A GENERAL RESULT ON THE SMARANDACHE STAR FUNCTION

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ABSTRACT: In this paper, the result (theorem-2.6) derived in REF. [2], the paper "Generalization Of Partition Function, Introducing 'Smarandache Factor Partition' which has been observed to follow a beautiful pattern has been generalized.

DEFINITIONS In [2] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows:

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots, p_r \) be arbitrarily chosen distinct primes then \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \) called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \) is defined as the number of ways in which the number

\[
N = \frac{\alpha_1}{p_1} \frac{\alpha_2}{p_2} \frac{\alpha_3}{p_3} \ldots \frac{\alpha_r}{p_r}
\]

could be expressed as the product of its' divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) = F'(N) \), where

\[
N = p_1 \frac{\alpha_1}{p_1} p_2 \frac{\alpha_2}{p_2} p_3 \frac{\alpha_3}{p_3} \ldots p_r \frac{\alpha_r}{p_r} \ldots p_n \frac{\alpha_n}{p_n}
\]

and \( p_r \) is the \( r \)th prime. \( p_1 = 2, p_2 = 3 \) etc.

Also for the case

240
\[ \alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1 \]

we denote

\[ F(1,1,1,1, \ldots) = F(1\#n) \]

\[ \leftarrow n \text{- ones} \rightarrow \]

**Smarandache Star Function**

1. \[ F^-(N) = \sum_{d \mid N} F'(d_r) \quad \text{where} \quad d_r \mid N \]
2. \[ F'''(N) = \sum_{d_r \mid N} F''(d_r) \]

\( d_r \) ranges over all the divisors of \( N \).

If \( N \) is a square free number with \( n \) prime factors, let us denote

\[ F'''(N) = F''(1\#n) \]

Here we generalise the above idea by the following definition

**Smarandache Generalised Star Function**

3. \[ F^{n*}(N) = \sum_{d_r \mid N} F^{(n-1)*}(d_r) \quad n > 1 \]

and \( d_r \) ranges over all the divisors of \( N \).

For simplicity we denote

\[ F'(Np_1p_2 \ldots p_n) = F'(N@1\#n) \quad \text{where} \]

\[ (N, p_i) = 1 \text{ for } i = 1 \text{ to } n \quad \text{and each } p_i \text{ is a prime.} \]

\( F'(N@1\#n) \) is nothing but the Smarandache factor partition of (a number \( N \) multiplied by \( n \) primes which are coprime to \( N \)).
In [3] a proof of the following result is given:

\[ F'(Np_1p_2p_3) = F'^*(N) + 3F'^2*(N) + F'^3*(N) \]

The present paper aims at generalising the above result.

**DISCUSSION:**

**THEOREM (3.1)**

\[ F'(N@1#n) = F'(Np_1p_2 \ldots p_n) = \sum_{m=0}^{n} \left[ a_{(n,m)} F'^m(N) \right] \]

where

\[ a_{(n,m)} = \left( \frac{1}{m!} \right) \sum_{k=1}^{m} (-1)^{m-k} \cdot mC_k \cdot k^n \]

**PROOF:**

Let the divisors of \( N \) be \( d_1, d_2, \ldots, d_k \)

Consider the divisors of \( (Np_1p_2 \ldots p_n) \) arranged as follows

- \( d_1, d_2, \ldots, d_k \) ------say type (0)
- \( d_1p_i, d_2p_i, \ldots, d_kp_i \) ------say type (1)
- \( d_1p_ip_j, d_2p_ip_j, \ldots, d_kp_ip_j \) ------say type (2)
- \( d_1p_ip_j, \ldots, d_2p_ip_j, \ldots, d_kp_ip_j \) ------say type (t)

(there are \( t \) primes in the term \( d_1p_ip_j \ldots \) apart from \( d_1 \))

\( d_1p_1p_2 \ldots p_n, d_2p_1p_2 \ldots p_n, d_np_1p_2 \ldots p_n \) ------say type (n)

There are \( ^nC_0 \) divisors sets of the type (0)

There are \( ^nC_1 \) divisors sets of the type (1)

There are \( ^nC_2 \) divisors sets of the type (2) and so on

There are \( ^nC_t \) divisors sets of the type (t)
There are \( ^nC_n \) divisors sets of the type \( (n) \)

Let \( Np_1p_2\ldots p_n = M \). Then

\[
F^*(M) = \sum \left[ ^nC_0 \text{[sum of the factor partitions of all the divisors of row (0)]} \right] \\
+ \sum \left[ ^nC_1 \text{[sum of the factor partitions of all the divisors of row (1)]} \right] \\
+ \sum \left[ ^nC_2 \text{[sum of the factor partitions of all the divisors of row (2)]} \right] \\
+ \ldots \sum \left[ ^nC_t \text{[sum of the factor partitions of all the divisors of row (t)]} \right] \\
+ \sum \left[ ^nC_n \text{[sum of the factor partitions of all the divisors of row (n)]} \right]
\]

Let us consider the contributions of divisor sets one by one.

