Q-Smarandache Fuzzy Implicative Ideal of QSmarandache BH-Algebra

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Q-Smarandache Fuzzy Implicative Ideal of Q-Smarandache BH-Algebra

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Abstract

In this paper, The notions of Q-Smarandache fuzzy implicative ideal and Q-Smarandache fuzzy sub implicative ideal of a Q-Smarandache BH-Algebra introduced, examples are given, and related properties investigated the relationships among these notions and other types of Q-Smarandache fuzzy ideal of a Q-Smarandache BH-Algebra are Studies.

Keywords: BCK-algebra, BH-algebra, BH-algebra, Q-Smarandache a filter of Smarandache BH-algebra.

1 Introduction

licative, medial) BH-algebra and sub-implicative ideal of a BH-algebra[7]. In 2015, H.H.Abbass and H.K.Gatea introduced the notion Q-Smarandache implicative ideal of a Q-Smarandache BH-Algebra[8]. In this paper we introduce the notion of Q-Smarandache fuzzy implicative ideal and Q-Smarandache fuzzy sub implicative ideal of a Q-Smarandache BH-Algebra. Note in this paper, X is Q-Smarandache BH-Algebra.

2. Preliminaries

In this section, we give some basic concepts about a BCI-algebra, a BCK-algebra, a BCH-algebra, a BH-algebra, a Q-Smarandache BH-algebra, and a Q-Smarandache ideal of a BH-algebra.

**Definition 2.1.** [9]. A BCI-algebra is an algebra \((X, \ast, 0)\), where \(X\) is a nonempty set, \(\ast\) is a binary operation and 0 is a constant, satisfying the following axioms: for all \(x, y, z \in X\):

i. \((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0,

ii. \((x \ast (x \ast y)) \ast y = 0,

iii. \(x \ast x = 0,

iv. \(x \ast y = 0\) and \(y \ast x = 0\) imply \(x = y\).

**Definition 2.2.** [3]. BCK-algebra is a BCI-algebra satisfying the axiom:

\(0 \ast x = 0\) for all \(x \in X\).

**Definition 2.3.** [10]. A BH-algebra is a nonempty set \(X\) with a constant 0 and a binary operation \(\ast\) satisfying the following conditions:

i. \(x \ast x = 0, \forall x \in X\).

ii. \(x \ast y = 0\) and \(y \ast x = 0\) imply \(x = y, \forall x, y \in X\).

iii. \(x \ast 0 = x, \forall x \in X\).

**Remark 2.4.** [10].

i. Every BCK-algebra is a BCI-algebra.

ii. Every BCK-algebra is a BCH/ BH-algebra.

**Remark 2.5.** [11] Let \(X\) and \(Y\) be BH-algebras. A mapping \(f : X \to Y\) is called a homomorphism if \(f(x \ast y) = f(x) \ast f(y)\) \(\forall x, y \in X\). A homomorphism \(f\) is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). For any
homomorphism \( f : X \rightarrow Y \) the set \( \{ x \in X : f(x)=0 \} \) called the kernel of \( f \), denoted by \( \text{Ker}(f) \), and the set \( \{ f(x) : x \in X \} \) is called the image of \( f \), denoted by \( \text{Im}(f) \). Notice that \( f(0) = 0 \).

**Definition 2.6.** [12] BCK-algebra \((X, *, 0)\) is said to be Bounded BCK-algebra satisfying the identity: \( x *(y *x) = x \forall x, y \in X \).

**Definition 2.7.** [13] A BH-algebra \( X \) is called BH*-algebra if \( (x*y)*x = 0, \forall x, y \in X \).

**Definition 2.8.** [6] A Smarandache BH-algebra is defined to be a BH-algebra \( X \) in which there exists a proper subset \( Q \) of \( X \) such that

i. \( 0 \in Q \) and \( |Q| \geq 2 \).

ii. \( Q \) is a BCK-algebra under the operation of \( X \).

**Definition 2.9.** [8] A Q-Smaradache BH-algebra is said to be a Q-Smaradache implicative BH-algebra if it satisfies the condition, \((x* (x * y)) * (y * x) = y * (y * x) \) \( \forall x, y \in Q \).

**Definition 2.10.** [8] A Q-Smarandache BH-algebra \( X \) is called a Q-Smarandache medial BH-algebra if \( x * (x*y) = y, \forall x, y \in Q \).

