Some Results on 4-Total Difference Cordial Graphs

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Abstract: Let $G$ be a graph. Let $f: V(G) \rightarrow \{0, 1, 2, \cdots, k-1\}$ be a map where $k \in \mathbb{N}$ and $k > 1$. For each edge $uv$, assign the label $|f(u) - f(v)|$. $f$ is called $k$-total difference cordial labeling of $G$ if $|t_{df}(i) - t_{df}(j)| \leq 1$, $i,j \in \{0, 1, 2, \cdots, k-1\}$ where $t_{df}(x)$ denotes the total number of vertices and the edges labeled with $x$. A graph with admits a $k$-total difference cordial labeling is called $k$-total difference cordial graphs.

Key Words: Difference cordial labeling, Smarandachely difference cordial labeling, star, path, cycle, bistar, crown, comb.

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§1. Introduction

We consider here finite, simple and undirected graphs only. Ponraj et al., has been introduced the concept of $k$-total difference cordial graph in [4]. In [4,5], 3-total difference cordial labeling path, complete graph, comb, armed crown, crown, wheel, star etc have been investigate and also we prove that every graph is a subgraph of a connected $k$-total difference cordial graphs in .In this paper we investigate 4-total difference of cordial labeling of some graphs like star, path, cycle, bistar, crown, comb, etc.

§2. $K$-Total Difference Cordial Labeling

Definition 2.1 Let $G$ be a graph. Let $f: V(G) \rightarrow \{0, 1, 2, \cdots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge $uv$, assign the label $|f(u) - f(v)|$. $f$ is called $k$-total difference cordial labeling of $G$ if $|t_{df}(i) - t_{df}(j)| \leq 1$, $i,j \in \{0, 1, 2, \cdots, k-1\}$ where $t_{df}(x)$ denotes the total number of vertices and the edges labelled with $x$. A graph with a $k$-total difference cordial labeling is called $k$-total difference cordial graph. Otherwise, if there is a pair $\{i,j\} \subset \{0, 1, 2, \cdots, k-1\}$ such that $|t_{df}(i) - t_{df}(j)| > 1$, such a labeling is called a Smarandachely $k$-total difference cordial labeling of $G$.

Remark 2.2 ([6]) 2-total difference cordial graph is 2-total product cordial graph.

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§3. Preliminaries

**Definition 3.1** The corona of $G_1$ with $G_2, G_1 \odot G_2$ is the graph obtained by taking one copy of $G_2$ and $p_i$ copies of $G_2$ and joining the $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$.

**Definition 3.2** Armed crown $AC_n$ is the graph obtained from the cycle $C_n: u_1u_2 \cdots u_n u_1$ with $V(AC_n) = V(C_n) \cup \{v_i, w_i: 1 \leq i \leq n\}$ and $E(AC_n) = E(C_n) \cup \{u_iv_i, v_iw_i: 1 \leq i \leq n\}$.

**Definition 3.3** An edge $x = uv$ of $G$ is said to be subdivided if it is replaced by the edges $uw$ and $wv$ where $w$ is a vertex not in $V(G)$. If every edge of $G$ is subdivided, the resulting graph is called the subdivision graph $S(G)$.

**Definition 3.4** Jelly fish graphs $J(m,n)$ obtained from a cycle $C_4: uxvyu$ by joining $x$ and $y$ with an edge and appending $m$ pendent edges to $u$ and $n$ pendent edges to $v$.

**Definition 3.5** Triangular snake $T_n$ is obtained from the path $P_n : u_1u_2 \cdots u_n$ with $V(T_n) = V(P_n) \cup \{v_i: 1 \leq i \leq n-1\}$ and $E(T_n) = E(P_n) \cup \{u_iv_i, u_{i+1}v_i: 1 \leq i \leq n-1\}$.

