Smarandache $\mathcal{N}$–subalgebras (resp. filters) of $CI$–algebras

Akbar Rezaei and Arsham Borumand Saeid

Abstract. In this paper, we introduce the notions of $\mathcal{N}$-subalgebras and $\mathcal{N}$-filters based on Smarandache $CI$-algebra and give a number of their properties. The relationship between $\mathcal{N}(Q, f)$-subalgebras (filters) and $\mathcal{N}$-subalgebras (filters) are also investigated.

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1 Introduction

Some recent researchers led to generalizations of the notion of fuzzy set that introduced by Zadeh in 1965 [15]. The generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the point $\{1\}$ into the interval $[0, 1]$. In order to provide a mathematical tool to deal with negative information, Jun et. al. introduced $\mathcal{N}$-structures, based on negative-valued functions [6]. In 1966, Y. Imai and K. Iseki [3] introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. H. S. Kim and Y. H. Kim defined a $BE$-algebra [5]. Biao Long Meng, defined notion of $CI$-algebra as a generalization of a $BE$-algebra [9]. It is known that any $BE$-algebra is a $CI$-algebra. Hence, every $BE$-algebra is
a weaker structure than $CI$-algebra, thus we can consider in any $CI$-algebra a weaker structure as $BE$-algebra. Jun et. al. discussed the notion of $N$-structures in $BCH/BCK/BCI$-algebras and investigated their properties in [6, 7]. They introduced the notions of $N$-ideals of subtraction algebras and $N$-closed ideals in $BCI$-algebras. We introduce the notions of $N$-subalgebras and $N$-filters in $CI$-algebras and give a number of their properties and The relationship between $N$-subalgebras and $N$-filters was discussed in [14]. Also, we discuss on Smarandache $CI$-algebra and investigated some of their useful properties in [2]. Beside, we introduced the notion of anti fuzzy set and stated the relationship with the $N$-function of $CI$-algebra $X$. We showed that every anti fuzzy filter is an anti fuzzy subalgebra in [1]. K. J. Lee and Y. B. Jun introduced the notion of $N$-subalgebras and $N$-ideals based on a sub-$BCK$-algebra of a $BCI$-algebras and their relations/properties are investigated in [8].

In the present paper, we continue study of $CI$-algebras and apply the $N$-structures to the filter theory in $CI$-algebras and Smarandache $CI$-algebras, also investigate the relationship between $N$-subalgebra and $N$-filters based on Smarandache $CI$-algebras. We show that any $N(Q, f)$-closed filter is an $N(Q, g)$-subalgebra. We give some conditions for $N$-subalgebras(filters) to be $N(Q, g)$-subalgebras(resp. filters).

2 Preliminaries

In this section we review the basic definitions and some elementary aspects that are necessary for this paper.

**Definition 2.1.** [9] An algebra $(X; *, 1)$ of type $(2, 0)$ is called a $CI$-algebra if it satisfying the following axioms:

\[(CI1)\quad x \ast x = 1,\]
\[(CI2)\quad 1 \ast x = x,\]
\[(CI3)\quad x \ast (y \ast z) = y \ast (x \ast z), \quad \text{for all } x, y, z \in X.\]

A $CI$–algebra $X$ satisfying the condition $x \ast 1 = 1$ is called a $BE$-algebra. In any $CI$-algebra $X$ one can define a binary relation “$\leq$” by $x \leq y$ if and only if $x \ast y = 1$.

A $CI$-algebra $X$ has the following properties:

\[(i)\quad y \ast ((y \ast x) \ast x) = 1,\]
\( (ii) \) \( (x \ast 1) \ast (y \ast 1) = (x \ast y) \ast 1, \)

\( (iii) \) if \( 1 \leq x \), then \( x = 1 \), for all \( x, y \in X \).

A non-empty subset \( S \) of a CI-algebra \( X \) is called a subalgebra of \( X \) if \( x \ast y \in S \) whenever \( x, y \in S \). A mapping \( f : X \to Y \) of CI-algebra is called a homomorphism if \( f(x \ast y) = f(x) \ast f(y) \), for all \( x, y \in X \). A non-empty subset \( F \) of CI-algebra \( X \) is called a filter of \( X \) if (1) \( 1 \in F \), (2) \( x \in F \) and \( x \ast y \in F \) implies \( y \in F \). A filter \( F \) of CI-algebra \( X \) is said to closed if \( x \in F \) implies \( x \ast 1 \in F \).