**Definition 2.11.** [6] A nonempty subset \( I \) of \( X \) is called a Q-Smarandache ideal of \( X \), denoted by a Q-S.I of \( X \) if it satisfies:

\[(J_1) \ 0 \in I \]
\[(J_2) \ \forall y \in I \text{ and } x*y \in I \Rightarrow x \in I, \ \forall x \in Q. \]

**Definition 2.12.** [8] A Q-Smarandache ideal \( I \) of \( X \) is called a Q-Smarandache implicative ideal of \( X \), denoted by a Q-S.I.I of \( X \) if:

\[(x*(y*x))*z \in I \text{ and } z \in I \implies x \in I, \ \forall x, y \in Q. \]

**Definition 2.13.** [8] A nonempty subset \( I \) of \( X \) is called a Q-Smarandache P-ideal of \( X \) if it satisfies \((J_3) \) and :

\[(J_3) \ (x*z) *(y*z) \in I \text{ and } y \in I \implies x \in I, \ \forall x, z \in Q. \]

**Definition 2.14.** [2] A fuzzy set \( A \) in a BH-algebra \( X \) is said to be a fuzzy subalgebra of \( X \) if it satisfies: \( A(x*y) \geq \min \{ A(x) , A(y) \} \), \( \forall x, y \in X \).

**Definition 2.15.** [14] A fuzzy subset \( A \) of a BH-algebra \( X \) is said to be a fuzzy ideal if and only if:

\[(I_1) \ A(0) \geq A(x), \ \forall x \in X. \]

\[(I_2) \ A(x) \geq \min \{ A(x*y), A(y) \}, \ \forall x, y \in X. \]

**Definition 2.16.** [15] A fuzzy subset \( A \) of a BH-algebra \( X \) is called a fuzzy implicative ideal of \( X \), denoted by a F.I.I if it satisfies\((I_1)\)and
A(x) ≥ \min\{A((x * (y * x)) * z), A(z)\}, \forall x, y, z \in X.

**Definition 2.17.** [16]. A fuzzy subset \(A\) of a BH-algebra \(X\) is called a fuzzy sub implicative ideal of \(X\), denoted by a F.S.I.I if it satisfies (I1) and

\[
(I_4) \ A(y * (y * x)) \geq \min \{A(((x * (x * y)) * (y * x) * z), A(z)), \forall x, y, z \in X.
\]

**Definition 2.18.** [17]. Let \(A\) be a fuzzy set in, \(\forall \alpha \in [0, 1]\), the set \(A_\alpha = \{ x \in X, A(x) \geq \alpha \}\) is called a level subset of \(A\).

Note that, \(A_\alpha\) is a subset of \(X\) in the ordinary sense.

**Definition 2.19.** [17].

Let \(X\) and \(Y\) be any two sets, \(A\) be any fuzzy set in \(X\) and \(f: X \to Y\) be any function. The set \(f^{-1}(y) = \{x \in X | f(x) = y\}, \forall y \in Y\). The fuzzy set \(B\) in \(Y\) defined by \(B(y) = \sup\{A(x) | x \in f^{-1}(y)\}\); \(\sup\) if \(f^{-1}(y) \neq \emptyset\), \(\forall y \in Y\), is called the image of \(A\) under \(f\) and is denoted by \(f(A)\).

**Definition 2.18.** [12].

Let \(X\) and \(Y\) be any two sets \(f: X \to Y\) be any function and \(B\) be any fuzzy set in \(f(A)\). The fuzzy set \(A\) in \(X\) defined by: \(A(x) = B(f(x)) \ \forall x \in X\) is called the primage of \(B\) under \(f\) and is denoted by \(f^{-1}(B)\).

**Definition 2.21.** [6]. A fuzzy subset \(A\) of \(X\) is said to be a Q-Smarandache fuzzy ideal of \(X\), denoted by a Q-S.F.I of \(X\):

\[
(F_1) \ A(0) \geq A(x), \forall x \in X
\]

\[
(F_2) \ A(x) \geq \min \{A((x * y)) A( y)\}, \forall x \in Q, y \in X
\]

3. Main results

In this section, we introduce the concepts of a Q-Smarandache fuzzy implicative ideal and Q-Smarandache fuzzy sub implicative ideal of a Q-Smarandache BH-algebra, also we study some properties of it with examples.

**Definition 3.1.** A fuzzy subset \(A\) of \(X\) is called a Q-Smarandache fuzzy implicative ideal of \(X\), denoted by a Q-S.F.I.I of \(X\) if it satisfies \((F_1)\) and,

\[
(F_3) \ A(x) \geq \min\{A(((x * (y * x)) * z) ), A(z)\}, \text{ for all } x, y \in Q, z \in X.
\]

**Example 3.2.**

Consider \(X = \{0, 1, 2\}\) with binary operation \(\ast\) defined by the following table:

<table>
<thead>
<tr>
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<th>0</th>
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<tbody>
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<tr>
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<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
where $Q = \{0, 2\}$ is a BCK-algebra. The fuzzy subset $A$ defined by $A(0) = 0.7$, $A(1) = 0.5$ and $A(2) = 0.2$ by calculation we knew that $A$ is Q-S.F.I.I.

