Smarandache’s Cevian Triangle Theorem in
The Einstein Relativistic Velocity Model of
Hyperbolic Geometry

Cătălin Barbu

"Vasile Alecsandri" College - Bacău, str. Iosif Cocea, nr. 12, sc. A, ap. 13,
Romania
kafka_mate@yahoo.com

Abstract

In this note, we present a proof of Smarandache’s cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

2000 Mathematical Subject Classification: 51K05, 51M10, 30F45, 20N99, 51B10

Keywords and phrases: hyperbolic geometry, hyperbolic triangle, Smarandache’s cevian triangle, gyrovector, Einstein relativistic velocity model
1. Introduction

Hyperbolic geometry appeared in the first half of the 19th century as an attempt to understand Euclid’s axiomatic basis for geometry. It is also known as a type of non-Euclidean geometry, being in many respects similar to Euclidean geometry. Hyperbolic geometry includes such concepts as: distance, angle and both of them have many theorems in common. There are known many main models for hyperbolic geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-Euclidian geometry. Here, in this study, we present a proof of Smarandache’s cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry. Smarandache’s cevian triangle theorem states that if \( A_1B_1C_1 \) is the cevian triangle of point \( P \) with respect to the triangle \( ABC \), then

\[
\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} = \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B_1C_1A_1} \quad [1].
\]

Let \( D \) denote the complex unit disc in complex \( z \)-plane, i.e.

\[ D = \{ z \in \mathbb{C} : |z| < 1 \}. \]

The most general Möbius transformation of \( D \) is

\[
z \rightarrow e^{i\theta} \frac{z_0 + z}{1 + z_0 \overline{z}} = e^{i\theta} (z_0 \oplus z),
\]

which induces the Möbius addition \( \oplus \) in \( D \), allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

\[
z \rightarrow z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}
\]

followed by a rotation. Here \( \theta \in \mathbb{R} \) is a real number, \( z, z_0 \in D \), and \( \overline{z_0} \) is the complex conjugate of \( z_0 \). Let \( \text{Aut}(D, \oplus) \) be the automorphism group
of the grupoid \((D, \oplus)\). If we define

\[
gyr : D \times D \to \text{Aut}(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + ab}{1 + ab},
\]

then is true gyrocommutative law

\[
a \oplus b = gyr[a, b](b \oplus a).
\]

A gyrovector space \((G, \oplus, \otimes)\) is a gyrocommutative gyrogroup \((G, \oplus)\) that obeys the following axioms:

1. \(gyr[u, v]a \cdot gyr[u, v]b = a \cdot b\) for all points \(a, b, u, v \in G\).

2. \(G\) admits a scalar multiplication, \(\otimes\), possessing the following properties. For all real numbers \(r, r_1, r_2 \in \mathbb{R}\) and all points \(a \in G\):

   (G1) \(1 \otimes a = a\)

   (G2) \((r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a\)

   (G3) \((r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a)\)

   (G4) \(\frac{|r| \otimes a}{|r| \otimes a} = \frac{a}{|a|}\)

   (G5) \(gyr[u, v](r \otimes a) = r \otimes gyr[u, v]a\)

   (G6) \(gyr[r_1 \otimes v, r_1 \otimes v] = 1\)

3. Real vector space structure \((\|G\|, \oplus, \otimes)\) for the set \(\|G\|\) of onedimensional "vectors"

\[
\|G\| = \{ \pm \|a\| : a \in G \} \subset \mathbb{R}
\]

with vector addition \(\oplus\) and scalar multiplication \(\otimes\), such that for all \(r \in \mathbb{R}\) and \(a, b \in G\),

(G7) \(\|r \otimes a\| = |r| \otimes \|a\|\)

(G8) \(\|a \oplus b\| \leq \|a\| \oplus \|b\|\)
Theorem 1 (The Hyperbolic Theorem of Ceva in Einstein Gyrovector Space) Let \( a_1, a_2, \) and \( a_3 \) be three non-gyrocollinear points in an Einstein gyrovector space \((V_s, \oplus, \otimes)\). Furthermore, let \( a_{123} \) be a point in their gyroplane, which is off the gyrolines \( a_1a_2, a_2a_3, \) and \( a_3a_1 \).

If \( a_1a_{123} \) meets \( a_2a_3 \) at \( a_{23} \), etc., then

\[
\frac{\gamma_{\ominus a_1 \oplus a_{12}}} {\gamma_{\ominus a_2 \oplus a_{12}}} = 1,
\]

(here \( \gamma_v = \frac{1}{\sqrt{1 - \|v\|^2}} \) is the gamma factor).