Row (0) or type (0) contributes

\[
F'(d_1) + F'(d_2) + F'(d_3) + \ldots + F'(d_n) = F^*(N)
\]

Row (1) or type (1) contributes

\[
[F'(d_1p_1) + F'(d_2p_1) + \ldots F'(d_kp_1)] \\
= [F^*(d_1) + F^*(d_2) + \ldots + F^*(d_k)] \\
= F'^2(N)
\]

Row (2) or type (2) contributes

\[
[F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_kp_1p_2)] \\
\text{Applying theorem (5) on each of the terms}
\]

\[
F'(d_1p_1p_2) = F^*(d_1) + F^{**}(d_1) \quad \text{-----(1)} \\
F'(d_2p_1p_2) = F^*(d_2) + F^{**}(d_2) \quad \text{-----(2)} \\
\vdots \\
F'(d_kp_1p_2) = F^*(d_k) + F^{**}(d_k) \quad \text{-----(k)}
\]

on summing up \( (1), (2), \ldots \) upto \( (n) \) we get

\[
F'^2*(N) + F'^3*(N)
\]

At this stage let us denote the coefficients as \( a_{(n,r)} \) etc. say
Consider row (t), one divisor set is 
\[ d_1 p_1 p_2 ... p_t, d_2 p_1 p_2 ... p_t, \ldots d_k p_1 p_2 ... p_t, \]

and we have
\[
F'(d_1@1#t) = a_{(t,1)}F'*(d_1) + a_{(t,2)}F'^2*(d_1) + \ldots + a_{(t,t)}F'^t*(d_1)
\]
\[
F'(d_2@1#t) = a_{(t,1)}F'*(d_2) + a_{(t,2)}F'^2*(d_2) + \ldots + a_{(t,t)}F'^t*(d_2)
\]
\[
\vdots
\]
\[
F'(d_k@1#t) = a_{(t,1)}F'*(d_k) + a_{(t,2)}F'^2*(d_k) + \ldots + a_{(t,t)}F'^t*(d_k)
\]

Summing up both the sides columnwise we get for row (t) or divisors of type (t) one of the \( ^nC_t \) divisor sets contributes
\[ a_{(t,1)}F'^2*(N) + a_{(t,2)}F'^3*(N) + \ldots + a_{(t,t)}F'^{t+1}*(N) \]

Similarly for row (n) we get
\[ a_{(n,1)}F'^2*(N) + a_{(n,2)}F'^3*(N) + \ldots + a_{(n,n)}F'^{n+1}*(N) \]

All the divisor sets of type (0) contribute
\[ ^nC_0 a_{(0,0)}F'*(N) \] factor partitions.

All the divisor sets of type (1) contribute
\[ ^nC_1 a_{(1,1)}F'^2*(N) \] factor partitions.

All the divisor sets of type (2) contribute
\[ ^nC_2 \{a_{(2,1)}F'^2*(N) + a_{(2,2)}F'^3*(N)\} \] factor partitions.