**Proposition 3.3.** Every Q-S.F.I.I is Q-S.F.I. of $X$.

**Proof.** Let $A$ be a Q-S.F.I.I, To prove that $A$ is Q-S.F.I. by Definition (3.1) the condition $(F_1)$ is satisfied. Now let $x, y \in Q$ and $z \in X$, we have $A(x) \geq \min\{A((x \ast (0)) \ast y), A(y)\}, (\text{since } A \text{ is a Q-S.F.I.I})$ it follows that $A(x) \geq \min\{A((x \ast (0)) \ast y), A(y)\}, (\text{since } x \ast x = 0, \forall x, y \in Q)$ implies that $A(x) \geq \min\{A((x \ast y), A(y)\}$ (since $x \ast 0 = x, \forall x \in Q$). Hence $A$ is Q-S.F.I of $X$.

**Remark 3.4.** A Q-S.I of $X$ may not be a Q-S.F.I.I of $X$ as in the following example.

**Example 3.5.** Consider $X = \{0, 1, 2, 3\}$ with binary operation $\ast$ defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
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<td>1</td>
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<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

where $Q = \{0, 1\}$ is a BCK-algebra. The fuzzy subset $A$ defined by $A(0) = A(2) = 0.5$ and $A(1) = A(3) = 0.2$ is Q-S.F.I of $X$ but it is not a Q-S.F.I.I of $X$. Since if $x = 1, y = 0, z = 2$, then $A(1) < \min\{A((1 \ast (0 \ast 1)) \ast 2), A(2)\}$.

**Theorem 3.6.** Let $A$ be a Q-S.F.I of $X$. Then $A$ is a Q-S.F.I.I of $X$ if and only if the level subset $A_{\alpha}$ is a Q-S.I.I of $X, \forall \alpha \in [0, A(0)]$, such that $A(0) = Sup_{x \in X} A(x)$.

**Proof.** Let $A$ be a Q-S.F.I of $X$. To prove $A_{\alpha}$ is a Q-S.I.I of $X$. It is clear that $A(0) \geq \alpha$. So $0 \in A_{\alpha}$. Hence $A_{\alpha}$ satisfies I1. Now let $x, y \in Q, z \in X$ such that $((x \ast (y \ast x)) \ast z) \in A_{\alpha}$ and $z \in A_{\alpha}$ it follows that $A((x \ast (y \ast x)) \ast z) \geq \alpha$ and $A(z) \geq \alpha$ Thus $\min\{A((x \ast (y \ast x)) \ast z), A(z)\} \geq \alpha$. But $A(x) \geq \min\{A((x \ast (y \ast x)) \ast z), A(z)\}$ [Since $A$ is a Q-S.F.I.I of $X$. By definition 3.1(F3)] So $A(x) \geq \alpha \Rightarrow x \in A_{\alpha}$ Therefore, $A_{\alpha}$ is a Q-S.I.I of $X$.

Conversely, Let $A_{\alpha}$ be a Q-S.I.I of $X, \forall \alpha \in [0, A(0)]$ and $\alpha = Sup_{x \in X} A(x)$. To prove that $A$ is a Q-S.F.I.I of $X$. $0 \in A_{\alpha}$. [Since $A_{\alpha}$ is a Q-S.I.I. of $X$].
imply \( A(0) \geq \alpha \) we get \( A(0) \geq A(x) \). Let \( x, y \in Q, z \in X \) such that \( \min\{A((x \ast (y \ast x)) \ast z), A(z)\} = \alpha \) then \( A((x \ast (y \ast x)) \ast z) \geq \alpha \) and \( A(z) \geq \alpha \).

It follows that \( ((x \ast (y \ast x)) \ast z) \in A_{\alpha} \) and \( z \in A_{\alpha} \). Since \( A_{\alpha} \) be an Q-S.I.I of \( X \) imply \( A(x) \geq \alpha \) we get \( A(x) \geq \min\{A(((x \ast (y \ast x)) \ast z), A(z)\} \).

Therefore, \( A \) is a Q-S.F.I.I of \( X \).

**Corollary 3.6.1.** A fuzzy subset \( A \) is a Q-S.F.I.I of \( X \) if and only if the set \( X_A \) is an Q-S.I.I of \( X \), where \( X_A = \{ x \in X \mid A(x) = A(0) \} \)

**Proof.** Let \( A \) be a Q-S.F.I.I of \( X \). To prove \( X_A \) is a Q-S.I.I of \( X \).

i. If \( x = 0 \) then \( A(0) = A(0) \Rightarrow 0 \in X_A \)

ii. Let \( x, y \in Q, z \in X \) such that \( (x \ast (y \ast x)) \ast z \in X_A \) and \( z \in X_A \).

