SMARANDACHE – R-MODULE AND COMMUTATIVE AND BOUNDED BE-ALGEBRAS

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ABSTRACT

In this paper we introduced Smarandache – 2 – algebraic structure of R-Module namely Smarandache – R-Module. A Smarandache – 2 – algebraic structure on a set N means a weak algebraic structure A₀ on N such that there exist a proper subset M of N, which is embedded with a stronger algebraic structure A₁, stronger algebraic structure means satisfying more axioms, by proper subset one understands a subset different from the empty set, from the unit element if any, from the whole set. We define Smarandache – R-Module and obtain some of its characterization through Commutative and Bounded BE-Algebras. For basic concepts we refer to Florentin smarandache[2] and Raul Padilla[9].


1. INTRODUCTION

New notions are introduced in algebra to study more about the congruence in number theory by Florentin smarandache[2]. By <proper subset> of a set A, We consider a set P included in A and different from A, different from the empty set, and from the unit element in A – if any they rank the algebraic structures using an order relationship.

The algebraic structures S₁ ≪ S₂ if :both are defined on the same set :: all S₁ laws are also S₂ laws; all axioms of S₁ law are accomplished by the corresponding S₂ law; S₂ law strictly accomplishes more axioms than S₁ laws, or in other words S₂ laws has more laws than S₁.

For example : semi group ≪ monoid ≪ group ≪ ring ≪ field, or Semi group ≪ commutative semi group, ring ≪ unitary ring, etc. they define a General special structure to be a structure SM on a set A, different from a structure SN, such that a proper subset of A is an SN structure, where SM ≪ SN.
2. Prerequisites

**Definition 2.1:** An algebra (A; *, 1) of type (2, 0) is called a BE-algebra if for all x, y and z in A,

1. \((BE1) \ x * x = 1\)
2. \((BE2) \ x * 1 = 1\)
3. \((BE3) \ 1 * x = x\)
4. \((BE4) \ x * (y * z) = y * (x * z)\)

In A, a binary relation “≤” is defined by \(x ≤ y\) if and only if \(x * y = 1\).

**Definition 2.2:** A BE-algebra \((X; *, 1)\) is said to be self-distributive if \(x * (y * z) = (x * y) * (x * z)\) for all \(x, y, z \in A\).

**Definition 2.3:** A dual BCK-algebra is an algebra \((A; *, 1)\) of type (2,0) satisfying \((BE1)\) and \((BE2)\) and the following axioms for all \(x, y, z \in A\).

1. \((dBCK1) \ x * y = y * x = 1\) implies \(x = y\)
2. \((dBCK2) \ (x * y) * ((y * z) * (x * z)) = 1\)
3. \((dBCK3) \ x * ((x * y) * y) = 1\).

**Definition 2.4:** Let A be a BE-algebra or dual BCK-algebra. A is said to be commutative if the following identity holds:

\[ x \vee_B y = y \vee_B x \text{ where } x \vee_B y = (y * x) * x \text{ for all } x, y \in A.\]

**Definition 2.5:** Let A be a BE-algebra. If there exists an element 0 satisfying \(0 \leq x\) (or \(0 * x = 1\)) for all \(x \in A\), then the element “0” is called unit of A. A BE-algebra with unit is called a bounded BE-algebra.

**Note:** In a bounded BE-algebra \(x * 0\) denoted by \(xN\).

**Definition 2.6:** In a bounded BE-algebra, the element \(x\) such that \(xNN = x\) is called an involution.

Let \(S(A) = \{x \in A \mid xNN = x\}\) where A is a bounded BE-algebra. \(S(A)\) is the set of all involutions in A. Moreover, since \(1NN = (1 * 0) * 0 = 0 * 0 = 1\) and \(0NN = (0 * 0) * 0 = 1* 0 = 0\), we have \(0, 1 \in S(A)\) and so \(S(A) \neq \emptyset\).

**Definition 2.7:** Each of the elements \(a\) and \(b\) in a bounded BE-algebra is called the complement of the other if \(a \vee b = 1\) and \(a \wedge b = 0\).

**Definition 2.8:** Now we have introduced our concept smarandache – R – module: “Let R be a module, called R-module. If R is said to be smarandache – R – module, then there exist a proper subset A of R which is an algebra with respect to the same induced operations of R.”
3. Theorem

Theorem 3.1: Let R be a smarandache-R-module, if there exists a proper subset A of R in which (BE1) to (BE4) are hold, then the following conditions are satisfied,

(i) $1_N = 0$, $0_N = 1$

(ii) $x \leq xNN$

(iii) $x * yN = y * xN$

(iv) $0 \lor x = xNN$, $x \lor 0 = x$.