A nonempty subset \( S \) of a CI-algebra \( X \) is called a subalgebra of \( X \) if \( x \ast y \in S \), for all \( x, y \in S \). For our convenience, the empty set \( \emptyset \) is regarded as a subalgebra of \( X \). Denote by \( Q(X, [-1, 0]) \) the collection of functions from a set \( X \) to \( [-1, 0] \). We say that an element of \( Q(X, [-1, 0]) \) is a negative-valued function from \( X \) to \( [-1, 0] \) (briefly, \( N \)-function on \( X \)). By an \( N \)-structure we mean an ordered pair \((X, f)\) of \( X \) and an \( N \)-function \( f \) on \( X \).

In what follows, let \( X \) denote a CI-algebra and \( f \) an \( N \)-function on \( X \) unless otherwise specified.

**Definition 2.2.** [14] By a subalgebra of \( X \) based on \( N \)-function \( f \) (briefly, \( N \)-subalgebra of \( X \)), we mean an \( N \)-structure \((X, f)\) in which \( f \) satisfies the following assertion:

\[
f(x \ast y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in X.
\]

**Definition 2.3.** [14] By a filter of \( X \) based on \( N \)-function \( f \) (briefly, \( N \)-filter of \( X \)), we mean an \( N \)-structure \((X, f)\) in which \( f \) satisfies the following conditions:

1. \( f(1) \leq f(y) \),
2. \( f(y) \leq \max\{f(x \ast y), f(x)\}, \text{ for all } x, y \in X \).

**Definition 2.4.** [2] A Smarandache CI-algebra \( X \) is defined to be a CI-algebra \( X \) in which there exists a proper subset \( Q \) of \( X \) such that satisfies the following conditions:

1. \( \text{(S1)} \ 1 \in Q \) and \( |Q| \geq 2 \),
2. \( \text{(S2)} \ Q \) is a BE-algebra under the operation of \( X \).
Example 2.1. [2] Let $X := \{1, a, b, c, d\}$ be a set with the following table.

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</table>

Then $X$ is a $CI$-algebra and $Q = \{1, a, b, c\}$ is a $BE$-algebra.

Definition 2.5. [2] A nonempty subset $F$ of $CI$-algebra $X$ is called a Smarandache filter of $X$ related to $Q$ (or briefly, $Q$-Smarandache filter of $X$) if it satisfies:

- (SF1) $1 \in F$,
- (SF2) $(\forall y \in Q)(\forall x \in F)(x \ast y \in F \Rightarrow y \in F)$.

Definition 2.6. [11] A fuzzy set $\mu : X \rightarrow [0, 1]$ is called an anti fuzzy subalgebra of $X$ if it satisfies:

$$\mu(x \ast y) \leq \max\{\mu(x), \mu(y)\}, \text{ for all } x, y \in X.$$ 

Definition 2.7. [1] A fuzzy set $\mu : X \rightarrow [0, 1]$ is called an anti fuzzy filter of $X$ if it satisfies:

- (AFF1) $\mu(1) \leq \mu(x)$,
- (AFF2) $\mu(y) \leq \max\{\mu(x \ast y), \mu(x)\}, \text{ for all } x, y \in X$.

3 Smarandache $N$-subalgebras

Definition 3.1. Let $X$ be a $Q$-Smarandache $CI$-algebra and $\varrho \in [-1, 0]$. An $N$-structure $(X, f)$ is called an $N$-subalgebra of $X$ based on $Q$ and $\varrho$ (briefly, $N(Q, \varrho)$-subalgebra of $X$) if it is an $N$-subalgebra of $X$ such that satisfies the following condition:

- (type 1) $(\forall x \in Q)(\forall y \in X \setminus Q) (f(x) \leq \varrho \leq f(y)),$
- (type 2) $(\forall x \in Q)(\exists y \in X \setminus Q) (f(x) \leq \varrho \leq f(y)),$
- (type 3) $(\exists x \in Q)(\forall y \in X \setminus Q) (f(x) \leq \varrho \leq f(y)),$
\begin{itemize}
\item (type 4) ($\exists x \in Q$) ($\exists y \in X \setminus Q$) ($f(x) \leq g \leq f(y)$).
\end{itemize}

**Note.** If $\varrho := 0$, then $f(y) = 0$, for all $y \in X \setminus Q$. So, $(Q, f)$ is an $N$-subalgebra. If $\varrho := -1$, then $f(x) = -1$, for all $x \in Q$. And so $(X, f) = N(Q, f)$.

