SMARANDACHE HYPER-K-ALGEBRAS

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Abstract. We introduce the notion of an extension of hyper K-algebras and Smaranda-
che hyper (\(\mathcal{I}, \mathcal{E}\))-ideal in hyper K-algebras, and investigate its properties.

1. Introduction

Generally, in any human field, a Smarandache Structure on a set A means a weak struc-
ture W on A such that there exists a proper subset B of A which is embedded with a
strong structure S. In [10], W. B. Vasudeva Kaudasmey studied the concept of Smaran-
dache substructures, subgroups, ideal of groups, semi-normal subgroups, Smarandache
Subgroups and strong Subgroups and obtained many interesting results about them.
Smarandache semigroups are very important for the study of congruences, and it was stud-
ed by R. Padilla [9].

In this paper, we introduce the notion of an extension of hyper K-algebras and Smarandache
hyper (\(\mathcal{I}, \mathcal{E}\))-ideal in hyper K-algebras, and investigate its properties.

2. Preliminaries

We include some elementary aspects of hyper K-algebras that are necessary for this
paper, and for more details we refer to [3] and [11]. Let \(H\) be a non-empty set endow-
ed with a hyper operation \(\circ\); that is, \(\circ\) is a function from \(H \times H\) to \(P(H) \setminus \{\}\). For
two subsets \(A\) and \(B\) of \(H\), denote by \(A \circ B\) the set \(\bigcup_{a \in A, b \in B} \{a \circ b\}\).

By a hyper BCK-algebra we mean a non-empty set \(H\) endowed with a hyperoperation
\(\circ\) and a constant \(0\) satisfying the following axioms:

\[(K1)\] \(x \circ z \subseteq (y \circ z) \subseteq x \circ y,\)
\[(K2)\] \(x \circ (y \circ z) = (x \circ y) \circ z,\)
\[(K3)\] \(x \circ 0 = x,\)
\[(K4)\] \(x \circ y = y \subseteq x \Rightarrow x = y,\)
for all \(x, y, z \in H\), when \(x \subseteq y\) is defined by \(0 \subseteq x \circ y\) and for every \(A, B \subseteq H, A \subseteq B\) is
defined by \(\forall a \in A, \exists b \in B\) such that \(a \subseteq b\).

By a hyper I-algebra we mean a non-empty set \(H\) endowed with a hyper operation \(\circ\) and
a constant \(1\) satisfying the following axioms:

\[(H1)\] \(x \circ 0 = 0 \subseteq x \circ y,\)
\[(H2)\] \(x \circ y \subseteq z \Rightarrow x \circ (y \circ z),\)
\[(H3)\] \(x \subseteq x,\)
\[(H4)\] \(x \subseteq y \Rightarrow x \circ y = x,\)
for all \(x, y, z \in H\).

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K(hyper BCK)-algebra, hyper \(K\)-product of hyper K-algebras, hyper BCK-product of hyper BCK-algebras,
Smarandache hyper \((\mathcal{I}, \mathcal{E})\)-ideal of hyper \(K\)-algebras.

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for all \( x, y, z \in H \), where \( x < y \) is defined by \( 0 \in x \circ y \) and for every \( A, B \subseteq H \), \( A < B \) is defined by \( 3a \in A \) and \( 3b \in B \) such that \( a < b \). If a hyper \( I \)-algebra \((H, \circ, 0)\) satisfies an additional condition:

\[ (H) \quad 0 < x \text{ for all } x \in H, \]

then \((H, \circ, 0)\) is called a hyper \( K\)-algebra (see [1]).

Every hyper \( BCK\)-algebra is a hyper \( K\)-algebra. We know that there exists a proper hyper \( K\)-algebra that is, there exists a hyper \( K\)-algebra which is not a hyper \( BCK\)-algebra (see [1, Theorem 2.5]).

In a hyper \( I\)-algebra \( H \), the following hold (see [1, Proposition 3.4]):

(a1) \((A \circ B) \circ C = (A \circ C) \circ B\),

(a2) \(A \circ B < C \iff A \circ C < B\),

(a3) \(A \subseteq B\) implies \( A < B\)

for all non-empty subsets \( A, B \) and \( C \) of \( H \).

In a hyper \( K\)-algebra \( F \), the following holds (see [1, Proposition 3.6]):

(a4) \(x \in x \circ 0\) for all \( x \in H\).

