The Dynamical Properties of Modified of Bogdanov Map

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\section{Abstract}

In the work, we study the general properties of modified of the Bogdanov map in the form $F_{a,m,k} \cdot (x, y) = (x + y, y + ay + k \cos x - kx + mxy)$

We prove it has sensitive dependence on initial conditions and positive Lyapunov exponent with some conditions. So we give estimate of topological entropy of modified Bogdanov map.

MSC. 41A25; 41A35; 41A36.

\section{1. Introduction}

Chaos theory is a branch of mathematics concern with studying the properties of deterministic systems that depend on their behavior on a set of elementary conditions, making their study somewhat complex using traditional mathematical tools. Mathematicians use chaos theory to model these systems in ways different in order to arrive at a specific mathematical description of it and its behavior depending on all possible initial conditions. The Bogdanov map provides a good approximation to the dynamics of the Poincaré map of periodically forced oscillators, first considered by Bogdanov [5].

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Aboites & etal., studied the dynamic behavior among periodic orbits high periodicity and chaos of Bogdanov map were observed through bifurcation[6]

The Bogdanov map is 2-dimension and discrete dynamical system, its form $F_{a,m,k}$

$$
\begin{align*}
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x + y \\ y + ay + kx^2 - kx + mxy \end{pmatrix},
\end{align*}
$$

we changed this form of Bogdanov map to modified Bogdanov map by replacing $x^2$ to $\cos(x)$ and denoted to $F_{a,m,k}^1$. We find important chaotic properties of it. One of this properties sensitive dependence on initial condition and Lyapunov exponents.

2. Basic Definitions

Any pair $\begin{pmatrix} k \\ h \end{pmatrix}$ for which $f \begin{pmatrix} k \\ h \end{pmatrix} = k$, $g \begin{pmatrix} k \\ h \end{pmatrix} = h$ is called fixed point of 2-D dynamical system [3]. Let $V$ be a subset of $\mathbb{R}^2$ and $v_0 = \begin{pmatrix} x \\ y \end{pmatrix}$ be any element in $V$. Consider $F_{a,m,k} : \mathbb{R} \rightarrow \mathbb{R}^2$ a map. Furthermore [4], assume that the first partials of the coordinate maps $F_{a,m,k(1)}$ and $F_{a,m,k(2)}$ of $F$ exist at $v_0$ is the linear map $D F_{a,m,k}(v_0)$ defined on $\mathbb{R}^2$ by

$$
DF_{a,m,k}(v_0) = \begin{pmatrix} \frac{\partial F_{a,m,k}^1}{\partial x}(v_0) & \frac{\partial F_{a,m,k}^1}{\partial y}(v_0) \\ \frac{\partial F_{a,m,k}^2}{\partial x}(v_0) & \frac{\partial F_{a,m,k}^2}{\partial y}(v_0) \end{pmatrix},
$$

for all $v_0$ in $V$ the determine of $DF_{a,m,k}(v_0)$ is said the Jacobin of $F_{a,m,k}$ at $v_0$ and is denoted by $J=\det DF_{a,m,k}(v_0)$, if $|J|F_{a,m,k}(v_0)|<1$ then $F_{a,m,k}(v_0)$ is area contracting at $v_0$ and if $|JF_{a,m,k}(v_0)| > 1$ then $F_{a,m,k}^1$ is area expanding at $v_0$ [1].

3. Some properties of modified Bogdanov map

In the section, we study modified Bogdanov map from fixed point, diffeomorphism (one- to-one, onto, invertible, $C^\infty$). Also we determined the contracting and expanding area.

Proposition(3-1)

If $k \neq 0$ Then $F_{a,m,k}$ has unique fixed point

Proof

By definition of a fixed point

$$
F_{a,m,k} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y + ay + k \cos(x) - kx + mxy \end{pmatrix},
$$
y=0 so $k \cos(x) = kx$ then $x=0$ therefore $p=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is unique fixed point.

Proposition(3-2)

Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the $F_{a,m,k}$ then the Jacobin of $F_{a,m,k}$ is $1 + a + k$

Proof

$$
DF_{a,m,k} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ k \sin(x) - k + my \\ 1 + a + mx \end{pmatrix} so JF_{a,m,k} = \det \begin{pmatrix} 1 & 1 \\ -k & 1 + a \end{pmatrix} = 1 + a + k
$$
Proposition (3-3)

\( F_{a,m,k} \) is area contracting map if \(|1 + a + k| < 1\) and if \(|1 + a + k| > 1\) then \( F_{a,m,k} \) is area expanding.