**Definition 3.6** Double Triangular snake $D(T_n)$ is obtained from the path $P_n : u_1u_2 \cdots u_n$ with $V(D(T_n)) = V(P_n) \cup \{v_i, w_i: 1 \leq i \leq n-1\}$ and $E(D(T_n)) = E(P_n) \cup \{u_iv_i, u_iw_i, 1 \leq i \leq n-1\} \cup \{w_{i+1}, v_{i+1}, u_{i+1}w_{i+1}: 1 \leq i \leq n-1\}$.

§4. Main Results

**Theorem 4.1** Any star $K_{1,n}$ is 4-total difference cordial.

**Proof** Let $V(K_{1,n}) = \{u, u_i: 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i: 1 \leq i \leq n\}$.

**Case 1.** $n \equiv 0 \pmod{4}$.

Let $n = 4r, r \in N$. Assign the label 1 to the central vertex. Next assign the label 0 to the vertices $u_1, u_2, \ldots, u_{2r}$ and assign the label 3 to the remaining vertices.

**Case 2.** $n \equiv 1 \pmod{4}$.

Let $n = 4r + 1, r \in N$. Assign the label 1 to the central vertex $u$. We now move to the pendent vertices. Assign the label 0 to the vertices $u_1, u_2, \ldots, u_{2r}$ and assign the label 3 to the next remaining vertices $u_{2r+1}, u_{2r+2}, \ldots, u_{4r}$ and $u_{4r+1}$.

**Case 3.** $n \equiv 2 \pmod{4}$.

Let $n = 4r + 2, r \in N$. In this case assign the label 0 to the vertices $u_1, u_2, \ldots, u_{2r}$ and $u_{2r+1}$. Next assign the label 3 to the vertices $u_{2r+2}, u_{2r+3}, \ldots, u_{4r+2}$. Finally assign 1 to the central vertex $u$.

**Case 4.** $n \equiv 3 \pmod{4}$.

As in case (3) assign the label to $u, u_1, u_2, \ldots, u_{n-1}$. Next assign the label 3 to the vertex.
Some Results on 4-Total Difference Cordial Graphs

Table 1 given below establish that this vertex labeling pattern is a 4-total difference cordial labeling.

<table>
<thead>
<tr>
<th>Values of n</th>
<th>( t_{df}(0) )</th>
<th>( t_{df}(1) )</th>
<th>( t_{df}(2) )</th>
<th>( t_{df}(3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{4} )</td>
<td>( 2r )</td>
<td>( 2r + 1 )</td>
<td>( 2r )</td>
<td>( 2r )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{4} )</td>
<td>( 2r )</td>
<td>( 2r + 1 )</td>
<td>( 2r + 1 )</td>
<td>( 2r + 1 )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{4} )</td>
<td>( 2r + 1 )</td>
<td>( 2r + 2 )</td>
<td>( 2r + 1 )</td>
<td>( 2r + 1 )</td>
</tr>
<tr>
<td>( n \equiv 3 \pmod{4} )</td>
<td>( 2r + 1 )</td>
<td>( 2r + 2 )</td>
<td>( 2r + 2 )</td>
<td>( 2r + 2 )</td>
</tr>
</tbody>
</table>

Table 1

A 4-total difference cordial labeling of \( K_{1,n} (n = 1, 2, 3) \) is given in Table 2.

<table>
<thead>
<tr>
<th>Values of n</th>
<th>( u )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2

This completes the proof. \( \square \)

**Theorem 4.2** The path \( P_n \) is 4-total difference cordial for all values of \( n \).

**Proof** Let \( P_n \) be the path \( u_1, u_2, \ldots, u_n \).

**Case 1.** \( n \equiv 0 \pmod{4} \) \( n > 3 \).

Let \( n = 4r, r \in \mathbb{N} \), Assign the labels 3, 1, 1 and 3 respectively to the vertices \( u_1, u_2, u_3, u_4 \). Next assign the labels 3, 1, 1 and 3 to the next 4 vertices \( u_5, u_6, u_7, u_8 \) respectively. Proceeding like this until we reach the vertex \( u_n \). That is in this process the last 4 vertices \( u_{n-3}, u_{n-2}, u_{n-1} \) and \( u_n \) receive the labels 3, 1, 1 and 3.

