Smarandache’s Cevian Triangle Theorem in
The Einstein Relativistic Velocity Model of
Hyperbolic Geometry

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Abstract

In this note, we present a proof of Smarandache’s cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

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1. Introduction

Hyperbolic geometry appeared in the first half of the 19th century as an attempt to understand Euclid’s axiomatic basis for geometry. It is also known as a type of non-Euclidean geometry, being in many respects similar to Euclidean geometry. Hyperbolic geometry includes such concepts as: distance, angle and both of them have many theorems in common. There are known many main models for hyperbolic geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-Euclidian geometry. Here, in this study, we present a proof of Smarandache’s cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry. Smarandache’s cevian triangle theorem states that if \( A_1B_1C_1 \) is the cevian triangle of point \( P \) with respect to the triangle \( ABC \), then \( \frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B_1B_1C \cdot C_1A} \) [1].

Let \( D \) denote the complex unit disc in complex \( z \)-plane, i.e.
\[
D = \{ z \in \mathbb{C} : |z| < 1 \}.
\]
The most general Möbius transformation of \( D \) is
\[
z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z}_0 z} = e^{i\theta}(z_0 \oplus z),
\]
which induces the Möbius addition \( \oplus \) in \( D \), allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation
\[
z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z}_0 z}
\]
followed by a rotation. Here \( \theta \in \mathbb{R} \) is a real number, \( z, z_0 \in D \), and \( \overline{z}_0 \) is the complex conjugate of \( z_0 \). Let \( Aut(D, \oplus) \) be the automorphism group.
of the grupoid \((D, \oplus)\). If we define
\[
gyr : D \times D \to Aut(D, \oplus), \ gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + \overline{ab}}{1 + \overline{ab}},
\]
then is true gyrocommutative law
\[
a \oplus b = \ gyr[a, b](b \oplus a).
\]

A gyrovector space \((G, \oplus, \otimes)\) is a gyrocommutative gyrogroup \((G, \oplus)\) that obeys the following axioms:

(1) \(\ gyr[u, v]a \cdot \ gyr[u, v]b = a \cdot b \) for all points \(a, b, u, v \in G\).

(2) \(G\) admits a scalar multiplication, \(\otimes\), possessing the following properties. For all real numbers \(r, r_1, r_2 \in \mathbb{R}\) and all points \(a \in G\):
\[
\begin{align*}
(G1) \quad & 1 \otimes a = a \\
(G2) \quad & (r_1 + r_2) \otimes a = r_1 \otimes a + r_2 \otimes a \\
(G3) \quad & (r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a) \\
(G4) \quad & |\frac{r \otimes a}{|r| \otimes a}| = \frac{|a|}{|a|} \\
(G5) \quad & gyr[u, v](r \otimes a) = r \otimes gyr[u, v]a \\
(G6) \quad & gyr[r_1 \otimes v, r_1 \otimes v] = 1
\end{align*}
\]

(3) Real vector space structure \((\|G\|, \oplus, \otimes)\) for the set \(\|G\|\) of onedimensional ”vectors”
\[
\|G\| = \{ \pm |a| : a \in G\} \subset \mathbb{R}
\]

with vector addition \(\oplus\) and scalar multiplication \(\otimes\), such that for all \(r \in \mathbb{R}\) and \(a, b \in G\),
\[
\begin{align*}
(G7) \quad & \|r \otimes a\| = |r| \otimes \|a\| \\
(G8) \quad & \|a \oplus b\| \leq \|a\| \oplus \|b\|
\end{align*}
\]
Theorem 1 (The Hyperbolic Theorem of Ceva in Einstein Gyrovector Space) Let $a_1, a_2,$ and $a_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V_s, \oplus, \otimes)$. Furthermore, let $a_{123}$ be a point in their gyroplane, which is off the gyrolines $a_1 a_2, a_2 a_3,$ and $a_3 a_1$. If $a_1 a_{123}$ meets $a_2 a_3$ at $a_{23}$, etc., then

$$\frac{\gamma_{a_1 a_{12}} \parallel a_1 \oplus a_{12}}{\gamma_{a_2 a_{12}} \parallel a_2 \oplus a_{12}} \frac{\gamma_{a_2 a_{23}} \parallel a_2 \oplus a_{23}}{\gamma_{a_3 a_{23}} \parallel a_3 \oplus a_{23}} \frac{\gamma_{a_3 a_{13}} \parallel a_3 \oplus a_{13}}{\gamma_{a_1 a_{13}} \parallel a_1 \oplus a_{13}} = 1,$$

(see [2, pp 461])

Theorem 2 (The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space) Let $a_1, a_2,$ and $a_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V_s, \oplus, \otimes)$. If a gyroline meets the sides of gyrotriangle $a_1 a_2 a_3$ at points $a_{12}, a_{13}, a_{23}$, then

$$\frac{\gamma_{a_1 a_{12}} \parallel a_1 \oplus a_{12}}{\gamma_{a_2 a_{12}} \parallel a_2 \oplus a_{12}} \frac{\gamma_{a_2 a_{23}} \parallel a_2 \oplus a_{23}}{\gamma_{a_3 a_{23}} \parallel a_3 \oplus a_{23}} \frac{\gamma_{a_3 a_{13}} \parallel a_3 \oplus a_{13}}{\gamma_{a_1 a_{13}} \parallel a_1 \oplus a_{13}} = 1$$

(see [2, pp 463])

For further details we refer to the recent book of A.Ungar [2].