All the divisor sets of type (3) contribute
\[ ^nC_3\{a_{(3,1)}F'^2*(N) + a_{(3,2)}F'^3*(N) + a_{(3,3)}F'^4*(N)\} \] factor partitions.
All the divisor sets of row (t) or type (t) contribute
\[ {nC_t \{ a_{(t,1)}F^t \}^2(N) + a_{(t,2)}F^t \}^3(N) + \ldots + a_{(t,t)}F^t \}^{(t+1)}(N) \]
\[ \ldots \]
All the divisor sets of row (n) or type (n) contribute
\[ {nC_n \{ a_{(n,1)}F^n \}^2(N) + a_{(n,2)}F^n \}^3(N) + \ldots + a_{(n,n)}F^n \}^{(n+1)}(N) \]
Summing up the contributions from the divisor sets of all the types and considering the coefficient of \( F^m \) for \( m = 1 \) to \((n+1)\) we get, coefficient of \( F^1 \cdot (N) = a_{(0,0)} = 1 = a_{(n+1,1)} \)
coefficient of \( F^2 \cdot (N) \)
\[ = {nC_1 a_{(1,1)} + nC_2 a_{(2,1)} + \ldots + nC_t a_{(t,1)} + \ldots + nC_n a_{(n,1)} \]
\[ = a_{(n+1,2)} \]
coefficient of \( F^3 \cdot (N) \)
\[ = {nC_2 a_{(2,2)} + nC_3 a_{(3,2)} + \ldots + nC_t a_{(t,2)} + \ldots + nC_n a_{(n,2)} \]
\[ = a_{(n+1,3)} \]
coefficient of \( F^m \cdot (N) = \)
\[ a_{(n+1,m)} = {nC_{m-1} a_{(m-1,m-1)} + nC_m a_{(m,m-1)} + \ldots + nC_{n} a_{(n,m-1)} \]
coefficient of \( F^{(n+1)} \cdot (N) = \)
\[ a_{(n+1,n+1)} = {nC_n a_{(n,n)} = nC_n \cdot n^{-1} C_{n-1} \cdot a_{(n-1,n-1)} = nC_n \cdot n^{-1} C_{n-1} \ldots \]
\[ 2C_2 \cdot a_{(1,1)} \]
\[ = 1 \]
Consider \( a_{(n+1,2)} \)
\[ = {nC_1 a_{(1,1)} + nC_2 a_{(2,1)} + \ldots + nC_t a_{(t,1)} + \ldots + nC_n a_{(n,1)} } \]
\[ = \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} \]
\[ = 2^n - 1 \]
\[ = \frac{(2^{n+1} - 2)}{2} . \]

Consider \( a_{(n+1,3)} \)
\[ = \binom{n}{2} a_{(2,2)} + \binom{n}{3} a_{(3,2)} + \binom{n}{4} a_{(4,2)} + \ldots + \binom{n}{t} a_{(t,2)} + \ldots + \binom{n}{n} a_{(n,2)} \]
\[ = \binom{n}{2} (2^1 - 1) + \binom{n}{3} (2^2 - 1) + \binom{n}{4} (2^3 - 1) + \ldots + \binom{n}{n} (2^{n-1} - 1) \]
\[ = \binom{n}{2} 2^1 + \binom{n}{3} 2^2 + \ldots + \binom{n}{n} 2^{n-1} - \left\{ \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} \right\} \]
\[ = \left(\frac{1}{2}\right) \left\{ \binom{n}{2} 2^2 + \binom{n}{3} 2^3 + \ldots + \binom{n}{n} 2^n \right\} - \left\{ \sum_{r=0}^{n} \binom{n}{r} - \binom{n}{1} - \binom{n}{0} \right\} \]
\[ = \left(\frac{1}{2}\right) \left\{ \sum_{r=0}^{n} \binom{n}{r} 2^r - \binom{n}{1} \cdot 2^1 - \binom{n}{0} \cdot 2^0 \right\} - \{ 2^n - n - 1 \} \]
\[ = \left(\frac{1}{2}\right) \left\{ 3^n - 2n - 1 \right\} - 2^n + n + 1 \]
\[ = \left(\frac{1}{2}\right) \left\{ 3^n - 2^{n+1} + 1 \right\} \]
\[ = \left(\frac{1}{3!}\right) \left\{ (1) \cdot 3^{n+1} - (3) \cdot 2^{n+1} + (3) \cdot (1)^{n+1} - (1) (0)^{n+1} \right\} \]