**Case 2.** \( n \equiv 1 \pmod{4} \) \( n > 3 \).

Let \( n = 4r + 1, r \in \mathbb{N} \). As in Case 1, assign the label to the vertices \( u_1, u_2, \ldots, u_{n-1} \). Next assign the label 3 to the vertex \( u_n \).

**Case 3.** \( n \equiv 2 \pmod{4} \), \( n > 3 \).

Let \( n = 4r + 2, r \in \mathbb{N} \). Assign the label to the vertices \( u_1, u_2, \ldots, u_{n-1} \) as in Case 2. Next assign the label 1 to the vertices \( u_n \).

**Case 4.** \( n \equiv 3 \pmod{4} \), \( n > 3 \).

Let \( n = 4r + 3, r \in \mathbb{N} \). Assign the label to the vertices \( u_1, u_2, \ldots, u_{n-1} \) as in Case 3. Next assign the label 1 to the vertex \( u_n \). This vertex labels is a 4-total difference cordial labels follows from Table 3 for \( n > 3 \).
Values of $n$  \hspace{1cm} $t_{df}(0)$  \hspace{1cm} $t_{df}(1)$  \hspace{1cm} $t_{df}(2)$  \hspace{1cm} $t_{df}(3)$
\hline
$n \equiv 0 \pmod{4}$ & $2r - 1$ & $2r$ & $2r$ & $2r$ \\
$n \equiv 1 \pmod{4}$ & $2r$ & $2r$ & $2r$ & $2r + 1$ \\
$n \equiv 2 \pmod{4}$ & $2r$ & $2r + 1$ & $2r + 1$ & $2r + 1$ \\
$n \equiv 3 \pmod{4}$ & $2r + 1$ & $2r + 2$ & $2r + 1$ & $2r + 1$ \\
\hline
\textbf{Table 3}

A 4-total difference cordial labeling of $P_n$ ($n = 1, 2, 3$) is given in Table 4.

<table>
<thead>
<tr>
<th>Values of $n$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

\textbf{Table 4}

This completes the proof. \hfill \Box

\textbf{Theorem 4.3} The cycle $C_n$ is 4-total difference cordial if $n \equiv 0, 1, 3 \pmod{4}$

\textbf{Proof} Let $C_n$ be the cycle $u_1u_2 \cdots u_nu_1$. Assign the label to the vertices $u_1, u_2, \cdots, u_n$ as in Theorem 4.2. Table 5 given below shows that this labeling of $C_n$ is a 4-total difference cordial.

<table>
<thead>
<tr>
<th>Values of $n$</th>
<th>$t_{df}(0)$</th>
<th>$t_{df}(1)$</th>
<th>$t_{df}(2)$</th>
<th>$t_{df}(3)$</th>
</tr>
</thead>
</table>
| $n \equiv 0 \pmod{4}$ & $2r$ & $2r$ & $2r$ & $2r$ \\
| $n \equiv 1 \pmod{4}$ & $2r$ & $2r$ & $2r$ & $2r + 1$ \\
| $n \equiv 3 \pmod{4}$ & $2r + 1$ & $2r + 2$ & $2r + 1$ & $2r + 1$ \\

\textbf{Table 5}

This completes the proof. \hfill \Box

\textbf{Theorem 4.4} The bistar $B_{n,n}$ is 4-total different cordial for all $n$.

\textbf{Proof} Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uu_i, vv_i, uv: (1 \leq i \leq n)\}$. Clearly $B_{n,n}$ has $2n + 2$ vertices and $2n + 1$ edges. Assign the label 1 to the central vertices $u$ and $v$. Assign the label 3 to the vertices $u_1, u_2, \cdots, u_n$ and $v_1$. We now assign the label 1 to the vertices $v_2, v_3, \cdots, v_n$. Clearly $t_{df}(0) = n$, $t_{df}(1) = t_{df}(2) = t_{df}(3) = n + 1$. Therefore $f$ is a 4-total difference cordial labeling. \hfill \Box

\textbf{Theorem 4.5} The crown $C_n \odot K_1$ is 4-total difference cordial labeling for all values of $n$.

