{a} + \frac{1}{b} = 1 \text{ and } a < b\}.
\]

Obviously, SDRPS(2) = \( \emptyset \) (the empty set).

The main objective of this study is to develop the method to calculate SDRPS(\( n \)). This paper also derives the expressions of SDRPS(3) and SDRPS(4). These are done in results and discussion part. Background materials and the method of this study are included in the materials and methods section. Conclusions portion concludes the research output and limitation, and provides the future research direction.

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MATERIALS AND METHODS

**Lemma 1:** In the set SDRPS(n), \( n \geq 3, \ a_1 \leq n - 1, \ a_n \) is not a prime.

**Proof:** Let, on the contrary \( a_1 \geq n \), so that
\[
n \leq a_1 < a_2 < \ldots < a_n.
\]
But then
\[
\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < 1,
\]
which contradicts the condition of the definition.

Next, let \( a_n = p \), where \( p \) is a prime. Letting
\[
\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{n-1}} = \frac{A}{a_1 a_2 \ldots a_{n-1}}
\]
We get
\[
pA = (p-1) a_1 a_2 \ldots a_{n-1},
\]
and we reach to a contradiction, since none of \( a_1, a_2, \ldots, a_{n-1} \) is divisible by \( p \).

Thus, instudying SDRPS(n), it is sufficient to consider the case when \( n \geq 3 \). This background materials and deep thinking along with the literature review provide some methods to calculate SDRPS(n) given in the results and discussion section.

RESULTS AND DISCUSSION

This results and discussion section is divided into two sub sections namely “Main Results” and “Remarks”.

**Main Results**

First, we prove the following results.

**Lemma 2:** For \( n \geq 3 \), there always exist integers \( a_1, a_2, \ldots, a_n \) satisfying the condition
\[
2 = a_1 < a_2 < \ldots < a_n
\]
such that
\[
\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.
\]

**Proof:** The proof is by induction on \( n \).

When \( n = 3 \), choosing
\[
a_2 = 3, \ a_3 = 6,
\]
we get
\[
\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 1.
\]
Thus, the result is true for \( n = 3 \). To proceed by induction, we assume that the result is true for some integer \( n \), that is, we assume that there are \( n \) integers \( 2 = a_1, a_2, \ldots, a_n \) with
\[
a_1 < a_2 < \ldots < a_n
\]
such that
To prove the result for $n+1$, we define the integers $b_1, b_2, \ldots, b_n, b_{n+1}$ as follows:

$$b_1 = a_1, \quad b_2 = a_2, \quad \ldots \quad b_{n-1} = a_{n-1}, \quad b_n = a_n + 1, \quad b_{n+1} = a_n(a_{n+1}).$$

Clearly, $b_1 < b_2 < \ldots < b_n < b_{n+1}$; moreover, $b_1, b_2, \ldots, b_n, b_{n+1}$ satisfy the condition:

$$\frac{1}{b_1} + \frac{1}{b_2} + \ldots + \frac{1}{b_n} + \frac{1}{b_{n+1}} = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.$$

This proves the validity of the result for $n+1$, thereby establishing the lemma.

The result in Lemma 2 is interesting. It proves that, for any set $SDRPS(n)$ for $n \geq 3$, we may have $a_1 = 2$. The proof of Lemma 2 gives a procedure of obtaining an element of the set $SDRPS(n+1)$, starting with the element $(a_1, a_2, \ldots, a_n)$ of $SDRPS(n)$.

To find $SDRPS(3)$, we first note that, by Lemma 1, we must have $a_1 = 2$. We now prove the following result.

**Lemma 3:** $a=3, \ b=6$ is the only solution of the Diophantine equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2}; \quad 0 < a < b. \quad (1)$$

**Proof:** We recast the equation in (1) in the following equivalent form:

$$2(a+b) = ab.$$  

Then, $a$ must divide $b$, so that

$$b = ka$$

for some integer $k \geq 2$.