2. Main result

In this section, we present a proof of Smarandache’s cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry.
Theorem 3 If $A_1B_1C_1$ is the cevian gyrotriangle of gyropoint $P$ with respect to the gyrotriangle $ABC$, then

$$\frac{\gamma_{[PA]}|PA| \cdot \gamma_{[PB]}|PB| \cdot \gamma_{[PC]}|PC|}{\gamma_{[PA_1]}|PA_1| \cdot \gamma_{[PB_1]}|PB_1| \cdot \gamma_{[PC_1]}|PC_1|} = \frac{\gamma_{[AB]}|AB| \cdot \gamma_{[BC]}|BC| \cdot \gamma_{[CA]}|CA|}{\gamma_{[AB_1]}|AB_1| \cdot \gamma_{[BC_1]}|BC_1| \cdot \gamma_{[CA_1]}|CA_1|}.$$ 

**Proof.** If we use a theorem 2 in the gyrotriangle $ABC$ (see Figure), we have

(1) $\gamma_{[AC_1]}|AC_1| \cdot \gamma_{[BA_1]}|BA_1| \cdot \gamma_{[CB_1]}|CB_1| = \gamma_{[AB_1]}|AB_1| \cdot \gamma_{[BC_1]}|BC_1| \cdot \gamma_{[CA_1]}|CA_1|$.

If we use a theorem 1 in the gyrotriangle $AA_1B$, cut by the gyroline $CC_1$, we get

(2) $\gamma_{[AC_1]}|AC_1| \cdot \gamma_{[BC]}|BC| \cdot \gamma_{[A_1P]}|A_1P| = \gamma_{[AP]}|AP| \cdot \gamma_{[A_1C]}|A_1C| \cdot \gamma_{[BC_1]}|BC_1|$.

If we use a theorem 1 in the gyrotriangle $BB_1C$, cut by the gyroline $AA_1$, we get

(3) $\gamma_{[BA_1]}|BA_1| \cdot \gamma_{[CA]}|CA| \cdot \gamma_{[B_1P]}|B_1P| = \gamma_{[BP]}|BP| \cdot \gamma_{[B_1A]}|B_1A| \cdot \gamma_{[CA_1]}|CA_1|$.

If we use a theorem 1 in the gyrotriangle $CC_1A$, cut by the gyroline $BB_1$, we get

(4) $\gamma_{[CB_1]}|CB_1| \cdot \gamma_{[AB]}|AB| \cdot \gamma_{[C_1P]}|C_1P| = \gamma_{[CP]}|CP| \cdot \gamma_{[C_1B]}|C_1B| \cdot \gamma_{[AB_1]}|AB_1|$.

We divide each relation (2), (3), and (4) by relation (1), and we obtain

(5) $\frac{\gamma_{[PA]}|PA|}{\gamma_{[PA_1]}|PA_1|} = \frac{\gamma_{[BC]}|BC|}{\gamma_{[BA_1]}|BA_1|} \cdot \frac{\gamma_{[B_1A]}|B_1A|}{\gamma_{[B_1C]}|B_1C|}$,

(6) $\frac{\gamma_{[PB]}|PB|}{\gamma_{[PB_1]}|PB_1|} = \frac{\gamma_{[CA]}|CA|}{\gamma_{[CB_1]}|CB_1|} \cdot \frac{\gamma_{[C_1B]}|C_1B|}{\gamma_{[C_1A]}|C_1A|}$. 

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Multiplying (5) by (6) and by (7), we have

\[
\frac{\gamma_{[PC]}|PC|}{\gamma_{[PC_1]}|PC_1|} = \frac{\gamma_{[AB]}|AB|}{\gamma_{[AC_1]}|AC_1|} \cdot \frac{\gamma_{[A_1C]}|A_1C|}{\gamma_{[A_1B]}|A_1B|}
\]

(7)

Multiplying (5) by (6) and by (7), we have

\[
\frac{\gamma_{[PA]}|PA|}{\gamma_{[PA_1]}|PA_1|} \cdot \frac{\gamma_{[PB]}|PB|}{\gamma_{[PB_1]}|PB_1|} \cdot \frac{\gamma_{[PC]}|PC|}{\gamma_{[PC_1]}|PC_1|} = \frac{\gamma_{[AB]}|AB| \cdot \gamma_{[BC]}|BC| \cdot \gamma_{[CA]}|CA|}{\gamma_{[AB_1]}|AB_1| \cdot \gamma_{[BC_1]}|BC_1| \cdot \gamma_{[CA_1]}|CA_1|}
\]

(8)

From the relation (1) we have

\[
\frac{\gamma_{[B_1A]}|B_1A| \cdot \gamma_{[C_1B]}|C_1B| \cdot \gamma_{[A_1C]}|A_1C|}{\gamma_{[A_1B]}|A_1B| \cdot \gamma_{[B_1C]}|B_1C| \cdot \gamma_{[C_1A]}|C_1A|} = 1,
\]

(9)

so

\[
\frac{\gamma_{[PA]}|PA|}{\gamma_{[PA_1]}|PA_1|} \cdot \frac{\gamma_{[PB]}|PB|}{\gamma_{[PB_1]}|PB_1|} \cdot \frac{\gamma_{[PC]}|PC|}{\gamma_{[PC_1]}|PC_1|} = \frac{\gamma_{[AB]}|AB| \cdot \gamma_{[BC]}|BC| \cdot \gamma_{[CA]}|CA|}{\gamma_{[AB_1]}|AB_1| \cdot \gamma_{[BC_1]}|BC_1| \cdot \gamma_{[CA_1]}|CA_1|}.
\]

References