Definition 2.1. (1) Let \((H, \circ, 0)\) be a hyper \( K\)-algebra and let \( S \) be a subset of \( H \) containing \( 0 \). If \( S \) is a hyper \( K\)-algebra with respect to the hyperoperation \("\circ\" on \( H \), we say that \( S \) is a hyper \( K\)-subalgebra of \( H\).

Note that if \( S \) is a non-empty subset of a hyper \( K\)-algebra \((H, \circ, 0)\), then \( S \) is a hyper \( K\)-subalgebra of \( H \) if and only if \( x \circ y \subseteq S \) for all \( x, y \in S \) (see [3, Theorem 4.12]).

Definition 2.2. (3, Theorem 3.4) \( \beta\), Smarandache hyper \( K\)-algebra is defined to be a hyper \( K\)-algebra \((H, \circ, 0)\) in which there exists a proper subset \( \Omega \) of \( H \) such that \((\Omega, 0, \circ)\) is a non-trivial hyper \( BCK\)-algebra.

Example 2.3. ([3, Example 3.5]) Let \( H = \{ 0, a, b, c \} \) and define an hyper operation \("\circ\" on \( H \) by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
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<tr>
<td>a</td>
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<td>0</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>a</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3a

Then \((H, 0, \circ)\) is a Smarandache hyper \( K\)-algebra because \((\Omega = \{ 0, a, b \}, 0, \circ)\) is a hyper \( BCK\)-algebra.

Example 2.4. ([3, Example 3.6]) Let \( H = \{ 0, a, b \} \) and define an hyper operation \("\circ\" on \( H \) by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>a</td>
</tr>
</tbody>
</table>

Table 3b

Then \((H, 0, \circ)\) is not a Smarandache hyper \( K\)-algebra since \((\Omega_1 = \{ 0, a \}, 0, \circ)\) and \((\Omega_2 = \{ 0, b \}, 0, \circ)\) are not hyper \( BCK\)-algebras.
Definition 2.5. ([3, Definition 3.1]) Let $H$ be a Smarandache hyper $K$-algebra and $\theta$ be a non-trivial hyper $BCK$-algebra which is properly contained in $H$. Then a non-empty subset $I$ of $H$ is called a Smarandache hyper $\langle\varepsilon, e\rangle$-ideal of $H$ related to $\theta$ (or briefly, $\theta$-Smarandache hyper $\langle\varepsilon, e\rangle$-ideal of $H$) if it satisfies:

(i) $0 \in I$.

(ii) $(x \in I) \rightarrow (x \circ y \in I)$ for all $x, y \in I$.

If $I$ is a Smarandache hyper $\langle\varepsilon, e\rangle$-ideal of $H$ related to every hyper $BCK$-algebra contained in $H$, we simply say that $I$ is a Smarandache hyper $\langle\varepsilon, e\rangle$-ideal of $H$.

Definition 2.6. ([3, Definition 3.1]) Let $H$ be a Smarandache hyper $K$-algebra and $\Omega$ be a non-trivial hyper $BCK$-algebra which is properly contained in $H$. Then a non-empty subset $I$ of $H$ is called a Smarandache hyper $\langle\varepsilon, e\rangle$-ideal of $H$ related to $\Omega$ (or briefly, $\Omega$-Smarandache hyper $\langle\varepsilon, e\rangle$-ideal of $H$) if it satisfies:

(i) $0 \in I$.

(ii) $(x, y \in I) \rightarrow (x \circ y \in I)$ for all $x, y \in I$.

If $I$ is a Smarandache hyper $\langle\varepsilon, e\rangle$-ideal of $H$ related to every hyper $BCK$-algebra contained in $H$, we simply say that $I$ is a Smarandache hyper $\langle\varepsilon, e\rangle$-ideal of $H$.

3. Main Result

Proposition 3.1. Let $(H, \cdot, 0)$ be a hyper $K$-algebra with $|H| \geq 3$. Then the following statements hold:

(i) If there exists a hyper $K$-subalgebra $S$ of $H$ such that $1 \in [S] < |H|$ and $|x \circ y| = 1$ for all $x, y \in S$, then $H$ is a Smarandache hyper $K$-algebra.