**Example 3.1.** a) In Example 2.1, an $N$-structure $(X, f)$ in which $f$ is defined by $f(1) = f(a) = -0.7$, $f(b) = -0.4$, $f(c) = -0.6$ and $f(d) = -0.3$ is an $N(Q, g)$-subalgebra of all types on $X$, for $\varrho \in [-0.4, -0.3]$ and $Q = \{1, a, b, c\}$.

b) In Example 2.1, an $N$-structure $(X, g)$ in which $g$ is defined by $g(1) = g(a) = -0.7$, $g(b) = -0.2$, $g(c) = -0.6$ and $g(d) = -0.3$ is not an $N(Q, g)$-subalgebra of $X$ because $g(d) = -0.3 \neq g(b) = -0.2$.

c) In Example 2.1, an $N$-structure $(X, f)$ in which $f$ is defined by $f(1) = f(a) = -0.7$, $f(b) = -0.4$, $f(c) = -0.5$ and $f(d) = -0.3$ is an $N(Q, g)$-subalgebra of type 2, type 3 and type 4 on $X$, for $\varrho \in [-0.4, -0.3]$ and $Q = \{1, a, b\}$, but it is not of type 1, because $f(c) \not\leq g$.

d) In Example 2.1, an $N$-structure $(X, f)$ in which $f$ is defined by $f(1) = f(a) = -0.7$, $f(b) = -0.2$, $f(c) = -0.3$ and $f(d) = -0.1$ is an $N(Q, g)$-subalgebra of type 3 and type 4 on $X$, for $\varrho \in [-0.7, -0.3]$ and $Q = \{1, a, b\}$, but it is not of type 1 and type 2 on $X$, because $f(b) \not\leq g$.

e) In Example 2.1, an $N$-structure $(X, f)$ in which $f$ is defined by $f(1) = f(a) = -0.7$, $f(b) = -0.2$, $f(c) = -0.5$ and $f(d) = -0.3$ is an $N(Q, g)$-subalgebra of type 4 on $X$, for $\varrho \in [-0.7, -0.3]$ and $Q = \{1, a, b\}$, but it is not of type 1, type 2, type 3 on $X$.

Now, in the following diagram we summarize the results of this definition. The mark $A \rightarrow B$, means that $A$ implies $B$.

\begin{center}
\begin{tikzpicture}
\node (type1) {type 1};
\node (type2) [right of=type1] {type 2};
\node (type3) [below of=type1] {type 3};
\node (type4) [right of=type3] {type 4};
\draw (type1) -- (type2);
\draw (type1) -- (type3);
\draw (type2) -- (type4);
\draw (type3) -- (type4);
\end{tikzpicture}
\end{center}

In this paper, we focus on $N(Q, g)$-subalgebra of type 1 and from now on $X$ is a $Q$-Smarandache $CI$-algebra.

The following example shows that there exists an $N$-structure $(X, f)$ in $X$ such that it satisfies the condition (type 1), but it is not an $N$-subalgebra of $X$.

**Example 3.2.** In Example 2.1, an $N$-structure $(X, f)$ in which $f$ is defined by $f(1) = -0.7$, $f(a) = -0.2$, $f(b) = -0.4$, $f(c) = -0.6$ and $f(d) = -0.3$. 
Then \((X, f)\) satisfies the condition (2.1) for \(\varrho \in [-0.2, -0.1]\), but it is not an \(N\)-subalgebra. Because

\[ f(b \ast c) = f(a) = -0.2 \not< -0.4 = \max\{f(b), f(c)\}. \]

**Proposition 3.1.** If an \(N\)-structure \((X, f)\) satisfies the following condition:

\[ (\forall x \in Q)(\forall y \in X \setminus Q)(f(x) \leq f(y)), \]

then \((X, f)\) is an \((Q, \varrho)\)-subalgebra of \(X\), for every \(\varrho \in \left[ \bigvee_{x \in Q} f(x), \bigwedge_{y \in X \setminus Q} f(y) \right]\).