Therefore, we get

$$2(1+k) = ka.$$  

Then, $a$ must divide $1+k$ (since $k$ does not divide $1+k$).

Now, when $k=2, a=k+1=3$ (and $b=6$), which we intended to prove.

Lemma 3, together with the fact that in $SDRPS(3)$ $a_1$ cannot be greater than 2, proves the lemma below.

**Lemma 4:** $SDRPS(3) = \{(2, 3, 6)\}$ is the singleton set.

Next, we find $SDRPS(4)$. To do so, we need some preliminary results given below.

**Lemma 5:** The only solutions of the Diophantine equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{6}; \quad 0 < a < b, \quad (2)$$

are (i) $a=7, \ b=42$, (ii) $a=8, b=24$, and (iii) $a=9, b=18$. 

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**Proof:** We rewrite (2) as

$$6(a+b)=ab.$$  

Now, $a$ must divide $b$, say

$$b=ka$$

for some integer $k \geq 2$.

Then

$$6(1+k)=ka.$$  

Thus, $a$ must divide $1+k$ (since $k$ does not divide $1+k$), and $k$ must divide 6.

Now, when $k=2$, $a=3(k+1)=9$ (and $b=18$), when $k=3$, then $a=2(1+k)=8$ (and $b=24$), and when $k=6$, $a=1+k=7$ (so that $b=42$). Hence, we get the desired result.

**Lemma 6:** The only solutions of the Diophantine equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{4} ; \ 0 < a < b, \quad (3)$$

are (i) $a=5$, $b=20$, and (ii) $a=6, b=12$.

**Proof:** Rewriting (3) as

$$6(a+b)=ab,$$

We see that, $a$ must divide $b$, so that

$$b=ka$$

for some integer $k \geq 2$.

Then

$$4(1+k)=ka,$$

so that $a$ must divide $1+k$, and 4 must be divisible by $k$.

Then, the desired solutions correspond to $k=2$ and $k=4$ respectively.

**Lemma 7:** The following Diophantine equation has no solution:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2} ; \ 5 \leq a < b < c.$$  

**Proof:** First, let $a=5$. Then,

$$\frac{1}{6} + \frac{1}{7} > \frac{3}{10}.$$  

Since

$$\frac{1}{6} + \frac{1}{8} < \frac{3}{10},$$

it follows that $a \neq 5$. So, let $a \geq 6$. But then, for $c>b>6$,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{6} + \frac{1}{7} + \frac{1}{8} < \frac{3}{10}.$$  

All these complete the proof of the lemma.

**Lemma 8:** The Diophantine equation below has no solution.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{2}{3} ; \ 4 \leq a < b < c.$$
Proof: Since, for \( c > b > a \geq 4 \),

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{4} + \frac{1}{5} + \frac{1}{6} < \frac{2}{3},
\]

the result follows.

The set SDRPS(4) is given in Lemma 9.

Lemma 9: SDRPS(4) is given as follows:

\[
SDRPS(4) = \{(2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15), (2, 4, 5, 20), (2, 4, 6, 12)\}.
\]

Proof: To find SDRPS(4), first note that, by Lemma 1, \( a_1 \) is either 2 or 3. By Lemma 7, \( a_1 \neq 3 \).

Hence, \( a_1 \) must be 2. When \( a_1 = 2 \), by Lemma 7, \( a_2 \) cannot be greater than 5. Thus, the only possible values of \( a_2 \) are \( a_2 = 3, 4 \).

With \( a_1 = 2, a_2 = 3 \), by Lemma 5, there are three solutions, and with \( a_1 = 2, a_2 = 4 \), by Lemma 6, there are two solutions.

It may be mentioned here that, in Murthy and Ashbacher (2005), SDRPS(3) is given without the proof that it is, in fact, a singleton set; also, only 5 elements of SDRPS(4) are listed there. We prove in Lemma 8 that SDRPS(4) has precisely 6 elements.