(ii) If there exists $x \in H$ such that $x \circ x \subseteq \{0, x\}$, then $H$ is a Smarandache hyper $K$-algebra.

Proof. (i) Let $S$ be a hyper $K$-subalgebra of $H$ such that $2 \leq |S| < |H|$ and $|x \circ y| = 1$ for all $x, y \in S$. Then it can be easily verified that $(S, 0, 0)$ is a hyper $BCK$-algebra. Therefore $H$ is a Smarandache hyper $K$-algebra.

(ii) Let $x \in H$ be such that $x \circ x \subseteq \{0, x\}$. Note that, $(\{0, x\}, 0, 0)$ is a hyper $BCK$-algebra, and so $H$ is a Smarandache hyper $K$-algebra.

Example 3.2. The condition $|x \circ y| = 1$ for all $x, y \in S$ in the Proposition 3.1(i) is necessary.

To see this, we consider $H = \{0, 1, 2\}$ in Example 2.4. Then $(S = \{0, 1\}, 0, 0)$ is a hyper $K$-algebra, but $(\{0, 0\}, 0, 0)$ is not a Smarandache hyper $K$-algebra.

Definition 3.3. Let $(H, \cdot, 0)$ be a hyper $K$-algebra. By an extension of $H$ we mean a hyper $K$-algebra $\langle\varepsilon, x\rangle$ such that:

(i) $H \subseteq L$.

(ii) $(x, y \in H) \rightarrow (x \circ y \in \varepsilon)$.

Example 3.4. ([1, Theorem 3.7]) Let $(H_1, \cdot_1, 0_1)$ and $(H_2, \cdot_2, 0_2)$ be hyper $K$-algebras (resp. hyper $BCK$-algebras) such that $H_1 \cap H_2 = \{0\}$ and $H = H_1 \cup H_2$. Then $(H, \cdot, 0)$ is a hyper $K$-algebra (resp. hyper $BCK$-algebra), where the hyperoperation $\cdot$ can be defined as follows:

$$x \circ y :=
\begin{cases}
0_1, & x \in H_1, y \in H_2,
0_2, & x \in H_2, y \in H_1,
\varepsilon, & \text{otherwise}
\end{cases}$$

for all $x, y \in H$.

We use the notation $H_1 \oplus H_2$ for the union of two hyper $K$-algebras (resp. hyper $BCK$-algebra) $H_1$ and $H_2$. 
Theorem 3.5. If \( H \) is a Smarandache hyper \( K \)-algebra, then every extension of \( H \) is also a Smarandache hyper \( K \)-algebra.

Proof. Straightforward.

The following example show that there exists a non-Smarandache hyper \( K \)-algebra \( H \) such that an extension \( L \) of \( H \) is a Smarandache hyper \( K \)-algebra.

Example 3.6. Let \((H = \{0, \bar{0}, y\}, \circ, 0)\) be a hyper \(BCK\)-algebra and let \((K = \{1, 1, y\}, \circ, 0)\) be a hyper \(K\)-algebra with the following Cayley tables:

\[
\begin{array}{c|c|c|c}
\mathbf{e} & 0 & y \\
\hline
0 & 0 & 0 \\
y & y & 0 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\mathbf{e} & 0 & y \\
\hline
0 & 0 & 0 \\
y & y & 0 \\
\end{array}
\]

Then \((L = H \oplus K, \circ, 0)\) is a Smarandache hyper \(K\)-algebra and it is an extension of \(H\). But \(H\) is not a Smarandache hyper \(K\)-algebra since does not exist a proper subset \(\Omega\) of \(H\) such that \((\Omega, \circ, 0)\) is a non-trivial hyper \(BCK\)-algebra.

Lemma 3.7. ([1, Theorem 3.9]) Let \((H_1, \circ_1, 0)\) and \((H_2, \circ_2, 0)\) be hyper \(K\)-algebras (resp. hyper \(BCK\)-algebras) and \(H = H_1 \times H_2\). We define a hyperoperation \(\circ^*\) on \(H\) is defined as follows,

\[(a_1, b_1) \circ^* (a_2, b_2) = (a_1 \circ_1 a_2, b_1 \circ_2 b_2)\]

for all \((a_1, b_1), (a_2, b_2) \in H\), where for \(A \subseteq H_1\) and \(B \subseteq H_2\) by \((A, B)\) we mean \((A, B) = \{(a, b) : a \in A, b \in B\}, 0 = (0_1, 0_2)\) and

\[(a_1, b_1) < (a_2, b_2) \Leftrightarrow a_1 < a_2, b_1 < b_2.\]

Then \((H, \circ, 0)\) is a hyper \(K\)-algebra (resp. hyper \(BCK\)-algebra), and it is called the hyper \(K\)-product (resp. hyper \(BCK\)-product) of \(H_1\) and \(H_2\).