\textbf{Proof} Let $C_n$ be the cycle $u_1u_2 \cdots u_nu_1$. Let $V(C_n \odot K_1)V(C_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_1) = E(C_n) \cup \{v_iv_i : 1 \leq i \leq n\}$. Assign the label 1 to the cycle vertices $u_1, u_2, \cdots, u_n$. Next we move to the pendent vertices $v_i$. Assign the label 3 to all pendent vertices $v_1, v_2, \cdots, v_n$. 

This completes the proof.
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Clearly \( t_{df}(0) = t_{df}(1) = t_{df}(2) = t_{df}(3) = n \). Hence \( f \) is a 4-total difference cordial labeling. □

**Corollary 4.1** All combs are 4-total difference cordial labeling.

*Proof* Clearly the vertex labeling in theorem 4.5 is also a 4-total difference cordial labeling of \( P_n \odot K_1 \). □

**Theorem 4.6** The armed crown \( AC_n \) is 4-total difference cordial for all \( n \).

*Proof* Clearly \( AC_n \) has 3 vertices and \( 3n \) edges. Let the vertex set and edge set as in Definition 3.2. Assign the label 1 to the all the cycle vertices \( u_1, u_2, \cdots, u_n \). Next we assign the label 3 to the vertices \( v_1, v_2, \cdots, v_n \).

**Case 1.** \( n \) is even.

In this case assign the label 3 to the pendent vertices \( w_1w_2 \cdots w_n \) and 1 to the remaining pendent vertices \( w_2+1, w_4+2, \cdots, w_n \).

**Case 2.** \( n \) is odd.

Assign the label 3 to the vertices \( w_1, w_2, \cdots, w_n \) and 1 to the vertices \( w_{n+2}, w_{n+5}, \cdots, w_n \). The table 6 given below establish that this vertex labeling pattern is a 4-total difference cordial labeling.

<table>
<thead>
<tr>
<th>Values of n</th>
<th>( t_{df}(0) )</th>
<th>( t_{df}(1) )</th>
<th>( t_{df}(2) )</th>
<th>( t_{df}(3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>n is even</td>
<td>( \frac{3n}{2} )</td>
<td>( \frac{3n}{2} )</td>
<td>( \frac{3n}{2} )</td>
<td>( \frac{3n}{2} )</td>
</tr>
<tr>
<td>n is odd</td>
<td>( \frac{3n+1}{2} )</td>
<td>( \frac{3n-1}{2} )</td>
<td>( \frac{3n-1}{2} )</td>
<td>( \frac{3n+1}{2} )</td>
</tr>
</tbody>
</table>

Table 6

This completes the proof. □

**Theorem 4.7** The double triangular snake \( DT_n \) is 4-total difference cordial for all \( n \).

*Proof* Let the vertex set and edge set as in Definition 3.6.

**Case 1.** \( n \equiv 0 \) (mod 3).

Assign the labels 3, 2, 3 to the path vertices \( u_1, u_2, u_3 \). Next assign the labels 3, 2, 3 to the next 3 vertices \( u_4, u_5, u_6 \) respectively. Proceeding like this until we reach the vertices \( u_n \). That is in the process the last three vertices \( u_{n-2}, u_{n-1}, u_n \) receive the label 3, 2, 3. Next assign the label 0 to the vertices \( v_1, v_2, \cdots, v_n \) and assign the label 2 to the vertices \( w_1, w_2, \cdots, w_n \).

**Case 2.** \( n \equiv 1 \) (mod 3).

In this case assign the labels to the vertices \( u_i, (1 \leq i \leq n-1), v_i, w_i, (1 \leq i \leq n-1) \) as in Case 1. Next assign the labels 3, 0, 2 respectively to the vertices \( u_n, v_{n-1}, w_n \).