\( A((x \ast (y \ast x)) \ast z) = A(0) \) and \( A(z) = A(0) \). we have \( A(x) \geq \min\{A((x \ast (y \ast x)) \ast z), A(z)\} \) [Since \( A \) is a Q-S.F.I.I of \( X \)] it follows that \( A(x) \geq A(0) \) Hence \( A(x) = A(0) \) [ Since \( A \) is a Q-S.F.I.I of \( X, A(x) \geq A(0) \)] we get \( x \in X_A \). Therefore, \( X_A \) is a Q-S.I.I of \( X \).

Conversely,

Let \( X_A \) be a Q-S.I.I of \( X \). To prove \( A \) is a Q-S.F.I.I of \( X \).

Since \( X_A = A(0) \)

Therefore, \( A \) is a Q-S.F.I.I of \( X \) [ By Theorem 3.6].

**Proposition 3.7.** Let \( A \) be a fuzzy subset of \( X \) defined by

\[
A(x) = \begin{cases} 
\alpha_1 & \text{if } x \in X_A \\
\alpha_2 & \text{otherwise,}
\end{cases}
\]

where \( \alpha_1, \alpha_2 \in [0, 1] \) such that \( \alpha_1 > \alpha_2 \).

Then \( A \) is a Q-S.F.I.I of \( X \) if and only if \( X_A \) is an Q-S.I.I of \( X \).

**Proof.** Let \( A \) be a Q-S.F.I.I of \( X \). To prove \( X_A \) is an Q-S.I.I of \( X \).

i. \( A(0) = \alpha_1 \Rightarrow 0 \in X_A \) [ Since \( A(0) \geq A(x); \forall x \in X \). By definition 3.1(F1)].

ii. Let \( x, y \in Q, z \in X_A \) such that \( (x \ast (y \ast x)) \ast z \in X_A \) and \( z \in X_A \).

we obtain \( A((x \ast (y \ast x)) \ast z) = A(0) = \alpha_1 \) and \( A(z) = A(0) = \alpha_1 \) it follows that \( A(x) \geq \min\{A((x \ast (y \ast x)) \ast z), A(z)\} = \alpha_1 \) [Since \( A \) is a Q-S.F.I.I of \( X \),
by definition $3.1(F_1)$, Thus $A(x) = \alpha_1 \Rightarrow x \in X_A$. Hence $X_A$ is a Q-S.I.I of $X$.

Conversely, Let $X_A$ be an Q-S.I.I of $X$. To prove $A$ is a Q-S.F.I.I of $X$.

i. Since $0 \in X_A$, then $A(0) = \alpha_1 \Rightarrow A(0) = \alpha_1 \geq A(x)$, we get $A(0) \geq A(x), \forall x \in X$.

ii. Let $x, y \in Q$, $z \in X$. Then we have four cases:

**Case 1:** If $(x \ast (y \ast x)) \ast z \in X_A$ and $z \in X_A$, it follows that $x \in X_A$[ Since $X_A$ is an Q-S.I.I of $X$], we get $A((x \ast (y \ast x)) \ast z) = A(z) = A(x) = \alpha_1$. Hence $A(x) \geq \min\{A ((x \ast (y \ast x)) \ast z), A(z)\}$.

**Case 2:** If $(x \ast (y \ast x)) \ast z \in X_A$ and $z \notin X_A$ it follows that $A((x \ast (y \ast x)) \ast z) = \alpha_1$ and $A(z) = \alpha_2$. we get $\min\{A ((x \ast (y \ast x)) \ast z), A(z)\} = \alpha_2$. Hence $A(x) \geq \min\{A ((x \ast (y \ast x)) \ast z), A(z)\}$.

**Case 3:** If $(x \ast (y \ast x)) \ast z \notin X_A$ and $z \in X_A$ it follows that $A((x \ast (y \ast x)) \ast z) = \alpha_2$ and $A(z) = \alpha_1$. we get $\min\{A ((x \ast (y \ast x)) \ast z), A(z)\} = \alpha_2$. Hence $A(x) \geq \min\{A ((x \ast (y \ast x)) \ast z), A(z)\}$. Therefore, $A$ is a Q-S.F.I.I of $X$.

**Remark 3.8.** Let $A$ be a fuzzy subset of $X$ and $w \in X$. The set $\{x \in X|A(w) \leq A(x)\}$ is denoted by $\uparrow A(w)$.