Theorem 3.8. Let \((H_1, \circ_1, 0)\) and \((H_2, \circ_2, 0)\) be hyper \(K\)-algebras. If \((H_1, \circ_1, 0)\) is a Smarandache hyper \(K\)-algebra or \((H_2, \circ_2, 0)\) is a Smarandache hyper \(K\)-algebra, then the hyper \(K\)-product \(H = H_1 \times H_2\) of \(H_1\) and \(H_2\) is also a Smarandache hyper \(K\)-algebra.

Proof. We may assume that \((H_1, \circ_1, 0)\) is a Smarandache hyper \(K\)-algebra without loss of generality. Then there exists a non-trivial hyper \(BCK\)-algebra \(\Omega\) in \(H_1\). Let \(\Gamma = \Omega \times \{0_2\}\). Then \(\Gamma\) is a proper subset of \(H = H_1 \times H_2\) and obviously \((\Gamma, \circ, 0)\) is a non-trivial hyper \(BCK\)-algebra. Hence \(H = H_1 \times H_2\) is a Smarandache hyper \(K\)-algebra.

The following example shows that the converse of Theorem 3.8 is not true in general.

Example 3.9. Let \(H_1 = \{0, x\}\) and \(H_2 = \{0, y\}\) and define the hyperoperations \(\circ_{1*}\) and \(\circ_{2*}\) on \(H_1\) and \(H_2\) respectively as follows:

\[
\begin{array}{c|c|c|c}
\mathbf{e} & 0 & x \\
\hline
0 & 0 & 0 \\
x & x & 0 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\mathbf{e} & 0 & y \\
\hline
0 & 0 & 0 \\
y & y & 0 \\
\end{array}
\]

Then \(H_1\) is a hyper \(BCK\)-algebra and \(H_2\) is a hyper \(K\)-algebra. We know that \((H_1 \times H_2, \circ, 0 = (0_1, 0_2))\) is a hyper \(K\)-algebra with the following Cayley table:

\[
\begin{array}{c|c|c|c|c|c|c}
(0_1, 0_2) & (0_1, 0_2) & (0_1, 0_2) & (0_1, 0_2) & (0_1, 0_2) & (0_1, 0_2) & (0_1, 0_2) \\
(0_1, y) & (0_1, (0_1, y)) & (0_1, (0_1, y)) & (0_1, (0_1, y)) & (0_1, (0_1, y)) & (0_1, (0_1, y)) & (0_1, (0_1, y)) \\
(x, 0_2) & (x, 0_2) & (x, 0_2) & (x, 0_2) & (x, 0_2) & (x, 0_2) & (x, 0_2) \\
(x, y) & (x, (0_1, y)) & (x, (0_1, y)) & (x, (0_1, y)) & (x, (0_1, y)) & (x, (0_1, y)) & (x, (0_1, y)) \\
\end{array}
\]
Now a proper subset \( H_1 \subset (0,0) \) of \( H_1 \cup H_2 \) is a non-trivial hyper \( BCK\)-algebra. Thus \( H_1 \times H_2 \) is a \( \text{Smarandache} \) \( K\)-algebra. But we know that neither \( H_1 \) nor \( H_2 \) is a \( \text{Smarandache} \) \( K\)-algebra.

**Proposition 3.10.** Let \( (H_1, \alpha_1, 0) \) and \( (H_2, \alpha_2, 0) \) be \( K\)-algebras. If \( (H_1 \times H_2, \alpha, 0) \), the hyper \( K\)-product of \( H_1 \) and \( H_2 \), is a \( \text{Smarandache} \) \( K\)-algebra, then \( H_1 \times H_2 \) is a \( \text{Smarandache} \) \( K\)-algebra.