**Case 3.** \( n \equiv 2 \) (mod 3).

As in Case 2 assign the labels to the vertices \( u_1, u_2, \cdots, u_{n-1}, v_1, v_2, \cdots, v_{n-2} \) and \( w_1, w_2, \cdots, w_{n-2} \).
Finally assign the label 2, 0 and 2 to the vertices $u_n, v_{n-1}$ and $w_{n-1}$. Table 7 given below establish that this labeling scheme is a 4-total difference cordial labeling of $DT_n$.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$t_{df}(0)$</th>
<th>$t_{df}(1)$</th>
<th>$t_{df}(2)$</th>
<th>$t_{df}(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{3}$</td>
<td>$2n - 2$</td>
<td>$2n - 2$</td>
<td>$2n - 1$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>$n \equiv 1 \pmod{3}$</td>
<td>$2n - 2$</td>
<td>$2n - 2$</td>
<td>$2n - 2$</td>
<td>$2n - 1$</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{3}$</td>
<td>$2n - 2$</td>
<td>$2n - 2$</td>
<td>$2n - 1$</td>
<td>$2n - 2$</td>
</tr>
</tbody>
</table>

Table 7

This completes the proof.

**Example 4.1** A 4-total difference cordial labeling of $D(T_6)$ is shown in Figure 1.

**Theorem 4.8** The jelly fish $J(n,n)$ is 4-total difference cordial for all $n$.

**Proof** Let $C_4$ be a cycle $uxvy$. Let $V(J(n,n)) = V(C_4) \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $E(J(n,n)) = E(C_4) \cup \{xy, xu_i, yv_i : 1 \leq i \leq n\}$. Assign the label 1 to the all cycle vertices $u,x,y,v$. Next we move to the pendent vertices. Assign the label 3 to the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2$. Assign the label 1 to the $v_3, v_4, \ldots, v_n$. Since $t_{df}(0) = n + 3, t_{df}(1) = t_{df}(2) = t_{df}(3) = n + 2$. $f$ is a 4-total difference cordial labeling.

**Theorem 4.9** The subdivision of the bistar $B_{n,n}$, $S(B_{n,n})$ is 4-total different cordial for all $n$.

**Proof** Let $V(S(B_{n,n})) = \{u, w, v, u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(S(B_{n,n})) = \{uu_i, u_ix_i, uw, wv, vv_i, y_iw : 1 \leq i \leq n\}$. Assign the label 1 to the vertices $u,w$ and $v$. Next assign the label 3 to the vertices $u_1, u_2, \ldots, u_i, x_1, x_2, \ldots, x_i$ and $v_1$. We now assign the label 2 to the vertices $y_1, y_2, \ldots, y_n$ and $v_2$. Finally assign the label 1 to the vertices $v_3, v_4, \ldots, v_n$. Since $t_{df}(0) = t_{df}(1) = t_{df}(3) = 2n + 1, t_{df}(2) = 2n + 2$. The labeling $f$ is a 4-total difference cordial labeling.

**Theorem 4.10** $P_n \odot 2K_1$ is 4-total difference cordial for all $n$. 

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Proof Let $P_n$ be the path $u_1, u_2, \cdots, u_n$. Let $v_i, w_i$ be the pendent vertices adjacent to $u_i$ $(1 \leq i \leq n)$. Assign the label 1 to the path vertices $u_1, u_2, \cdots, u_n$.

Case 1. $n$ is even.

Assign the label 3 to all the vertices $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_{\frac{n}{2}}$. We now assign the label 1 to the vertices $w_{\frac{n}{2}+1}, w_{\frac{n}{2}+2}, \cdots, w_n$.

Case 2. $n$ is odd.

As in Case 1 assign the label to the vertices $u_i, v_i, w_i$ $(1 \leq i \leq n)$. Next assign the label 3 to the vertices $v_i$ and assign the label 1 to the vertex $w_n$.

Table 8 given below establish that this vertex labeling pattern is a 4-total difference cordial labeling.