**Theorem 3.2.** Let \(\varrho \in [-1, 0]\). If \((X, f)\) is an \(N(Q, \varrho)\)-subalgebra of \(X\), then

(i) \(Q \subseteq C(f; \varrho)\),

(ii) \((\forall \beta \in [-1, 0])(\beta < \varrho \Rightarrow C(f; \beta) \text{ is a subalgebra of } Q)\).

**Proof.** Let \((X, f)\) be a \(N(Q, \varrho)\)-subalgebra of \(X\). Obviously, \(Q \subseteq C(f; \varrho)\). If \(\beta \in [-1, 0]\) be such that \(\beta < \varrho\), then \(C(f; \beta) \subseteq Q\). Let \(x, y \in C(f; \beta)\). Then \(f(x) \leq \beta\) and \(f(x) \leq \beta\). Thus \(f(x \ast y) \leq \max\{f(x), f(y)\} \leq \beta\), and so \(x \ast y \in C(f; \beta)\). Thus \(C(f; \beta)\) is a subalgebra of \(Q\).

In the following theorem we give some conditions for an \(N\)-subalgebra to be an \(N(Q, \varrho)\)-subalgebra.

**Theorem 3.3.** Let \(\varrho \in [-1, 0]\). If \((X, f)\) is an \(N\)-subalgebra of \(X\) satisfies the conditions (i) and (ii) in Theorem 3.2, then \((X, f)\) is an \(N(Q, \varrho)\)-subalgebra of \(X\).

**Proof.** Let \(x \in Q\) and \(y \in X \setminus Q\). Then by Theorem 3.2(i), \(x \in C(f; \varrho)\), and so \(f(x) \leq \varrho\). Let \(f(y) = \beta\). If \(\beta < \varrho\), then by Theorem 3.2(ii), \(y \in C(f; \beta) \subseteq Q\), which is a contradiction. Hence \(f(x) \leq \varrho \leq \beta = f(y)\). Thus \((X, f)\) is an \(N(Q, \varrho)\)-subalgebra of \(X\).

\[ \square \]

4 Smarandache \(N\)-filters

**Definition 4.1.** Let \(X\) be a \(Q\)-Smarandache \(CI\)-algebra and \(\varrho \in [-1, 0]\). An \(N\)-structure \((X, f)\) is called an \(N\)-filter of \(X\) based on \(Q\) and \(\varrho\) (briefly, \(N(Q, \varrho)\)-filter of \(X\)) if it satisfies the following conditions:

(i) \((\forall x \in Q)(\forall y \in X \setminus Q)(f(1) \leq f(x) \leq \varrho \leq f(y))\).
Example 4.1. In Example 2.1, an $\mathcal{N}$–structure $(X, f)$ in which $f$ is defined by $f(1) = -0.6$, $f(a) = -0.4$, $f(b) = -0.5$, $f(c) = -0.4$ and $f(d) = -0.3$ is an $\mathcal{N}(Q, g)$–filter of $X$ for $g \in [-0.4, -0.3]$.

Theorem 4.1. Let $\{\mathcal{N}(Q_i, g_i) : i \in \Delta\}$ be a family of $\mathcal{N}(Q_i, g_i)$–subalgebras (filters) of $X$ where $\Delta \neq \emptyset$ and $g_i \in [-1, 0]$, for all $i \in \Delta$.
Then $\mathcal{N}(\cap Q_i, \min \{g_i\})_{i \in \Delta}$ is a subalgebra (filter) of $X$, too.

Theorem 4.2. Let $g \in [-1, 0]$. If $(X, f)$ is an $\mathcal{N}(Q, g)$–filter of $X$, then

(i) $Q \subseteq C(f; g)$,
(ii) $(\forall \beta \in [-1, 0]) (\beta < g \Rightarrow C(f; \beta) \text{ is a filter of } Q)$.