**Proof.** Let \( (H_1 \times H_2, \alpha, 0) \) be a \( \text{Smarandache} \) \( K\)-algebra. Then there exists a proper subset \( \Omega \) of \( H_1 \times H_2 \) such that \( (\Omega, \alpha, 0) \) is a non-trivial hyper \( BCK\)-algebra. Let \( \Omega_1 = \{ x \in H_1 : (x, b) \in \Omega \} \), for some \( b \in H_2 \), and \( \Omega_2 = \{ y \in H_2 : (a, y) \in \Omega \} \), for some \( a \in H_1 \). It is easily verified that \( \Omega = \Omega_1 \cup \Omega_2 \). Let \( x, y, z \in \Omega_1 \). Then there exist \( a, b, c \in H_2 \) such that \( (x, a), (y, b), (z, c) \in \Omega \). Now we show that \( (\Omega_1, \alpha_1, 0) \) is a hyper \( BCK\)-algebra.

\((\Omega_1, \Omega_2) \text{ satisfies the condition } (HI)\), we have:

\[ ((x, a), (x, c)) \circ ((y, b), (x, c)) \leq ((x, a), (y, b)), \]

that is,

\[ ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]) \leq ([x, a] \circ_2 [y, a] \circ_2 [c, b]). \]

Hence \( ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]) \leq ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]), \)

or \((\Omega_1, \alpha_1, 0) \) satisfies the condition \((HI)\). We have:

\[ (x, a) \circ_1 (y, b) \in (x, a) \circ_1 (y, b), \]

which implies that \( ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]) \leq ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]), \)

Hence, we get \( (x, a) \circ_1 (y, b) \leq (x, a) \circ_1 (y, b), \)

\[ ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]) \leq ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]), \]

or \((\Omega_1, \alpha_1, 0) \text{ satisfies the condition } (HI)\). Since \( (\Omega_1, \alpha_1, 0) \) satisfies the condition \((HI)\), we have \((x, a) \circ_1 (y, b) \leq (x, a) \circ_1 (y, b), \)

which implies that \( ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]) \leq ([x, a] \circ_1 [y, a] \circ_1 [a, c] \circ_1 [b, c]), \)

Hence, we get \( (x, a) \circ_1 (y, b) \leq (x, a) \circ_1 (y, b), \)

Hence, we get \( x \circ y \leq x \) and so \((\Omega_1, \alpha_1, 0) \) holds in \( (\Omega_1, \alpha_1, 0) \).

Thus, \((\Omega_1, \alpha_1, 0) \text{ is a hyper } BCK\)-algebra. In the similar way we can show that \((\Omega_2, \alpha_2, 0) \) is a hyper \( BCK\)-algebra. It follows from \( \Xi \neq (0,0) \), that \( \Omega_1 \neq \emptyset \) or \( \Omega_2 \neq \emptyset \). Without loss of generality we may assume that \( \Omega_1 \neq \emptyset \). Note that \( \Omega_1 \subset H_1 \), but \( \Omega_2 \neq H_1 \) since \( H_1 \) is a \( \text{hyper } BCK\)-algebra. Hence, \( \Omega_1 \) is a proper subset of \( H_1 \) such that \((\Omega_1, \alpha_1, 0) \) is a \( \text{non-trivial } BCK\)-algebra. Therefore \( H_1 \) is a \( \text{Smarandache} \) \( K\)-algebra.

**Proposition 3.11.** Let \( (H_1, \alpha_1, 0) \) and \( (H_2, \alpha_2, 0) \) be \( K\)-algebras such that \( H_1 \cap H_2 = \{ 0 \} \). If at least one of \( H_1 \) and \( H_2 \) is a \( \text{Smarandache } K\)-algebra, then \((H_1 \oplus H_2, \alpha, 0) \), the union of \( H_1 \) and \( H_2 \), is a \( \text{Smarandache } K\)-algebra.

**Proof.** Let \( (H_1, \alpha_1, 0) \) and \( (H_2, \alpha_2, 0) \) be \( K\)-algebras such that \( H_1 \cap H_2 = \{ 0 \} \). Without loss of generality we may assume that \( H_1 \) is a \( \text{Smarandache } K\)-algebra. Thus there exists a proper subset \( \Omega \) of \( H_1 \) such that \((\Omega, \alpha, 0) \) is a non-trivial hyper \( BCK\)-algebra. Since \( H_1 \subset H_1 \oplus H_2 \), \( \Omega \) is a proper subset of \( H_1 \oplus H_2 \). By the definition of hyperintersection \( \cap \) on \( H_1 \oplus H_2 \) and \( \cap \subset H_1 \), we have \((\Omega, \alpha, 0) \) is a \( \text{Smarandache } K\)-algebra.