(see [2, pp 461])

Theorem 2 (The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space) Let \( a_1, a_2, \) and \( a_3 \) be three non-gyrocollinear points in an Einstein gyrovector space \((V_s, \oplus, \otimes)\). If a gyroline meets the sides of gyrotriangle \( a_1a_2a_3 \) at points \( a_{12}, a_{13}, a_{23} \), then

\[
\frac{\gamma_{\ominus a_1 \oplus a_{12}}} {\gamma_{\ominus a_2 \oplus a_{12}}} = 1
\]

(see [2, pp 463])

For further details we refer to the recent book of A.Ungar [2].

2. Main result

In this section, we present a proof of Smarandache’s cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry.
Theorem 3  If $A_1B_1C_1$ is the cevian gyrotriangle of gyropoint $P$ with respect to the gyrotriangle $ABC$, then
\[
\frac{\gamma_{[PA]}[PA]}{\gamma_{[PA_1]}[PA_1]} \cdot \frac{\gamma_{[PB]}[PB]}{\gamma_{[PB_1]}[PB_1]} \cdot \frac{\gamma_{[PC]}[PC]}{\gamma_{[PC_1]}[PC_1]} = \frac{\gamma_{[AB]}[AB]}{\gamma_{[AB_1]}[AB_1]} \cdot \frac{\gamma_{[BC]}[BC]}{\gamma_{[BC_1]}[BC_1]} \cdot \frac{\gamma_{[CA]}[CA]}{\gamma_{[CA_1]}[CA_1]}
\]

Proof.  If we use a theorem 2 in the gyrotriangle $ABC$ (see Figure), we have

(1)  $\gamma_{[AC_1]}[AC_1] \cdot \gamma_{[BA_1]}[BA_1] \cdot \gamma_{[CB_1]}[CB_1] = \gamma_{[AB_1]}[AB_1] \cdot \gamma_{[BC_1]}[BC_1] \cdot \gamma_{[CA_1]}[CA_1]$.

If we use a theorem 1 in the gyrotriangle $AA_1B$, cut by the gyroline $CC_1$, we get

(2)  $\gamma_{[AC_1]}[AC_1] \cdot \gamma_{[BC_1]}[BC_1] \cdot \gamma_{[A_1P]}[A_1P] = \gamma_{[AP]}[AP] \cdot \gamma_{[A_1C]}[A_1C] \cdot \gamma_{[BC_1]}[BC_1]$.

If we use a theorem 1 in the gyrotriangle $BB_1C$, cut by the gyroline $AA_1$, we get

(3)  $\gamma_{[BA_1]}[BA_1] \cdot \gamma_{[CA_1]}[CA_1] \cdot \gamma_{[B_1P]}[B_1P] = \gamma_{[BP]}[BP] \cdot \gamma_{[B_1A]}[B_1A] \cdot \gamma_{[CA_1]}[CA_1]$.

If we use a theorem 1 in the gyrotriangle $CC_1A$, cut by the gyroline $BB_1$, we get

(4)  $\gamma_{[CB_1]}[CB_1] \cdot \gamma_{[AB_1]}[AB_1] \cdot \gamma_{[C_1P]}[C_1P] = \gamma_{[CP]}[CP] \cdot \gamma_{[C_1B]}[C_1B] \cdot \gamma_{[AB_1]}[AB_1]$.

We divide each relation (2), (3), and (4) by relation (1), and we obtain

(5)  $\frac{\gamma_{[PA]}[PA]}{\gamma_{[PA_1]}[PA_1]} = \frac{\gamma_{[BC]}[BC]}{\gamma_{[B_1A]}[B_1A] \cdot \gamma_{[B_1C]}[B_1C]}$

(6)  $\frac{\gamma_{[PB]}[PB]}{\gamma_{[PB_1]}[PB_1]} = \frac{\gamma_{[CA]}[CA]}{\gamma_{[C_1B]}[C_1B] \cdot \gamma_{[C_1A]}[C_1A]}$.
Multiplying (5) by (6) and by (7), we have

\[ \frac{\gamma_{[PC]}|PC|}{\gamma_{[PC_1]}|PC_1|} = \frac{\gamma_{[AB]}|AB|}{\gamma_{[AC]}|AC_1|} \cdot \frac{\gamma_{[A_1C]}|A_1C|}{\gamma_{[A_1B]}|A_1B|}. \]

From the relation (1) we have

\[ \frac{\gamma_{[AB]}|AB| \cdot \gamma_{[BC]}|BC| \cdot \gamma_{[CA]}|CA|}{\gamma_{[A_1B]}|A_1B| \cdot \gamma_{[B_1C]}|B_1C| \cdot \gamma_{[C_1A]}|C_1A|}. \]

References
