Research Article

On the Distribution Properties of the Smarandache Prime Part

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Received 27 March 2021; Accepted 5 August 2021; Published 17 August 2021

Academic Editor: Ghulam Shabbir

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1. Introduction

The properties of primes play a fundamental role in number theory; especially, the distribution of prime numbers in certain sequences has always been a hot topic. It is well known that if $(h, k) = 1$, the arithmetic progression $kn + h, \ n = 0, 1, 2, \ldots,$ contains infinitely many primes. This result is now known as Dirichlet’s theorem. Take another example; for any positive real irrational number $\alpha$, it follows from a classical result of Ivan Vinogradov that the number $\pi_{B(\alpha, \beta)} \sim (1/\alpha) \pi(x)$. In 2020, Janyarak Tongsomporn and Jorn Steuding showed that, for positive real irrational number $\alpha$ of finite type, arbitrary real number $\beta$, and every $b > 2$, there exist a positive constant $\delta_b > 0$ and arbitrarily large $x$ and $x'$ such that

\[
\frac{\pi_{B(\alpha, \beta)}(x + y) - \pi_{B(\alpha, \beta)}(x)}{\alpha \log x} > \frac{y}{1 + \delta_b}, \quad y, x > 1,
\]

where $y = y(x) = (\log x)^b$ and $y' = y(x')$.

In 1993, American-Romanian number theorist Florentin Smarandache published a book named *Only problems, not solutions!* In this book, he presented 105 unsolved arithmetical problems and conjectures about special sequences and functions, which can help us to analyze the properties of primes and the factors of integers. In 2019, Liu Miaohua studied the Smarandache dual function $S^*(m)$ is defined as $n$, that $n$ factorial divide $m$, namely, is

\[
S^*(m) = \max \{n : n \in \mathbb{N}, n! \mid m\}.
\]

They derived a formula of $\prod_{d\mid n} S^*(d)$; that is, for any positive integer $m$,

\[
\prod_{d\mid n} S^*(d) = \begin{cases} 
1, \\
2^\beta (\beta_1 + 1)(\beta_2 + 1) \ldots (\beta_r + 1), \\
2^\beta \beta_1(\beta_2 + 1)(\beta_3 + 1) \ldots (\beta_r + 1) \beta_1(\beta_2 + 1) \ldots (\beta_r + 1) \ldots (\beta_{d-1} + 1), \end{cases}
\]

where $m = p_1^{\beta_1} \ldots p_r^{\beta_r}$,

\[
\begin{align*}
& m = p_1^{\beta_1} \ldots p_r^{\beta_r}, \quad p_i \geq 3, \\
& m = 2^h p_1^{\beta_1} \ldots p_r^{\beta_r}, \quad p_i \geq 3, \\
& m = 2^h 3^h p_1^{\beta_1} \ldots p_r^{\beta_r}, \quad p_i \geq 5, \beta = 1 \text{ or } 2.
\end{align*}
\]
where $p_i$ is different odd prime numbers and $\beta_i$ and $\beta_0$ are the positive integers.

Based on this, we now consider the following Smarandache problems. As usual, let $p_n$ be the $n$th prime. For each integer $n$, denote by $P_p(n)$ the largest prime $\leq n$ and by $P_s(n)$ the largest prime $\geq n$ (see [1]), where $P_p(n)$ and $P_s(n)$ are called as the Smarandache inferior prime part and superior prime part of $n$, respectively. According to these definitions, the following are obvious:

(a) For each integer $n$, we have $P_p(n) \leq n \leq P_p(n)$.

(b) The integer $n$ is a prime if and only if $P_p(n) = P_p(n)$.

(c) We have

$$
\begin{cases}
P_p(m) = p_n, & (p_n < m < p_{n+1}).
\end{cases}
$$

Define

$$
I_n = \frac{1}{n} \sum_{m \leq n} P_p(m),
$$

$$
S_n = \frac{1}{n} \sum_{m \leq n} P_p(m).
$$

In [2], Problem 10, Kashihara proposed two problems about $I_n$ and $S_n$:

(A) To determinate whether $\lim_{n \to \infty} (I_n - S_n)$ converges or diverges. If it converges, find the limit.

(B) To determinate whether $\lim_{n \to \infty} I_n/S_n$ converges or diverges. If it converges, find the limit.

Yan [3] proved that

$$
I_n/S_n = 1 + O(n^{-1/3}),
$$

where we have made the convention that $p_0 = 0$. From (12) and (13), we can deduce that

$$
I_n - S_n = \frac{1}{n} \sum_{p_k \leq n} p_k (p_k - p_{k-1}) - \frac{1}{n} \sum_{p_k \leq n} p_{k-1} (p_k - p_{k-1})
$$

$$
= \frac{1}{n} \left( \sum_{p_k \leq n} p_k^2 - \sum_{p_k \leq n} p_k - 2 \sum_{p_k \leq n} p_k p_k - p_k \right)
$$

$$
= \frac{1}{n} \sum_{p_k \leq n} (p_k - p_{k-1})^2.
$$

According to Theorem 1, for any $\varepsilon > 0$, we have

$$
\lim_{n \to \infty} \frac{I_n}{S_n} = 1.
$$

In this short note, we shall continue a such study. Our aims is double. Firstly, we shall give a complete answer for question (A). Our proof shows that this problem is closely related to the quantity $D(X) = \sum_{X < p_k \leq 2X} (p_{k+1} - p_k)^2$. According to [4], it is conjectured that $D(X) - X \log X$ as $X \to \infty$. Our first result shows that $\lim_{n \to \infty} (I_n - S_n)$ diverges. Precisely, we have

This short note will focus on these two problems. The first result in this paper shows that $\lim_{n \to \infty} (I_n - S_n)$ diverges. Precisely, we have the following theorem.