**Proposition 3.9.** Let $A$ be a Q-S.F.I.I of $X$ and $w \in X$. If $A$ satisfies the condition

$\forall x, y \in Q \ A(x) \geq A(x \ast (y \ast x))$ (b2). Then $\uparrow A(w)$ is a Q-S.I.I of $X$.

Proof. Let $A$ be a Q-S.F.I of $X$. Then $A(0) \geq A(x), \forall x \in X$ [ By Definition 2.21(F_1)].

It follows that $A(0) \geq A(w)$ [ Since $w \in X$ we get $0 \in \uparrow A(w)$

Now, Let $x, y \in Q$, $z \in X$ such that $((x \ast (y \ast x)) \ast z) \in \uparrow A(w)$ and $z \in \uparrow A(w)$

Thus $A(w) \leq A((x \ast (y \ast x)) \ast z)$ and $A(w) \leq A(z)$ implies that

$A(w) \leq \min \{A((x \ast (y \ast x)) \ast z), A(z)\} \leq A(x \ast (y \ast x))$ [Since $A$ is a Q-S.F.I]

of $X$. But $A(x \ast (y \ast x)) \leq A(x)$ . [ By (b2)] we get $A(w) \leq A(x)$. Hence $x \in \uparrow A(w)$ Therefore, $A(w)$ is a Q-S.I.I of $X$.

**Proposition 3.10.** Let $w \in X$. If $A$ is a Q-S.F.I.I of $X$, then $\uparrow A(w)$ is a Q-S.I.I of $X$.
Proof. Let $A$ be a Q-S.F.I of $X$. Then $A(0) \geq A(x)$, $\forall x \in X$ it follows that $A(0) \geq A(w)$ [Since $w \in X$.] Hence $0 \in \uparrow A(w)$. Let $x, y \in Q$, $z \in X$ such that $(x*(x*y))^*z$ 

$z \in \uparrow A(w)$ and $z \in \uparrow A(w)$ Then $A(w) \leq A((x*(y*x))^*z)$ and $A(w) \leq A(z)$ it follows $A(w) \leq \min \{ A((x*(y*x))^*z) , A(z) \}$ But $\min A((x*(y*x))^*z) , A(z) \leq A(x)$ [By Definition 2.21(F2)] we get $A(w) \leq A(x)$. Hence $x \in \uparrow A(w)$. Therefore, $\uparrow A(w)$ is a Q-S.F.I of $X$.

**Proposition 3.11.**

Let $\{ A_i / i \in I \}$ be a family of Q-S.F.I. of $X$. Then $\bigcap_{i \in I} A_i$ is a Q-S.F.I. of $X$.

Let $\{ A_i / i \in I \}$ be a family of Q-S.F.I. of $X$.

i. Let $x \in X$. Then

$$\bigcap_{i \in I} A_i(0) = \inf \{ A_i(0) \mid i \in I \} \geq \inf \{ A_i(x) \mid i \in I \} = \bigcap_{i \in I} A_i(x)$$

(ii). Let $x, y \in Q$, $z \in X$. Then, we have

$$\bigcap_{i \in I} A_i(x) = \inf \{ A_i(x) \mid i \in I \} \geq \inf \{ \min \{ A_i((x*(y*x))^*z), A_i(z) \} \}$$

$$= \inf \{ \min \{ A_i((x*(y*x))^*z), A_i(z) \} \}$$

$$= \min \{ \inf \{ A_i((x*(y*x))^*z) \mid i \in I \}, \inf \{ A_i(z) \} \}$$

$$\Rightarrow \bigcap_{i \in I} A_i(x) \geq \min \{ \{ \bigcap_{i \in I} A_i((x*(y*x))^*z) \}, \{ \bigcap_{i \in I} A_i(z) \} \}$$

Therefore, $\bigcap_{i \in I} A_i(x)$ is a Q-S.F.I. of $X$.

**Remark 3.12.** The union of a Q-S.F.I. of $X$ may not be a Q-S.F.I. of $X$ as in The following example.

**Example 3.13.** Consider $X = \{0, 1, 2, 3, 4, 5\}$ with binary operation ”*” defined by the following table:

<table>
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<tr>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<th>5</th>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Where \( Q = \{0, 2\} \) is a BCK-algebra. The fuzzy subset \( A, B \) defined by
\[
A(0) = A(1) = 0.9, \quad A(2) = A(3) = A(4) = A(5) = 0.4 \quad \text{and} \quad \\
B(0) = B(5) = 0.9, \quad B(1) = B(2) = B(3) = B(4) = 0.4
\]
are two \( Q\)-S.F.I.I, but
\[
A \cup B(0) = A \cup B(1) = A \cup B(5) = 0.9 \quad \text{and} \quad \\
A \cup B(2) = A \cup B(3) = A \cup B(4) = 0.4
\]
is not a \( Q\)-S.F.I.I of \( X \). Since
\[
(A \cup B)(2) = 0.4 < \min \{ (A \cup B)((2*(0*2))*5), (A \cup B)(5) \}
\]