Proof. Let $(X, f)$ be an $\mathcal{N}(Q, g)$–filter of $X$. Obviously, $Q \subseteq C(f; g)$. Let $\beta \in [-1, 0]$ be such that $\beta < g$. If $x \in C(f; \beta)$, then $f(x) \leq \beta < g$, and so $x \in Q$. Hence $C(f; \beta) \subseteq Q$. By Definition 4.1(i), $f(1) \leq f(x) \leq \beta$ for all $x \in X$. Hence $f(1) \leq f(x) \leq \beta$ for all $x \in C(f; \beta)$, and so $1 \in C(f; \beta)$. Let $x, y \in Q$ be such that $x \ast y \in C(f; \beta)$ and $x \in C(f, \beta)$. Then $f(x \ast y) \leq \beta$ and $f(x) \leq \beta$. If $x, y \in C(f; \beta)$, then $f(x) \leq \beta$. Now by Definition 4.1(ii), $f(y) \leq \max\{f(x \ast y), f(x)\} \leq \beta$. Thus $y \in C(f; \beta)$. Therefore, $C(f; \beta)$ is a filter of $Q$.

For a $Q$–Smarandache $CI$–algebra $X$ and $g \in [-1, 0]$, the following example shows that an $\mathcal{N}$–filter $(X, f)$ of $X$ may not be an $\mathcal{N}(Q, g)$–filter of $X$.

Example 4.2. Let $X := \{1, a, b, c\}$ be a set with the following table.

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Then $X$ is a $CI$–algebra and $Q := \{1, a\}$ is a $BE$–algebra [13]. Define an $\mathcal{N}$–structure $(X, f)$ in which $f$ is defined by $f(1) = -0.7$, $f(a) = -0.2$, $f(b) = -0.4$, $f(c) = -0.2$. Then $(X, f)$ is an $\mathcal{N}$–filter of $X$. But it is not an $\mathcal{N}(Q, g)$ of $X$ for $g \in [-0.7, -0.3]$. Because $f(a) = -0.2 > g$.

In the following theorem we give conditions for an $\mathcal{N}$–filter to be an $\mathcal{N}(Q, g)$–filter.
Theorem 4.3. Let $\varrho \in [-1,0]$ and $(X,f)$ be an $N$-filter of $X$ satisfies the conditions (i) and (ii) of Theorem 4.2. Then $(X,f)$ is an $N(Q,\varrho)$-filter of $X$.

Proof. Let $x \in Q$ and $y \notin X \setminus Q$. Then by Theorem 4.2(i), $x \in C(f;\varrho)$, and so $f(x) \leq \varrho$. Let $f(y) = \beta$. If $\beta < \varrho$, then by Theorem 4.2(ii), $y \in C(f;\beta) \subseteq Q$, which is a contradiction. Hence $\varrho \leq \beta = f(y)$. Since $f$ is an $N$-filter of $X$, the condition (ii) of Definition 4.1 is obvious. Therefore, $(X,f)$ is an $N(Q,\varrho)$-filter of $X$.

The following example shows that an $N(Q,\varrho)$-subalgebra may not be an $N(Q,\varrho)$-filter.

Example 4.3. Let $X := \{1,a,b,c,d\}$ be a set with the following table.

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Then $X$ is a $CI$-algebra and $Q = \{1,a,b,c\}$ is a $BE$-algebra. Define an $N$-structure $(X,f)$ in which $f$ is defined by $f(1) = -0.7$, $f(a) = -0.3$ and $f(b) = -0.4$. Then $(X,f)$ is an $N$-subalgebra, but it is not an $N$-filter because

$$f(c) = -0.2 \not\leq -0.3 = \max\{f(b \ast c), f(b)\}.$$ 

Definition 4.2. An $N$-function on $X$ is called closed $N$-filter if $f$ satisfies:

$$f(x \ast 1) \leq f(x) \leq \max\{f(y \ast x, f(y))\}, \text{ for all } x,y \in X.$$ 

Example 4.4. Let $X := \{1,a,b\}$ be a set with the following table:

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Then $X$ is a $CI$-algebra [10]. Define an $N$-function $f : X \to [0,1]$ by $f(1) = -0.7$, $f(a) = -0.3$ and $f(b) = -0.4$. Then $(X,f)$ is an $N$-filter of $X$. But it is not an $N$-closed filter because

$$f(b \ast 1) = f(a) = -0.3 \not\leq f(b) = -0.4.$$
Example 4.5. In Example 4.4, if define $N$-function $f : X \to [0, 1]$ by $f(1) = -0.7$, $f(a) = -0.4$ and $f(b) = -0.4$. Then $(X, f)$ is an $N$-closed filter of $X$.

Proposition 4.4. Let $(X, f)$ be an $N$-closed filter. Then $f(1) \leq f(x)$, for all $x \in X$.