Proof

by proposition (3-2) then \(|1 + a + k| < 1\) hence \( k < a < 2 + k\)

therefore the \( F_{a,m,k}(x, y) \) is an area contracting map and if \(2 + k < a < k\) the \( F_{a,m,k} \) is an area expanding.

Proposition (3-4)

The eigenvalues of \( D F_{a,m,k}(x, y) \) at fixed point is \( \lambda_{1,2} = \frac{-(a+b)\pm\sqrt{(a^2+8a-4k)}}{2} \)

Proof

\[ \det(DF_{a,m,k}(v) - I\lambda) = \det \begin{pmatrix} 1 & \frac{1}{k} \\ -k & 1 + a \end{pmatrix} = (1 + a)(k) = 0 \text{ then } \lambda_{1,2} = \frac{-(a+b)\pm\sqrt{(a^2+8a-4k)}}{2}. \]

Proposition (3-5)

Let \( F: \mathbb{R}^2 \to \mathbb{R}^2 \) be modified bogdanov map, then \( F_{a,m,k} \) is diffeomorphism.

Proof

1. Let \( T(x, y) = (x + y, ay + k\cos x - kx + mxy) \)

\( T(1, 0) = (1, k\cos(1) - k) \)

\( T(0, 1) = (1, 1 + a + k) \)

Then, \( F_{a,m,k}(1 \  k\cos(1) - k) \ from \( \begin{pmatrix} 1 & k\cos(1) - k \\ 1 + a + k & 2a + 3k - 2k\cos(1) \end{pmatrix} \)

\( F_{a,m,k} \) has a pivot position in every column then \( F_{a,m,k} \) is one to one and has a pivot position in every row then \( F_{a,m,k} \) is onto.

2. \( F_{a,m,k} \) is \( C^\infty \): \( F_{a,m,k}(x, y) = \begin{pmatrix} x + y \\ y + ay + k\cos(x) - kx + mxy \end{pmatrix} \)

Note that \( \frac{\partial f^1(x,y)}{\partial x} = 1, \quad \frac{\partial^2 f^1(x,y)}{\partial x^2} = 0 \ldots \ldots \frac{\partial^n f^1(x,y)}{\partial x^n} = 0 \text{ for all } n \in \mathbb{N} \text{ and } n \geq 2 \)

\( \frac{\partial f^1(x,y)}{\partial y} = 1, \quad \frac{\partial^2 f^1(x,y)}{\partial y^2} = 0 \ldots \ldots \frac{\partial^n f^1(x,y)}{\partial x^n} = 0 \text{ for all } n \in \mathbb{N} \text{ and } n \geq 2 \)

\( \frac{\partial^2 f^2(x,y)}{\partial x} = -k\sin x + m\cos x, \quad \frac{\partial^2 f^2(x,y)}{\partial x^2} = k\cos x \text{ for all } n \in \mathbb{N} \)

\( \frac{\partial^2 f^2(x,y)}{\partial y} = 1 + m\cos x + a, \quad \frac{\partial^2 f^2(x,y)}{\partial y^2} = 0 \ldots \ldots \frac{\partial^n f^2(x,y)}{\partial x^n} = 0 \text{ for all } n \in \mathbb{N} \text{ and } n \geq 2 \)

Then the partial derivatives continuous and exist then \( F_{a,m,k} \) is \( C^\infty \)
4. Sensitive Dependence on initial conditions (S.D.I)

Definition (4-1)

Let \((X, k)\) be a metric space. A map \(F: (X, k) \rightarrow (X, k)\) is said to be sensitive dependence on initial conditions (S. D. I) if there exist \(\epsilon > 0 \exists\) for any \(x_0 \in X\) and any open set \(U \subseteq X\) containing \(x_0\) there exists \(y_0 \in U\) and \(n \in \mathbb{Z}^+\) such that \(d(f^n(x_0), f^n(y_0)) > \epsilon\) that is \(\exists \epsilon > 0, \forall x > 0, \exists y \in B_0(x), \exists n \in \mathbb{N}, d(f^n(x_0), f^n(y_0)) > \epsilon\).
5. Lyapunov exponent

The Lyapunov exponent plays a role in the field of calculus and integration and control theory, as it has many life applications in the field of Physics, Medicine, Engineering, Space Science and other Scientific fields.