**Proposition 3.14.**

Let \( \{A_i / i \in \Gamma \} \) be a chain of \( Q\)-S.F.I.I of \( X \). Then \( \bigcup_{i \in \Gamma} A_i (x) \) is a \( Q\)-S.F.I.I of \( X \).

**Proof.**

Let \( \{A_i | i \in \Gamma \} \) be a chain of Q-S.F.I.I of \( X \).

i: Let \( x \in X \). Then
\[
\bigcup_{i \in \Gamma} A_i (0) = \sup \{ A_i(0) | i \in \Gamma \} \geq \sup \{ A_i(x) | i \in \Gamma \} = \bigcup_{i \in \Gamma} A_i (x)
\]
[Since \( A_i \) is a \( Q\)-S.F.I.I of \( X \), \( i \in \Gamma \), by Definition 3.1(i) ]
\[
\Rightarrow \bigcup_{i \in \Gamma} A_i (0) \geq \bigcup_{i \in \Gamma} A_i (x)
\]

ii: Let \( x, y \in Q, z \in X \). Then, we have
\[
\bigcup_{i \in \Gamma} A_i (x) = \sup \{ A_i(x) | i \in \Gamma \} \geq \sup \{ \min \{ A_i(x*(y*x)*z), A_i(z) | i \in \Gamma \} \}
\]
[Since \( A_i \) is a Q-S.F.I.I of \( X \), \( i \in \lambda \) by Definition 3.1(i) ]
\[
\Rightarrow = \min \{ \sup \{ A_i(x*(y*x)*z), A_i(z) | i \in \Gamma \} \} \quad \text{[ since } A_i \text{ is a chain, } i \in \Gamma \}
\]
\[
\Rightarrow = \min \{ \sup \{ A_i(x*(y*x)*z) | i \in \Gamma \}, \sup \{ A_i(z) | i \in \Gamma \} \}
\]
\[
\Rightarrow = \min \{ \bigcup_{i \in \Gamma} A_i (x*(y*x)*z) | i \in \Gamma \}, \bigcup_{i \in \Gamma} A_i (z) | i \in \Gamma \}
\]
Therefore, \( \bigcup_{i \in \Gamma} A_i(x) \) is a Q-S.F.I. of \( X \).

**Theorem 3.15.** Let \( A \) be a Q-S.F.I. of \( X \). Then \( A \) is a Q-S.F.I.I of \( X \) if and only if \( A \) satisfies the following inequality: \( \forall x, y \in Q \ A(x) \geq A(x*(y*x)) \) (b2).

**Proof.** Let \( A \) be a Q-S.F.I.I of \( X \) and \( x, y \in Q \) then

\[ A(x) \geq \min \{ A((x*(y*x)) *0 ), A(0) \} \]

it follows that \( \geq \min \{ A(x*(y*x)) , A(0) \} \) [since \( x*(y*x)*0 = x*(y*x) \)]. Therefore the condition (b1) is satisfied.

Conversely,

Let \( A \) be a Q-S.F.I of \( X \). Then (F1) satisfied.

Now, let \( x, z \in Q \), then \( A(x*(y*z)) \geq \min \{ A((x*(y*z)) *z) , A(z) \} \) [Since \( A \) is a Q-S.F.I of \( X \). By (2.21)(F2)] we have \( A(x) \geq \min \{ A((x*(y*z)) *z) , A(z) \} \). Hence, \( A \) is a Q-S.F.I.I of \( X \).

**Definition 3.16.** A fuzzy subset \( A \) of \( X \) is called a Q-Smarandache fuzzy P-ideal of \( X \), denoted by a Q-S.F.P.I of \( X \) if satisfies (F1) and:

\( (F_4) \ A(x) \geq \min \{ A((x*z)*(y*z)) , A(y) \} \), for all \( x, z \in Q, y \in X \).

**Example 3.17.** Consider \( X = \{0, 1, 2, 3\} \) with binary operation "*" defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

where \( Q = \{0, 1\} \) is a BCK-algebra. The fuzzy subset \( A \) defined by

\[ A(0) = A(1) = A(2) = 0.8 \text{ and } A(3) = 0.2 \]

is a Q-S.F.P.I of \( X \).