Remarks

Given the set SDRPS(\( n \)) (with \( f_{DP}(n) \) elements), Murthy and Ashbacher (2005) considered the problem of extending it to get some of the elements of the set SDRPS(\( n+1 \)). In this context, Murthy and Ashbacher (2005) suggest different methods. One such method is stated in Lemma 10, which is, in fact, due to Maohua (2001).

Lemma 10: Let \( (a_1, a_2, \ldots, a_n) \in SDRPS(n) \). Then, \( (2, 2a_1, 2a_2, \ldots, 2a_n) \in SDRPS(n+1) \).

Proof: Simply, we can prove it by showing the sum of the reciprocals of \( 2, 2a_1, 2a_2, \ldots, 2a_n \) equal to 1.

We can also prove Lemma 11, which may be employed to find an element of the set SDRPS(\( n+2 \)), starting with an element of SDRPS(\( n \)).

Lemma 11: Let \( (a_1, a_2, \ldots, a_n) \in SDRPS(n) \). Then,

\[
(2, 3, 6a_1, 6a_2, \ldots, 6a_n) \in SDRPS(n+2).
\]

Proof: By assumption,

\[
2 \leq a_1 < a_2 < \ldots < a_n,
\]

with

\[
\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.
\]

Therefore,

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{6a_1} + \frac{1}{6a_2} + \ldots + \frac{1}{6a_n} = \frac{5}{6} + \frac{1}{6} \left( \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \right) = 1.
\]

Thus, the lemma is established.

Now, let \( (a_1, a_2, \ldots, a_n) \in SDRPS(n) \). Then, replacing \( a_i (1 \leq i \leq n) \) by \( b_{i1} \) and \( b_{i2} \), where

\[
b_{i1} = a_i + 1, \ b_{i2} = a_i(a_i+1), \quad (4)
\]
we see that \((a_1, a_2, \ldots, a_{i-1}, b_1, b_2, a_{i+1}, \ldots, a_n)\) (rearranging the numbers, if necessary), an element of SDRPS\((n+1)\).

The third method suggested by Murthy and Ashbacher (2005) is as follows. Let \(d_{i1}\) and \(d_{i2}\) be two distinct divisors of \(a_i\), so that \(a_i = d_{i1} \times d_{i2}\). Then, replacing \(a_i\) by \(c_{i1}\) and \(c_{i2}\), where
\[
c_{i1} = d_{i1}(d_{i1} + d_{i2}), \quad c_{i2} = d_{i2}(d_{i1} + d_{i2}),
\]
(and rearranging the terms, if necessary), we get an element of SDRPS\((n+1)\).

Thus, starting with \((2, 3, 6) \in \text{SDPRS}(3)\), by applying the three methods outlined above, we get the following elements of SDRPS\((4)\):
\[
(2, 4, 6, 12), (2, 3, 7, 42), (2, 3, 10, 15).
\]

This example shows that, the methods described above are overlapping, and do not generate all the elements of SDRPS\((4)\).

To find a lower bound for \(f_{\text{DR}}(n)\), we confine our attention to SDRPS\((4)\) with 6 elements. A closer look at the elements of SDRPS\((4)\) show that the second procedure described in (4) can be applied to only three (of the four components) in each of the first five elements of SDRSP\((4)\). Then, by the second method, we get the following 17 elements of SDRPS\((5)\): (2, 4, 7, 12, 42), (2, 3, 8, 42, 56), (2, 3, 7, 43, 1806), (2, 4, 8, 12, 24), (2, 3, 9, 24, 72), (2, 3, 8, 25, 600), (2, 4, 9, 12, 18), (2, 3, 10, 18, 90), (2, 3, 9, 19, 342), (2, 4, 10, 12, 15), (2, 3, 11, 15, 110), (2, 3, 10, 16, 240), (3, 4, 5, 6, 20), (2, 4, 6, 20, 30), (2, 4, 5, 21, 420), (2, 4, 6, 12, 20) and (2, 4, 6, 13, 156). Applying the third method, the following eleven elements of SDRPS\((5)\) are obtained: (2, 3, 7, 89, 91), (2, 3, 3, 88, 1338, (2, 3, 8, 28, 168), (2, 3, 9, 22, 99), (2, 3, 29, 49, 54), (2, 3, 14, 15, 35), (2, 3, 10, 12, 1420), (2, 4, 5, 24, 120), (2, 4, 5, 36, 45), (2, 4, 6, 21, 28). And finally, by Lemma 10, the following four elements of SDRPS\((5)\) result: (2, 4, 6, 14, 84), (2, 4, 16, 48), (2, 4, 6, 18, 36) and (2, 4, 8, 10, 40). Thus, the suggested three methods together give only 32 elements of SDRPS\((5)\).

