SMARANDACHE SEMIRINGS AND SEMIFIELDS

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Abstract

In this paper we study the notions of Smarandache semirings and semifields and obtain some interesting results about them. We show that not every semiring is a Smarandache semiring. We similarly prove that not every semifield is a Smarandache semifield. We give several examples to make the concept lucid. Further, we propose an open problem about the existence of Smarandache semiring S of finite order.

Keywords: semiring, semifield, semi-algebra, distributive lattice, Smarandache semirings.

Definition [1]:

A non-empty set S is said to be a semiring if on S is defined two binary closed operations + and \times such that (S, +) is an abelian semigroup with 0 and (S, \times) is a semigroup and multiplication distributes over addition from the left and from the right.

A semiring is a strict semiring if x + y = 0 implies x = y = 0. Semiring is commutative if (S, \times) is a commutative semigroup. A commutative semiring is a semifield if (S, \times) has a unit element and x \times y = 0 in S if and only if x = y = 0. For more properties of semirings please refer [1], [3], [4] and [5].

Definition 1:

The Smarandache semiring is defined [4] to be a semiring S such that a proper subset A of S is a semifield (with respect to the same induced operation). That is \emptyset \neq A \subset S.

Example 1: Let \(M_{n \times n} = \{(a_{ij})| a_{ij} \in \mathbb{Z}^+ \cup \{0\}\} \). Here, \(\mathbb{Z}^+ \) denotes the set of positive integers. Clearly \(M_{n \times n} \) is a semiring with the matrix addition and matrix multiplication. For consider \(A = \{(a_{ij}) | a_{ii} = 0, i \neq j \text{ and } a_{ij} \in \mathbb{Z}^+ \cup \{0\}\} \), that is all diagonal matrices with entries from \(\mathbb{Z}^+ \cup \{0\}\). Clearly, A is a semifield. Hence \(M_{n \times n} \) is a Smarandache semiring.
Example 2: Let $S$ be the lattice given by the following figure. Clearly $S$ is a semiring under min-max operation. $S$ is a Smarandache semiring for $A = \{1, b, g, h, 0\}$ is a semifield.

![Lattice Diagram]

Theorem 2:

Every distributive lattice with 0 and 1 is a Smarandache Semiring.

Proof: Any chain connecting 0 and 1 is a lattice which is a semifield for every chain lattice is a semiring which satisfies all the postulates of a semifield. Hence the claim.

Definition 3:

The Smarandache sub-semiring [4] is defined to be a Smarandache semiring $B$ which is a proper subset of the Smarandache semiring $S$.

Example 3: Let $M_{ax}$ be the semiring as in Example 1. Clearly $M_{ax}$ is a Smarandache semiring. Now,

$$B = \begin{bmatrix}
 a_{11} & 0 & \ldots & 0 & 0 \\
 0 & 0 & \ldots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 \vdots & \vdots & \ddots & 0 & 0 \\
 0 & 0 & \ldots & 0 & a_{rn}
\end{bmatrix}$$

$a_{11}$ and $a_{rn} \in \mathbb{Z}^+ \cup \{0\}$
is a Smarandache sub-semiring.

**Example 4:** Let \( M_{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}^* \cup \{0\} \right\} \). Clearly \( M_{2 \times 2} \) under the matrix addition and multiplication is a semiring which is not a semifield. But \( M_{2 \times 2} \) is a Smarandache semiring for \( N = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}^* \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \) is a semifield.

**Theorem 4:**

Not all semirings are Smarandache semirings.

**Proof:** Let \( S = \mathbb{Z}^* \cup \{0\} \). \((S, +, \times)\) is a semiring which has no proper semifield contained in it. Hence the claim.

**Definition 5:**

The *Smarandache semifield* \([4]\) is defined to be a semifield \((S, +, \times)\) such that a proper subset of \( S \) is a \( K \)-semi algebra (with respect with the same induced operations and an external operation).

**Example 5:** Let \( S = \mathbb{Z}^* \cup \{0\} \). Now, \((S, +, \times)\) is a semifield. Consider \( p \in S \), \( p \) any prime. \( A = \{0, p, 2p, \ldots\} \) is a \( k \)-semi algebra. So \((S, +, \times)\) is a Smarandache semifield.

**Consequence 1:**

There also exist semifields which are not Smarandache semifields. The following example illustrates the case.

**Example 6:** Let \( S = \mathbb{Q}^* \cup \{0\} \). \((S, +, \times)\) is a semifield but it is not a Smarandache semifield.

**Example 7:** Let \( S = \mathbb{Z}^* \cup \{0\} \). Now \((S, +, \times)\) is a semifield. Let \( S[x] \) be polynomial semiring in the variable \( x \). Clearly \( S[x] \) is a Smarandache semiring for \( S \) is a proper subset of \( S[x] \) is a semifield.

**Theorem 5:**

Let \( S \) be any semifield. Every polynomial semiring is a Smarandache semiring.

**Proof:** Obvious from the fact \( S \) is a semifield contained in \( S[x] \).

We now pose an open problem about the very existence of finite semirings and Smarandache semirings that are not distributive lattices.
Problem 1: Does there exist a Smarandache semiring $S$ of finite order? ($S$ is not a finite distributive lattice)?

Note:

We do not have finite semirings other than finite distributive lattices. Thus the existence of finite semirings other than finite distributive lattices is an open problem even in semirings.

References