Proof. Let $x \in X$. Now, by Definition 4.2, we have

$$f(1) \leq \max\{f(x * 1), f(x)\} \leq \max\{f(x), f(x)\} = f(x).$$

$\Box$

Theorem 4.5. Let $(X, f)$ be an closed $N$-filter and $g \in [-1, 0]$. Then every $N(Q_1, \varrho)$-filter is $N(Q, \varrho)$-subalgebra of $X$.

Proof. Let $(X, f)$ be $N(Q_1, \varrho)$-filter and $x, y \in X$. Then by (CI3) and Definition 4.2, we have

$$f(x * y) \leq \max\{f(y * (x * y)), f(y)\}$$
$$= \max\{f(x * (y * y)), f(y)\}$$
$$= \max\{f(x * 1), f(y)\}$$
$$\leq \max\{f(x), f(y)\}.$$ 

Therefore, $(X, f)$ is an $N$-subalgebra of $X$. $\Box$

Theorem 4.6. Let $(X, f)$ and $(X, g)$ be $N(Q_1, \varrho_1)$ and $N(Q_2, \varrho_2)$-subalgebra (filter) of $X$ respectively. Then $(X \times X, f \times g)$ is an $N(Q_1 \times Q_2, \max\{\varrho_1, \varrho_2\})$-subalgebra(filter) of $X \times X$.

Proof. Let $(x, y) \in (Q_1 \times Q_2)$ and $(z, t) \in (X \times X) \setminus (Q_1 \times Q_2)$. Then we have

$$(f \times g)(1, 1) = \max\{f(1), g(1)\} \leq \max\{f(x), g(y)\}$$
$$\leq \max\{\varrho_1, \varrho_2\}$$
$$\leq \max\{f(z), f(t)\} = (f \times g)(z, t).$$

Now, let $(x_1, x_2), (y_1, y_2) \in (Q_1 \times Q_2)$. Then

$$(f \times g)((x_1, x_2) * (y_1, y_2)) = (f \times g)((x_1 * y_1), (x_2 * y_2))$$
$$= \max\{f(x_1 * y_1), g(x_2 * y_2)\}$$
$$\leq \max\{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\}$$
$$= \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), g(y_2)\}\}$$
$$= \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}.$$ 

Hence $(X \times X, f \times g)$ is an $N(Q_1 \times Q_2, \max\{\varrho_1, \varrho_2\})$-subalgebra(resp. filter) of $X \times X$. $\Box$
Proposition 4.7. Let $Q_1$ and $Q_2$ be two BE-algebras which are properly contained in $X$, $Q_1 \subseteq Q_2$ and $\varrho \in [-1,0]$. Then every $N(Q_2, \varrho)$-subalgebra(filter) of $X$ is an $N(Q_1, \varrho)$-subalgebra(filter) of $X$.

Note. By the following example we show that the converse of above theorem is not correct in general.

Example 4.6. Let $X := \{1, a, b, c\}$ be a set with the following table.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>a</td>
<td>1</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

Then $Q_1 = \{1, a\}$, $Q_2 = \{1, a, b\}$ are BE-algebras which are properly contained in $X$ and $f(1) = -0.7$, $f(a) = -0.4$, $f(b) = -0.2$ and $f(c) = -0.1$. Then $(X, f)$ is an $N(Q_1, \varrho)$-subalgebra, for all $\varrho \in [-0.4, 0]$, but it is not an $N(Q_2, \varrho)$-subalgebra, because, if $\varrho := -0.3$, then $f(b) = -0.2 \not< -0.3$.

5 Conclusion

A Smarandache structure on a set $A$ means a week structure $W$ on $A$ such that there exist a proper subset $B$ of $A$ which is embedded with a strong structure $S$. It is that any BE-algebra is a CI-algebra. Hence, every BE-algebra is a weaker structure than CI-algebra, thus we can consider in any CI-algebra a weaker structure as BE-algebra.

In this paper, we have introduced the concept of $N$-subalgebra (filter) based on Smarandache CI-algebras and some related properties are investigated. We show that any $N(Q, f)$-closed filter is an $N(Q, f)$-subalgebra. We give some conditions for an $N$-subalgebras (filters) to be $N(Q, \varrho)$-subalgebras (filters).

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