Smarandache's function applied to perfect numbers

Sebastián Martín Ruiz
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Smarandache's function may be defined as follows:
$S(n)$ is the smallest positive integer such that $S(n)!$ is divisible by $n$. [1]
In this article we are going to see that the value this function takes when $n$ is a perfect number of the form $n = 2^{k-1} \cdot (2^k - 1)$, $p = 2^k - 1$ being a prime number.

Lemma 1 Let $n = 2^i \cdot p$ when $p$ is an odd prime number and $i$ an integer such that:
$$0 \leq i \leq E\left(\frac{p}{2}\right) + E\left(\frac{p}{2^2}\right) + \cdots + E\left(\frac{p}{2^{E\left(\log_2 p\right)}}\right) = e_2(p!)
$$
Where $e_2(p!)$ is the exponent of $2$ in the prime number decomposition of $p!$. $E(x)$ is the greatest integer less than or equal to $x$.
One has that $S(n) = p$.

Demonstration:
Given that $gcd(2^i, p) = 1$ ($gcd$ = greatest common divisor) one has that $S(n) = max\{s(2^i), S(p)\} \geq S(p) = p$. Therefore $S(n) \geq p$.
If we prove that $p!$ is divisible by $n$ then one would have the equality.
$$p! = p_1^{e_{p_1}(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_s^{e_{p_s}(p!)}$$
where $p_i$ is the $i$-th prime of the prime number decomposition of $p!$. It is clear that $p_1 = 2, \quad p_s = p, \quad e_{p_s}(p!) = 1$ for which:
$$p! = 2^{e_2(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)} \cdot p$$
From where one can deduce that:
$$\frac{p!}{n} = 2^{e_2(p!)} - i \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)}$$
is a positive integer since $e_2(p!) - i \geq 0$.
Therefore one has that $S(n) = p$.
Proposition 1 If \( n \) a perfect number of the form \( n = 2^{k-1} \cdot (2^k - 1) \) with \( k \) a positive integer, \( 2^k - 1 = p \) prime, one has that \( S(n) = p \).

Demonstration:
For the Lemma it is sufficient to prove that \( k - 1 \leq e_2(p!) \).

If we can prove that

\[
k - 1 \leq 2^{k-1} - \frac{1}{2}
\]

we will have proof of the proposition since:

\[
k - 1 \leq 2^{k-1} - \frac{1}{2} = \frac{2^k - 1}{2} = \frac{p}{2}
\]

As \( k-1 \) is an integer one has that \( k - 1 \leq E(\frac{p}{2}) \leq e_2(p!) \).

Proving (1) is the same as proving \( k \leq 2^{k-1} + \frac{1}{2} \) at the same time, since \( k \) is integer, is equivalent to proving \( k \leq 2^{k-1} \) (2).

In order to prove (2) we may consider the function: \( f(x) = 2^{x-1} - x \quad x \in \mathbb{R} \).

This function may be derived and its derivate is \( f'(x) = 2^{x-1} \ln 2 - 1 \).

\( f \) will be increasing when \( 2^{x-1} \ln 2 - 1 > 0 \) resolving \( x \):

\[
x > 1 - \frac{\ln(\ln 2)}{\ln 2} \approx 1.5287
\]

In particular \( f \) will be increasing \( \forall x \geq 2 \).

Therefore \( \forall x \geq 2 \quad f(x) \geq f(2) = 0 \) that is to say \( 2^{x-1} - x \geq 0 \quad \forall x \geq 2 \)

therefore

\[
2^{k-1} \geq k \quad \forall k \geq 2 \quad \text{integer}
\]

and thus is proved the proposition.

EXAMPLES:

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References:


Author:

Sebastián Martín Ruiz. Avda. de Regla 43. CHIPIONA (CADIZ) 11550 SPAIN.