<table>
<thead>
<tr>
<th>Values of $n$</th>
<th>$t_{df}(0)$</th>
<th>$t_{df}(1)$</th>
<th>$t_{df}(2)$</th>
<th>$t_{df}(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ is even</td>
<td>$\frac{3n}{2} - 1$</td>
<td>$\frac{3n}{2}$</td>
<td>$\frac{3n}{2}$</td>
<td>$\frac{3n}{2}$</td>
</tr>
<tr>
<td>$n$ is odd</td>
<td>$\frac{3n-1}{2}$</td>
<td>$\frac{3n+1}{2}$</td>
<td>$\frac{3n-1}{2}$</td>
<td>$\frac{3n-1}{2}$</td>
</tr>
</tbody>
</table>

This completes the proof. \[ \square \]

Theorem 4.11 $S(P_n \odot K_1)$ is 4-total difference cordial for all $n$.

Proof Let $P_n$ be the path $u_1, u_2, \cdots, u_n$. Let $V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_i : 1 \leq i \leq n\}$. Let $x_i$ be the vertex which subdivide the edge $u_i u_{i+1}, (1 \leq i \leq n-1)$ and $y_i$ be the vertex which subdivide $u_i v_i : (1 \leq i \leq n)$. Assign the label 3 to all the path vertices $u_1, u_2, \cdots, u_n$ and $x_1, x_2, \cdots, x_n$ and $v_2$. Next assign the label 1 to the vertices $y_1, y_2, \cdots, y_n$ and $v_1$. Finally we assign the label 2 to the remaining vertices $v_3, v_4, \cdots, v_n$. Clearly $t_{df}(0) = t_{df}(1) = t_{df}(2) = 2n - 1, t_{df}(3) = 2n$. Therefore, $f$ is a 4-total difference cordial labeling of $S(P_n \odot K_1)$. \[ \square \]

Theorem 4.12 $S(C_n \odot K_1)$ is 4-total difference cordial for all values of $n$.

Proof Let $C_n : u_1 u_2 \cdots u_n u_1$ be the cycle. Let $V(C_n \odot K_1) = V(C_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_1) = E(C_n) \cup \{u_i v_i : 1 \leq i \leq n\}$. Let $x_i, y_i$ be the vertices which subdivide the edges $u_i u_{i+1}, (1 \leq i \leq n-1)$, $u_i v_i (1 \leq i \leq n)$ respectively. First we assign the label 3 to the cycle vertices $u_1, u_2, \cdots, u_n$ and $x_1, x_2, \cdots, x_n$. Next we assign the label 1 to the $y_1, y_2, \cdots, y_n$. Finally assign the label 2 to the all pendent vertices $v_1, v_2, \cdots, v_n$. Clearly $t_{df}(0) = t_{df}(1) = t_{df}(2) = t_{df}(3) = 2n$. Therefore $f$ is a 4-total difference cordial labeling of $S(C_n \odot K_1)$. \[ \square \]

Theorem 4.13 $S(AC_n)$ is 4-total difference cordial for all $n$.

Proof Let the vertex set and edge set of $AC_n$ as in definition 3.2.let $x_i : (1 \leq i \leq n - 1), y_i : (1 \leq i \leq n - 1)$ and $z_i : (1 \leq i \leq n - 1)$ be the vertex which subdivide the edges $u_i u_{i+1}, (1 \leq i \leq n-1), u_i v_i : (1 \leq i \leq n-1)$ and $v_i w_i : (1 \leq i \leq n-1)$ respectively. Assign the label 3 to the vertices $u_1, u_2, \cdots, u_n$ and $x_1, x_2, \cdots, x_n$ and $w_1, w_2, \cdots, w_n$. Next assign
the label 1 to the vertices $y_1, y_2, \cdots, y_n$. Then assign the label 2 to the vertices $v_1, v_2, \cdots, v_n$ and $z_1, z_2, \cdots, z_n$. Obviously $t_{df}(0) = t_{df}(1) = t_{df}(2) = t_{df}(3) = 3n$. Therefore $f$ is a 4-total difference cordial labeling of $S(A C_n)$.

\[\square\]

Example 4.2 A 4-total difference cordial labeling of $S(A C_n)$ is shown in Figure 2.

![Figure 2](image)

Theorem 4.14 $S(T_n)$ is 4-total difference cordial.