**Theorem 3.18.** Every Q-S.F.P.I is a Q-S.F.I of \( X \).

**Proof.** Let \( A \) be a Q-S.F.P.I of \( X \). Then(F1) satisfied.

Now, let \( x, z \in Q \) and \( y \in X, z = 0 \) in (F4) we get:

\[ A(x) \geq \min \{ A((x*0) * (y*0)),A(y) \} \]

[Since \( X \) is a Q-Smarandache BH-algebra \( x * 0 = x \).] \( A(x) \geq \min \{ A(x*y), A(y) \} \)

Therefore, \( A \) is Q-S.F.I of \( X \).

**Theorem 3.19.** Every Q-S.F.P.I is a Q-S.F.I.I of \( X \).

**Proof.** Let \( A \) be a Q-S.F.P.I of \( X \). Then(F1) satisfied [By definition 3.16(F1)]. And Let \( a, c, x, y \in Q \) and \( d \in X \). Then
\[ A(a) \geq \min\{A((a * c) * (d * c)), A(d)\} \]  

By (F4). Put \( a = x, d = 0, c = y * x \), we get
\[ A(x) \geq \min\{A((x * (y * x)) * (0 * (y * x))), A(0)\} \]
\[ = \min\{A(x * (y * x)) * 0), A(0)\} \]  

Since Q is BCK \( 0 * x = 0 \]
\[ = \min\{A(x * (y * x)), A(0)\} \]

Since Q is BCK \( x * 0 = x \]
\[ = A(x * (y * x)) \]  

Since \( A(0) \geq A(x), \forall x \in X \]
Therefore, A is a \( Q\)-S.F.I.I of X [by Theorem 3.15]

**Remark 3.20.** In the following example, we see that the converse of Theorem (3.21) may not be true in general.

**Example 3.21.** Consider \( X = \{0, 1, 2\} \) with binary operation "\(*\)" defined by table
where Q = \{0, 2\} is a BCK-algebra. The fuzzy subset A defined by
\[ A(0) = 0.7, A(1) = 0.5 \text{ and } A(2) = 0.2 \]
Then A is \( Q\)-S.F.I.I of X, but A is not a \( Q\)-S.F.P.I of X, since if \( x = 2, y = 1, z = 2 \), then
\[ A(2) = 0.2 \leq \min\{A((2 * 2) * (1 * 2)), A(1)\} = 0.5 \]

**Theorem 3.22.** Let A be a \( Q\)-S.F.I, such that Q is a bounded BCK- algebra. Then A is a \( Q\)-S.F.I.I of X.

**Proof.** It’s clear that \( A(0) \geq A(X), \forall x \in X \)

Now, let \( x, y \in Q \) and \( z \in X \), Then
\[ A(x * (y * x)) \geq \min\{A((x * (y * x)) * z), A(z)\}, \]  

Since A is a \( Q\)-S.F.I of X, by 2.21(F2)]
implies that \( A(x) \geq \min\{A((x * (y * x)) * z), A(z)\} \)  

Since Q is bounded BCK- algebra, by 2.6] Therefore, A is a \( Q\)-S.F.I.I of X

**Definition 3.23.** A fuzzy subset A of X is called a \( Q\)-Smarandache fuzzy sub-implicative ideal of X, denoted by (a \( Q\)-S.F.S.I.I ) of X if it satisfies: (F1) and (F5) \[ A(y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * z), A(z)\} \]  

for all \( x, y \in Q, z \in X \)

**Example 3.24.**
Consider \( X = \{0, 1, 2, 3\} \) with binary operation "\(*\)" defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Where Q={0,2} is a BCK-algebra. The fuzzy subset A is defined by
\[ A(0) = A(1) = 0.9 \text{ and } A(2) = A(3) = 0.3 \]
it easy to check that A is \( Q\)-S.F.S.I.I of X
**Proposition 3.25.** Every Q-S.F.S.I.I is Q-S.F.I. of X.

Proof. Let A be a Q-S.F.S.I.I. Then (F1) it is satisfied. Now let \( x \in Q \) and \( y \in X \).

\[
A(x) = A(x \ast 0) = A(x \ast (x \ast x)) \geq \min\{A(((x \ast (x \ast y)) \ast (x \ast x)) \ast y),A(y)\}
\]

[Since A is a Q-S.F.S.I.I of X, by Definition 3.23 (F5)]

\[
= \min\{A(((x \ast 0) \ast y),A(y)) \} \ [\text{Since } x \ast x = 0]
\]

\[
= \min\{A(x \ast y),A(y)) \} \ [\text{Since } x \ast 0 = x]
\]

Thus \( A(x) \geq \min\{A(x \ast y),A(y)\} \)

Therefore, A is a Q-S.F.I of X.