Proof. Let R be a smarandache-R-module. Then by definition there exists a proper subset A of R which is an algebra. By hypothesis A holds for (BE1) to (BE4) then A is bounded BE-algebras.

(i) We have $1_N = 1 \cdot 0 = 0$ and $0_N = 0 \cdot 0 = 1$. by using (BE1) and (BE3)

(ii) Since $x * xNN = x * ((x * 0) * 0) = (x * 0) * (x * 0) = 1$

We get $x \leq x$ (by (BE1) and (BE4))

(iii) We have $x * yN = x * (y * 0)$ (by using (BE4))

$= y * (x * 0)$

$= y * xN$.

(iv) By routine operations, we have $0 \lor x = (x * 0) * 0 = xNN$ and $x \lor 0 = (0 * x) * x = 1 * x = x$.

Theorem 3.2: Let R be a smarandache-R-module, if there exists a proper subset A of R in which (BE1) to (BE4) are hold, then the following conditions are satisfied $x * y \leq (y \lor x) * y$ for all $x, y \in A$.

Proof. Let R be a smarandache-R-module. Then by definition there exists a proper subset A of R which is an algebra. By hypothesis A holds for (BE1) to (BE4) then A is bounded BE-algebras.

Since

$(x * y) * ((y \lor x) * y) = (y \lor x) * ((x * y) * y) = (y \lor x) * (y \lor x) = 1$

We have $x * y \leq (y \lor x) * y$.

Theorem 3.3: Let R be a smarandache-R-module, if there exists a proper subset A of R in which (BE1) to (BE4) are hold, In addition to that satisfy $x * (y * z) = (x * y) * (x * z)$ then the following conditions are satisfied for all $x, y, z \in A$

(i) $x * y \leq yN \leq xN$

(ii) $x \leq y$ implies $yN \leq xN$.

Proof. Since R be a smarandache-R-module. Then by definition there exists a proper subset A of R which is an algebra. By hypothesis A holds for (BE1) to (BE4) then A is bounded and Self-Distributive BE-algebras.

(i) Since $(x * y) * (yN * xN) = (x * y) * ((y * 0) * (x * 0))$

$= (y * 0) * ((x * y) * (x * 0))$ (by BE4)
\[(y \ast 0) \ast (x \ast (y \ast 0)) \text{ (by distributivity)} \]
\[= x \ast ((y \ast 0) \ast (y \ast 0)) \text{ (by BE4)} \]
\[= x \ast 1 \text{ (by BE1)} \]
\[= 1 \text{ (by BE2)} , \]

We have \(x \ast y \leq yN \ast xN\).

(ii) It is trivial by \(x \leq y\), We have \(z \ast x \leq z \ast y\) then \(y \ast z \leq x \ast z \) for all \(x, y, z \in A\).

**Theorem 3.4:** Let \(R\) be a smarandache-R-module, if there exists a proper subset \(A\) of \(R\) in which (BE1) to (BE4) are hold, In addition to that satisfy \(x \ast (y \ast z) = (x \ast y) \ast (x \ast z)\), then the following conditions are satisfied

(i) \((y \vee x) \ast y \leq x \ast y\).

(ii) \(x \ast (x \ast y) = x \ast y\).

**Proof.** Since \(R\) be a smarandache-R-module. Then by definition there exists a proper subset \(A\) of \(R\) which is an algebra. By hypothesis \(A\) holds for (BE1) to (BE4) then \(A\) is a Self-Distributive BE-algebras.

(i) Since
\[
\begin{align*}
x \ast (y \vee x) & = x \ast ((x \ast y) \ast y) \\
& = (x \ast y) \ast (x \ast y) \\
& = 1.
\end{align*}
\]
We have \(x \leq y \vee x\). By \(z \ast x \leq z \ast y\)

We have \((y \vee x) \ast y \leq x \ast y \) for all \(x, y, z \in A\)

(ii) By using self distributive definition, (BE1) and (BE3), we have
\[
\begin{align*}
x \ast (x \ast y) & = (x \ast x) \ast (x \ast y) \\
& = 1 \ast (x \ast y) \\
& = x \ast y.
\end{align*}
\]

**Theorem 3.5:** Let \(R\) be a smarandache-R-module, if there exists a proper subset \(A\) of \(R\) in which (BE1) to (BE4) are hold, In addition to that satisfy \(0 \leq x \text{ (or } 0 \ast x = 1\), then the following conditions are satisfied for all \(x, y \in A\)

(i) \(xNN = x\)

(ii) \(xN \land yN = (x \lor y)\)
(iii) $x\lor y = (x \land y)$

(iv) $x \land y = y \land x$.