This leads to the following conservative estimate of SDRPS\((n)\):
\[
f_{\text{DR}}(n+1) \geq (n-1)[f_{\text{DR}}(n) - 1] + n - 2 + f_{\text{DR}}(n) = nf_{\text{DR}}(n) - 1.
\]

In the above inequality, the number of elements of SDRPS\((n+1)\), obtained from the elements of SDRPS\((n)\) by the method of (4), is \((n-1)[f_{\text{DR}}(n) - 1] + n - 2\); and since \(a_n\) is not a prime, we may safely say that the number of elements of SDRPS\((n+1)\), arising from the elements of SDRPS\((n)\) by the third method, is \(f_{\text{DR}}(n)\).

Considering, \(A_1 = \{(2, 3, 10, 15)\}, B_1 = \{(4, 5, 6, 8, 9, 12, 20, 72)\}\),
we see that both \(A_1\) and \(B_1\) satisfy Definition 1 with \(A_1 \cap B_1 = \emptyset\) and no common component. Then, applying the procedure outlined in (4) for the last components of \(A_1\) and \(B_1\), we get
\[
A_2 = \{(2, 3, 10, 16, 15 \times 16)\}, B_2 = \{(4, 5, 6, 8, 9, 12, 20, 73, 72 \times 73)\},
\]
where \(A_2 \cap B_2 = \emptyset\) with no components common, with \(A_2 \in \text{SDPRS}(5)\) and \(B_2 \in \text{SDPRS}(9)\).

Applying the procedure (4) once more, we get
\[
A_3 = \{(2, 3, 10, 16, 240, 240 \times 241)\}, B_3 = \{(4, 5, 6, 8, 9, 12, 20, 73, 5257, 5256 \times 5257)\},
\]
with \(A_3 \cap B_3 = \emptyset\) with no common components and each of \(A_3\) and \(B_3\) satisfying (4). Continuing the process, we get two infinite sequences of sets \(\{A_i\}_{i=1}^{\infty}\) and \(\{B_i\}_{i=1}^{\infty}\) such that each \(A_i\) and \(B_i\) satisfies the condition in Definition 1 with \(A_i \cap B_i = \emptyset\) and no common components for any \(i \geq 1\). This example proves the conjecture of Murthy and Ashbacher (2005) that there are infinitely many disjoint sets \(A_i\) and \(B_i\) satisfying the condition of Definition 1.
CONCLUSIONS

This study has presented some methods to calculate Smarandache distinct reciprocal partition of unity in \( n \) partitions, \( \text{SDRPS}(n) \). The method by induction is the main process to find out \( \text{SDRPS}(n) \). We have found that \( \text{SDRPS}(3) = \{(2, 3, 6)\} \) is the singleton set and \( \text{SDRPS}(4) \) consists of 6 elements such as \( (2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15), (2, 4, 5, 20) \) and \( (2, 4, 6, 12) \). Thirty-two elements of \( \text{SDRPS}(5) \) are resultant from different methods. Infinitely many disjoint sets are found those satisfy the condition of the Smarandache distinct reciprocal partition of unity. One can extend this study by developing a direct method to calculate \( \text{SDRPS}(n) \) in lieu of induction process.

REFERENCES