The following example shows that the converse of Proposition 3.11 may not be true.

**Example 3.12.** Consider the hyper \( K\)-algebras \( H_1 = \{ 0, x \} \) and \( H_2 = \{ 0, y \} \) as in Example 3.9, where \( 0 = y_1 = y_2 \). It is easily verified that \((H_1 \oplus H_2, \alpha, 0) \) is a hyper \( K\)-algebra under
the following Cayley table.

\[
\begin{array}{ccc}
0 & a & b \\
0 & 0 & a & b \\
a & a & 0 & a \\
b & a & b & 0 \\
\end{array}
\]

Using the above table it is easily verified that \((0, a, b, 0)\) is a hyper BCK-algebra. Therefore, \(H_1 \oplus H_2\) is a Smarandache hyper K-algebra. But \(H_1\) and \(H_2\) are not Smarandache hyper K-algebras, since \(|H_1| = 2 = |H_2|\).

Proposition 3.13. Let \((H_1, 0_1, 0)\) and \((H_2, 0_2, 0)\) be hyper K-algebras such that \(H_1 \cap H_2 = \{0\}\). If \((H_1 \oplus H_2, 0)\) is a Smarandache hyper K-algebra, then at least one of \(H_1\) and \(H_2\) is a Smarandache hyper K-algebra.

Proof. Let \((H_1 \oplus H_2, 0)\) be a Smarandache hyper K-algebra. Then there exists a proper subset \(\Omega\) of \(H_1 \oplus H_2\) such that \((\Omega, 0, 0)\) is a non-trivial hyper BCK-algebra. Assume that \(\Omega_1 = \Omega \cap H_1\) and \(\Omega_2 = \Omega \cap H_2\). Then \(\Omega = \Omega_1 \cup \Omega_2\) and so \(\Omega_1 \neq \emptyset\) or \(\Omega_2 \neq \emptyset\). Without loss of generality we may assume that \(\Omega_1 \neq \emptyset\). Since \(x \circ y = x \circ y\) for all \(x, y \in \Omega_1\), we have \((\Omega_1, 0_1) = (\Omega_1, 0_0)\). Let \(x, y \in H_1\) and \(x, y \in \Omega\). Then \(x \circ y = x \circ y\) and \(x \circ y \in \Omega_1\). This shows that \(\Omega_1\) is a hyper subalgebra of \(\Omega\). Hence, \((\Omega_1, 0_1) = (\Omega_1, 0_0)\) is a non-trivial hyper BCK-algebra. Obviously \(\Omega_1\) is a proper subset of \(H_1\). Therefore \(H_1\) is a Smarandache hyper K-algebra.

Definition 3.14. Let \(H\) be a Smarandache hyper K-algebra, \(\Omega\) be a non-trivial hyper BCK-algebra which is properly contained in \(H\). Then a non-empty subset \(I\) of \(H\) is called a Smarandache hyper \((\cap, \circ)\)-ideal of \(H\) related to \(\Omega\) (or briefly, \(\Omega\)-Smarandache hyper \((\cap, \circ)\)-ideal) of \(H\) if it satisfies:

\((c_1)\) \(I \subseteq \Omega\),
\((c_2)\) \(\{y \in \Omega | (x \circ y) \cap I \neq \emptyset \Rightarrow x \in I\}\).

If \(I\) is a Smarandache hyper \((\cap, \circ)\)-ideal of \(H\) related to every hyper BCK-algebra contained in \(H\), then we say that \(I\) is a Smarandache hyper \((\cap, \circ)\)-ideal of \(H\).

Example 3.15. Let \(H = \{0, a, b, c\}\) and define the hyperoperation \(\circ\) on \(H\) by the following Cayley table:

\[
\begin{array}{ccc}
0 & a & b \\
0 & 0 & a & b \\
a & a & 0 & a \\
b & a & b & 0 \\
c & c & c & b \\
c & c & c & b \\
\end{array}
\]

Then \((H, 0, 0)\) is a Smarandache hyper K-algebra because \((\Omega = \{0, a, b, 0\})\) is a hyper BCK-algebra. Moreover, a subset \((0, a)\) is an \(\Omega\)-Smarandache hyper \((\cap, \circ)\)-ideal of \(H\).