**Proof.** Since $R$ be a smarandache-R-module. Then by definition there exists a proper subset $A$ of $R$ which is an algebra. By hypothesis $A$ holds for (BE1) to (BE4) then $A$ is a bounded and Commutative BE-algebras.

(i) It is obtained that

$$xN = (x \land 0) \land 0 \text{ (from BE3)}$$

$$= (0 \land x) \land x \text{ (by commutativity)}$$

$$= 1 \land x$$

$$= x.$$ 

(ii) By the definition of “$\land$” and (i) we have that

$$x \land y = (xNN \lor yNN)N = (x \lor y)N.$$ 

(iii) By the definition of “$\land$” and (i) we have that

$$(x \land y)N = (xN \lor yN)NN = xN \lor yN.$$ 

(iv) We have

$$xN \land yN = (x \land 0) \land (y \land 0)$$

$$= y \land ((x \land 0) \land 0)$$

$$= y \land (xNN) = y \land x.$$ 

**Theorem 3.6:** Let $R$ be a smarandache-R-module, if there exists a proper subset $A$ of $R$ in which (BE1) to (BE4) are hold, In addition to that, there exists a complement of any element of $A$ and then it is unique.

**Proof.** Since $R$ be a smarandache-R-module. Then by definition there exists a proper subset $A$ of $R$ which is an algebra. By hypothesis $A$ holds for (BE1) to (BE4) then $A$ is a bounded and Commutative BE-algebras. Let $x \in A$ and $a, b$ be two complements of $x$. Then we know that $x \land a = x \land b = 0$ and $x \lor a = x \lor b = 1$. Also since $x \lor a = (x \land a) \land a = 1$ and $a \land (x \land a) = x \land (a \land a) = x \land 1 = 1$,

We have $x \land a \leq a$ and $a \leq x \land a$. So we get $x \land a = a$.

Similarly

$$x \land b = b.$$ 

Hence

$$a \land b = (x \land a) \land (x \land b) = (aN \land xN) \land (bN \land xN) \text{ by Theorem 2.5 (iv)}$$

$$= bN \land ((aN \land xN) \land xN) \text{ by BE-4}$$

$$= bN \land (xN \lor aN)$$

$$= bN \land (x \land a)N \text{ by Theorem 2.5 (iii)}$$

$$= (x \land a) \land b \text{ by Theorem 2.5 (iii)}$$

$$= 0 \land b$$

$$= 1.$$ 

With similar operations, we have $b \land a = 1$.

Hence we obtain $a = b$ which gives that the complement of $x$ is unique.
Theorem 3.7: Let R be a smarandache-R-module, if there exists a proper subset A of R in which (BE1) to (BE4) are hold, In addition to that satisfy $0 \leq x$ (or $0 \cdot x = 1$), then the following conditions are equivalent for all $x, y \in A$

(i) $x \land xN = 0$
(ii) $xN \lor x = 1$
(iii) $xN * x = x$
(iv) $x * xN = xN$
(v) $x * (x * y) = x * y$.

Proof. Since R be a smarandache-R-module. Then by definition there exists a proper subset A of R which is an algebra. By hypothesis A holds for (BE1) to (BE4) then A is a Commutative and bounded BE-algebras.

(i) $\Rightarrow$ (ii) Let $x \land xN = 0$. Then it follows that

$$xN \lor x = (xN \lor x)$$

by Theorem 2.5 (i)

$$= (xN \land xN)$$

by Theorem 2.5 (ii)

$$= (x \land xN)$$

by Theorem 2.5 (i)

$$= 0N$$

$$= 1.$$ (ii) $\Rightarrow$ (iii) Let $xN \lor x = 1$. Then, since

$$(xN * x) * x = xN \lor x = 1$$

and

$$x * (xN * x) = xN * (x * x) = xN * 1 = 1$$

We get $xN * x = x$ by (dBCK1).

(iii)$\Rightarrow$ (iv) Let $xN * x = x$. Substituting $xN$ for $x$ and using Theorem 2.5 (i) We obtain the result.

(iv) $\Rightarrow$ (v) Let $x * xN = xN$. Then

We get $yN * (x * xN) = yN * xN$.

Hence we have $x * (yN * xN) = yN * xN$. Using Theorem 2.5 (iv) We obtain $x * (x * y) = x * y$.

(v) $\Rightarrow$ (ii) Let $x * (x * y) = x * y$. Then

We have $xN \lor x = (x * (xN)) * xN$

$$= (x * (x * 0)) * xN$$

$$= (x * 0) * (x * 0)$$

$$= 1.$$ (ii) $\Rightarrow$ (i) Let $xN \lor x = 1$. Then

We obtain $N \land x = xN \land xN$

$$= (x \lor xN)$$

by Theorem 2.5 (ii)

$$= 1N$$

$$= 0.$$


REFERENCES