Proof Let the vertex set and edge set of $T_n$ as in Definition 3.7. Let $x_i, y_i$ and $z_i$ be the vertices which subdivide the edges $u_i u_{i+1}, u_i v_i$ and $u_i v_{i+1}$, $(1 \leq i \leq n)$.

Case 1. $n \equiv 0 \pmod{4}$.

Assign the label 3 to the vertices $u_1, u_2, \cdots, u_n$ and $x_1, x_2, \cdots, x_{n-1}$. Assign the label 1 to the vertices $y_1, y_2, \cdots, y_{n-1}$. Next assign the label 2, 3, 1 and 3 to the vertices $z_1, z_2, z_3, z_4$ then assign the label 2, 3, 1 and 3 to the next 4 vertices $z_5, z_6, z_7, z_8$ respectively. Proceeding like this until we reach the vertices $z_{n-1}$. That is in the process the last four vertices are $z_{n-4}, z_{n-3}, z_{n-2}, z_{n-1}$ receive the label 2, 3, 1, 3. Next assign the label 2, 3, 1, 3 to the vertices $u_1, v_2, v_3, v_4$ then assign the label 1, 2, 3, 2 to the next 4 vertices $v_5, v_6, v_7, v_8$ respectively. Proceeding like this until we reach the vertices $v_{n-1}$. That is in the process the last 4 vertices $v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}$ receive the label 0, 2, 3, 2.

Case 2. $n \equiv 1 \pmod{4}$.

As in Case 1 assign the labels to the vertices $u_i, (1 \leq i \leq n-1), v_i, x_i, y_i, z_i, (1 \leq i \leq n-2)$. Next assign the labels 3, 0, 3, 1 and 2 respectively to the vertices $u_n, v_{n-1}, x_{n-1}, y_{n-1}$ and $z_{n-1}$.

Case 3. $n \equiv 2 \pmod{4}$.

In this case, as in Case 2 assign the labels to the vertices $u_i, (1 \leq i \leq n-1), v_i, x_i, y_i, z_i, (1 \leq i \leq n-2)$ as in Case 2. Finally assign the labels 3, 2, 3, 1 and 3 to the vertices $u_n, v_{n-1}, x_{n-1}, y_{n-1}$ and
Case 4. \( n \equiv 3 \pmod{4} \).

As in Case 3, assign the label to \( u_i, (1 \leq i \leq n-1), v_i, x_i, y_i, z_i, (1 \leq i \leq n-2) \). Next assign the labels 3, 3, 3, 1 and 1 to the vertices \( u_n, v_{n-1}, x_{n-1}, y_{n-1} \) and \( z_{n-1} \) respectively. Table 9 given below establish that this vertex labeling pattern is a 4-total difference cordial labeling.

<table>
<thead>
<tr>
<th>Nature of n ( \pmod{4} )</th>
<th>( t_{df}(0) )</th>
<th>( t_{df}(1) )</th>
<th>( t_{df}(2) )</th>
<th>( t_{df}(3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{4} )</td>
<td>( \frac{11n-8}{4} )</td>
<td>( \frac{11n-8}{4} )</td>
<td>( \frac{11n-12}{4} )</td>
<td>( \frac{11n-12}{4} )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{4} )</td>
<td>( \frac{11n-9}{4} )</td>
<td>( \frac{11n-9}{4} )</td>
<td>( \frac{11n-5}{4} )</td>
<td>( \frac{11n-9}{4} )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{4} )</td>
<td>( \frac{11n-10}{4} )</td>
<td>( \frac{11n-10}{4} )</td>
<td>( \frac{11n-10}{4} )</td>
<td>( \frac{11n-10}{4} )</td>
</tr>
<tr>
<td>( n \equiv 3 \pmod{4} )</td>
<td>( \frac{11n-9}{4} )</td>
<td>( \frac{11n-9}{4} )</td>
<td>( \frac{11n-9}{4} )</td>
<td>( \frac{11n-13}{4} )</td>
</tr>
</tbody>
</table>

This completes the proof. \( \square \)

References