**Proposition 3.26.** Let A be a Q-S.F.I of X. Then A is a Q-S.F.S.I.I of X if and only if A satisfies the following inequality: \( \forall x, y \in Q \, A((y \ast (y \ast x)) \geq A(((x \ast (x \ast y)) \ast (y \ast x)) \ast (y \ast x)) \).

(b3).

Proof. Let A be a Q-S.F.S.I.I. and \( x, y \in Q \), then

\[
A(y \ast (y \ast x)) \geq \min\{A(((x \ast (x \ast y)) \ast (y \ast x)) \ast 0),A(0)\} = \min\{A((x \ast (x \ast y)) \ast (y \ast x)),A(0)\} \ [\text{Since } Q \text{ is BCK; } x \ast 0 = x]
\]

By (b3) we have \( A((y \ast (y \ast x)) \geq A(((x \ast (x \ast y)) \ast (y \ast x))) \)

implies that \( A(y \ast (y \ast x)) \geq \min\{A(((x \ast (x \ast y)) \ast (y \ast x)) \ast z),A(z)\} \)

Therefore, A is Q-S.F.S.I.I of X.

**Theorem 3.27.** Let X be a Q-Smarandache implicative BH-algebra. Then every Q-S.F.I of X is a Q-S.F.S.I.I of X.

Proof. Let A be a Q-S.F.I of X. Then (F1) satisfied[By (2.21)] and let \( x, y \in Q \). Then

\[
A((x \ast (x \ast y)) \ast (y \ast x)) \geq \min\{A(((x \ast (x \ast y)) \ast (y \ast x)) \ast z),A(z)\} \ [\text{Since A is a Q-S.F.I of X by Definition 2.21}]
\]

By (b3) we have \( A((y \ast (y \ast x)) \geq A(((x \ast (x \ast y)) \ast (y \ast x))) \)

implies that \( A(y \ast (y \ast x)) \geq \min\{A(((x \ast (x \ast y)) \ast (y \ast x)) \ast z),A(z)\} \)

Therefore, A is Q-S.F.S.I.I of X.

**Corollary 3.27.2.** Let X be a Q-Smarandache implicative BH-algebra and A be Q-S.F.I.I of X. Then A is a Q-S.F.S.I.I of X.

Proof. Directly from proposition 3.3 and Theorem 3.27

**Proposition 3.28.** Let X be a Q-Smarandache medial BH-algebra and A be Q-S.F.I.
of $X$. Then $A$ is a $Q$-S.F.S.I of $X$.

Proof. Let $A$ be a $Q$-S.F.I of $X$. Then (F1) satisfied [By 2.21] and let $x, y \in Q$, and $z \in X$.

Then $A((x * (x * y)) * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * z)), A(z)\}$. We get

$A((y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * z)), A(z)\} [Since X is a Q-Smarandache medial BH-algebra]. Hence $A$ is a $Q$-S.F.S.I of $X$.

**Corollary 3.28.3.** Let $X$ be an $Q$-Smarandache medial BH-algebra and $A$ be a $Q$-S.F.S.I of $X$. Then $A$ is a $Q$.S.F.S.I of $X$.

Proof. Directly from proposition 3.3 and proposition 3.28.

**Theorem 3.29.** Let $X$ be a $Q$-Smarandache medial BH-algebra and $A$ be $Q$.S.F.S.I satisfies the condition $\forall x, y \in Q, A((x * (x * y)) * (y * x)) \geq A(x * (y * x)) (b_4)$. Then $A$ is $Q$.S.F.S.I.

Proof. Let $A$ be a $Q$-S.F.S.I of $X$. Then (F1) is satisfied

Now let $x, y \in Q$ and $z \in X$. Then By (b4) we have $A((x * (x * y)) * (y * x)) \geq A(x * (y * x))$. Thus, $A((y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * z)), A(z)\} [Since A is a Q-S.F.S.I of X] if $z = 0$, then $A(y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * 0)), A(0)\}$ we obtain $A(y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)), A(0)) [Since Q is a BCK-algebra, x * 0 = x]. It follows that $A(y * (y * x)) \geq A((x * (x * y)) * (y * x)) By (b_4), We have $A((x * (x * y)) * (y * x)) \geq A(x * (y * x)).Thus $A(y * (y * x)) \geq A(x * (y * x)), But $A(x) = A(x * (y * x)) [Since X is a medial, y * (y * x) = x]. So, $A(x) \geq A(y * (y * x))$ Hence, $A$ is a $Q$.S.F.S.I of $X$ [By 3.15(b_2)]

**References**