Proposition (5-1)

For all \((x, y)\) then \(F_{a,m,k}(x, y)\) has positive Lyapunov exponent.

Proof

\(X = (x, y) \in R^2\), the Lyapunov exponent of \(F_{a,m,k}\) is given by

\[X_i((x, y), v1) = \lim_{n \to \infty} \frac{1}{n} \ln \left\| D^n F_{a,m,k} (x, y), v1 \right\|\]

by (3-4) we have \(F_{a,m,k}\) has two eigenvalues \(\lambda_1\) and \(\lambda_2\). If

\(|\lambda_1| < 1\) then \(X_i((x, y), v1) = \lim_{n \to \infty} \frac{1}{n} \ln \left\| (DF^n_{a,m,k} (X, Y), v1)^n \right\| > \ln \left\| \frac{-(a+b)+\sqrt{(a^2+2+8a-4k)}}{2} \right\|\),

By hypothesis

\(L_1 > 0\) so if \(|\lambda_1| < 1\) then \(\lim_{n \to \infty} \frac{1}{n} \ln \left\| (DF^n_{a,m,k} (X, Y), v1)^n \right\| < \ln \left\| \frac{-(a+b)+\sqrt{(a^2+2+8a-4k)}}{2} \right\|\), SO

\(L_v = \max \{x1(x,v1),x2(x,v2)\}\) hence the Lyapunov exponent of modified Bogdanov map is positive.

Remark (5-2)

If the Lyapunov exponents are positive, then the sensitive dependence on initial condition exists. Therefore \(F_{a,m,k}\) has sensitive dependence on initial condition and Lyapunov exponent.
6. Topological entropy

In 1965, Adler, Konheim and McAndrew introduced the concept of topological entropy. Their definition was modeled after the definition of the Kolmogorov-Sinai, but the second definition before Dinaburg and Rufus Bowen and who explained the meaning of a topological entropy for a system given by an iterated map. Topological entropy is the measure of the complexity of the system and the topological entropy is a non-negative real number [5].

We give estimate of topological entropy of continuous map.

We recall the theorem (3.35) in [2] by

**Theorem (6.1)**

Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be a continuous map then

\[
h_{\text{top}}(f) \geq \log |\lambda|
\]

where \( \lambda \) is the largest eigenvalue of \( Df(v) \), where \( v \in \mathbb{R}^n \).

We give estimate of topological entropy of modified Bogdanov map

**Proposition (6.2)**

If \(|\lambda_1| > |\lambda_2|\) therefore \( h_{\text{top}}(F_{a,m,k}) \geq \log |\lambda_1| \)

**Proof**

By proposition (3-4) and by hypothesis hence

\[
h_{\text{top}}(F_{a,m,k}) \geq \log |\lambda_1| \text{ then } h_{\text{top}}(F_{a,m,k}) \geq \log \left| -\frac{(a+b)+\sqrt{(a^2+8a-4k)}}{2} \right|
\]

**Remark (6.3)**

In the same way if \(|\lambda_2| > |\lambda_1|\) then \( h_{\text{top}}(F_{a,m,k}) \geq \log |\lambda_2| \).

We recall the theorem (3.35) in [2] by:

**Theorem (6.4)**

Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be

\[
h_{\text{top}}(F_{a,m,k}(x)) \leq \log \max_{x \in \mathbb{R}^n} \max_{L \in \mathbb{T}_{X\mathbb{R}^n}} |\det(DF_{a,m,k}(x)L)|
\]

\[-\frac{(a+b)+\sqrt{(a^2+8a-4k)}}{2} \]
Proposition (6-5)

The upper estimate of topological entropy of modified Bogdanov map.

Proof

by theorem (3.35) on [2] we get

\[ H_{\text{top}}(F_{a, m, k}) \leq \log \max \max_{x \in \mathbb{R}^2} \left| \det \left( D \left( F_{a, m, k}(x) \right) \right) \right| \leq \log \max \max_{x \in \mathbb{R}^2} |1 + a + k| \]

References