Theorem 3.16. Let \(H\) be a Smarandache hyper K-algebra, \(\Omega\) be a non-trivial hyper BCK-algebra which is properly contained in \(H\). Then every \(\Omega\)-Smarandache hyper \((\cap, \circ)\)-ideal of \(H\) is an \(\Omega\)-Smarandache hyper \((\cap, \circ)\)-ideal of \(H\).

Proof. Let \(I\) be an \(\Omega\)-Smarandache hyper \((\cap, \circ)\)-ideal of \(H\) and let \(x \circ y \in I\) for all \(x, y \in \Omega\). Then \(x \circ \circ y \subseteq \Omega\) for all \(x, y \in \Omega\). Therefore for any \(a \subseteq \circ y \subseteq \Omega\) there exists \(i \in I\) such that \(a \subseteq \circ i\), which implies that \(0 \subseteq a \circ i\). Hence \((a \circ i) \cap I \neq \emptyset\) and so by \((c_2)\) we have \(a \subseteq I\). This implies that \((x \circ y) \subseteq I\) and so by \((c_2)\) we have \(x \subseteq I\).

The following example shows that the converse of Theorem 3.16 may not be true.
Example 3.17. Let $H = \{0, a, b, c\}$ and define the hyperoperation "$\circ$" on $H$ by the following Cayley table:

\[
\begin{array}{cccc}
0 & a & b & c \\
\hline
0 & [0] & [0] & [0] \\
a & [a] & [0, a] & [0, a] \\
b & [b] & [a, b] & [0, a, b] \\
c & [c] & [c] & [0, b, c] \\
\end{array}
\]

Then $(H, \circ, 0)$ is a Smarandache hyper $K$-algebra because $(H = \{0, a, b, c\}, \circ, 0)$ is a hyper $BCK$-algebra. Moreover, a subset $I = \{0, a\}$ is an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$. But it is not an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$, since $(i\circ a)\cap I \neq 8$ and $a \in I$, but $b \notin I$.

Corollary 3.18. Let $H$ be a Smarandache hyper hyper $K$-algebra, $\Omega$ be a non-trivial hyper $BCK$-algebra which is properly contained in $H$. Then every $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$ is an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$.

Proof. The result is obvious by Theorem 2.16 and Theorem 3.16 in [3].

Theorem 3.19. Let $H$ be a Smarandache hyper hyper $K$-algebra, $\Omega$ be a non-trivial hyper $BCK$-algebra which is properly contained in $H$ and let $I$ be an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$ such that

$$(\forall x \in I)(x \circ x \subseteq I \subseteq H).$$

Then the following implication is valid:

$$x \circ y \subseteq I \implies y \subseteq x \circ y \subseteq I.$$ 

Proof. Let $x, y \in H$ be such that $(x \circ y) \cap I \neq \emptyset$. Then there exists $i \in I$ such that $i \in (x \circ y) \cap I$. It follows from (H1) that $(x \circ y) \circ (x \circ y) \subseteq x \circ y$ so from hypothesis that $(x \circ y) \circ (x \circ y) \subseteq I$. This implies that $x \circ y \subseteq I$. For all $x \in H$ and hence $y \in H$ since $I$ is an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$ and $I \subseteq H$. Therefore $x \circ y \subseteq I$.

Theorem 3.20. Let $H$ be a Smarandache hyper hyper $K$-algebra, $\Omega$ be a non-trivial hyper $BCK$-algebra which is properly contained in $H$ and let $I$ be an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$ such that

$$(\forall x \in I)(x \circ x \subseteq I \subseteq H).$$

Then $I$ is an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$.

Proof. Let $x, y \in H$ be such that $(x \circ y) \cap I \neq \emptyset$ and $y \in I$. Then $x \circ y \subseteq I$ by Theorem 3.19, and so $x \circ y \subseteq I$. Since $I$ is an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$, it follows that $x \in I$. Therefore $I$ is an $\Omega$-Smarandache hyper $(\langle c, e \rangle, e)$-ideal of $H$.

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