**Theorem 1.** For any $\varepsilon > 0$, there is a positive constant $C_\varepsilon > 0$ depending on $\varepsilon$ such that the following inequalities

$$
(193/192 - \varepsilon) \log n - C_\varepsilon \leq I_n - S_n \leq C_\varepsilon n^{5/18 + \varepsilon},
$$

hold for $n \geq 1$. In particular, we have $\lim_{n \to \infty} (I_n - S_n) = \infty$.

The second result in this paper improves Yan’s equality (8) considerably.

**Theorem 2.** For any $\varepsilon > 0$, we have

$$
\frac{I_n}{S_n} = 1 + O(n^{-13/18 + \varepsilon}),
$$

as $n \to \infty$.

For comparison, we have $(13/18) = 0.722, \ldots$ and $(1/3) = 0.333, \ldots$.

**2. Proof of Theorem 1**

In view of facts (a), (b), and (c) mentioned above, we have

$$
\sum_{X \leq p_k \leq 2X} (p_k - p_{k-1})^2 \geq \left( \frac{193}{192} - \varepsilon \right) X \log X,
$$

for all $X \geq X_0(\varepsilon)$. Combining (14) and (15), we can derive that

$$
I_n - S_n \geq \frac{1}{n} \sum_{1 \leq d \leq \log \left( n/X_0(\varepsilon) \right) / \log 2} \sum_{n/2^{d-1} < p_k \leq n/2^d} (p_k - p_{k-1})^2
$$

$$
\geq \frac{1}{n} \sum_{1 \leq d \leq \log \left( n/X_0(\varepsilon) \right) / \log 2} \left( \frac{193}{192} - \varepsilon \right) n \log \frac{n}{2^d}
$$

$$
= \left( \frac{193}{192} - \varepsilon \right) (M \log n - E \log 2),
$$
In view of (14) and (19), we derive that

\[
M := \sum_{1 \leq d \leq \log (n^{(5/18)+\varepsilon}]} \frac{1}{Sn^d},
\]

(17)

\[
E := \sum_{1 \leq d \leq \log (n^{(5/18)+\varepsilon}]} d \frac{1}{Sn^d}.
\]

By elementary computation, we have

\[
M = 1 + O\left(\frac{n}{Sn}\right),
\]

\[
E \ll 1.
\]

Then, inserting (18) into (16), we can obtain the first inequality in (10).

On the contrary, formula (13) can be rewritten as

\[
\sum_{p_k \leq n} (P_k - P_{k-1})^2 \ll n^{(23/18)+\varepsilon},
\]

(19)

for all \( n \geq 1 \). The second inequality in (10) follows from (14) and (19).

The second assertion is an immediate consequence of (10). This completes the proof of Theorem 1.

3. Proof of Theorem 2

In view of (14) and (19), we derive that

\[
I_n \frac{Sn}{S_n} = 1 + O\left(n^{-13/18+\varepsilon}\right).
\]

(25)

This completes the proof.

4. Conclusion

The main results of this paper are two theorems involving Smarandache inferior prime part \( p_p (n) \) and superior prime part \( P_p (n) \). Theorem 1 establishes an inequality for \( I_n - S_n \) and shows that \( \lim_{n \to \infty} (I_n - S_n) = \infty \). Theorem 2 establishes an estimate on \( I_n/S_n \) and obtains that \( (I_n/S_n) = 1 + O\left(n^{-13/18+\varepsilon}\right) \) as \( n \to \infty \). These results represent new contributions to the research on the distribution of the Smarandache inferior prime part and improve the related results before. Of course, our methods can also be generalized to other problems of primes in integer sets. However, there are also many unsolved Smarandache problems to be explored, such as the properties of the determinant of Smarandache prime numbers. These open problems remain to be further studied.

Data Availability

All the data generated or analysed during this study are included within this published article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Both authors read and approved the final manuscript.

Acknowledgments

The authors would like to express their sincere appreciation to Wu Jie for his enthusiastic guidance and generous help. This work was supported by the N. S. F (11871193) of P. R. China, Young Talent-Training Plan for College Teachers (2019GGJS241) of Henan Province, Key Research Projects for Universities and Colleges (20A110027) of Henan Province, Startup Foundation for Introducing Talent (HYRC2019007) of Hetao College (CN), and N. S. F. (2021MS01003) of Inner Mongolia (CN).

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