Evaluating \( a_{(n+1,4)} \)
\[ a_{(n+1,4)} = \binom{n}{3} a_{(3,3)} + \binom{n}{4} a_{(4,3)} + \ldots + \binom{n}{n} a_{(n,3)} \]
\[ = \binom{n}{3} \{3^2 + 1 - 2^3 \}/2 + \binom{n}{4} \{3^3 + 1 - 2^4 \}/2 + \ldots + \binom{n}{n} \{3^{n-1} + 1 - 2^n \}/2 \]
\[ = \left(\frac{1}{2}\right) \left\{ 3^2 \cdot \binom{n}{3} + 3^3 \cdot \binom{n}{4} + \ldots + 3^{n-1} \cdot \binom{n}{n} \right\} \]
\[ = \left(\frac{1}{2}\right) \left\{ \sum_{r=0}^{n} \binom{n}{r} 3^r - 3^2 \binom{n}{2} - 3^n \binom{n}{1} - \binom{n}{0} \right\} \]
\[ = \left(\frac{1}{2}\right) \left\{ \frac{1}{3!} \left\{ (1) \cdot 3^{n+1} - (3) \cdot 2^{n+1} + (3) \cdot (1)^{n+1} - (1) (0)^{n+1} \right\} - \left\{ 2^n - n - 1 \right\} \right\} \]
\[ = \left(\frac{1}{2}\right) \left\{ \frac{1}{3!} \left\{ (1) \cdot 3^{n+1} - (3) \cdot 2^{n+1} + (3) \cdot (1)^{n+1} - (1) (0)^{n+1} \right\} \right\} - \left\{ 2^n - n - 1 \right\} \]
\[ = (1/2) [(1/3)\{ 4^n - 9n(n-1)/2 - 3n - 1 \} + \{ 2^n - n(n-1)/2 - n - 1 \} \]

246
\[ a_{(n+1,4)} = \frac{1}{4!} \left[ (1) \ 4^{n+1} - (4) \ 3^{n+1} + (6) \ 2^{n+1} - (4) \ 1^{n+1} + 1(0)^{n+1} \right] \]

Observing the pattern we can explore the possibility of

\[ a_{(n,r)} = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^n \]  

-------(3.2)

which is yet to be established. Now we shall apply induction.

Let the following proposition (3.3) be true for \( r \) and all \( n > r \).

\[ a_{(n+1,r)} = \frac{1}{r!} \sum_{k=1}^{r} (-1)^{r-k} \cdot C_k \cdot k^{n+1} \]  

-------(3.3)

Given (3.3) our aim is to prove that

\[ a_{(n+1,r+1)} = \frac{1}{(r+1)!} \sum_{k=1}^{r+1} (-1)^{r+1-k} \cdot C_k \cdot (k)^{n+1} \]

we have

\[ a_{(n+1,r+1)} = \binom{n}{r} a_{(r,r)} + \binom{n}{r+1} a_{(r+1,r+1)} + \binom{n}{r+2} a_{(r+2,r+1)} + \ldots + \binom{n}{n} a_{(n,r)} \]

\[ a_{(n+1,r+1)} = \binom{n}{r} \left\{ \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^r \right\} + \binom{n}{r+1} \left\{ \frac{1}{r+1} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^{r+1} \right\} \]

\[ + \ldots + \binom{n}{n} \left\{ \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \cdot C_k \cdot k^n \right\} \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \left\{ \binom{n}{r} k^r + \binom{n}{r+1} k^{r+1} + \ldots + \binom{n}{n} k^n \right\} \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \left\{ \sum_{q=0}^{n} \binom{n}{q} k^q - \sum_{q=0}^{r-1} \binom{n}{q} k^q \right\} \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k (1+k)^n \]  

\[ - \frac{1}{r!} \sum_{q=0}^{r-1} (-1)^{r-k} \cdot C_k \left\{ \sum_{q=0}^{n} \binom{n}{q} k^q \right\} \]

247
If we denote the 1st and the second term as $T_1$ and $T_2$, we have

$$a_{(n+1,r+1)} = T_1 - T_2 \quad \text{---------(3.4)}$$

consider $T_1 = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k (1+k)^n]$

$$= \frac{1}{(r+1)!} \sum_{k=0}^{r} [(-1)^{r-k} \{ r!/(k!(r-k)!) \} (1+k)^n]$$

$$= \frac{1}{(r+1)!} \sum_{k=0}^{r} [(-1)^{r-k} ((r+1)!/(k!(r-k)!))(1+k)^{n+1}]$$

$$= \frac{1}{(r+1)!} \sum_{k=0}^{r} [(-1)^{r-k} C_{k+1} (1+k)^{n+1}]$$

$$= \frac{1}{(r+1)!} \sum_{k=0}^{r} [(-1)^{(r+1)-(k+1)} C_{k+1} (1+k)^{n+1}]$$

$$\text{Let } k+1 = s, \text{ we get } s = 1 \text{ at } k = 0 \text{ and } s = r + 1 \text{ at } k = r$$

$$= \frac{1}{(r+1)!} \sum_{s=1}^{r+1} [(-1)^{(r+1)-s} C_{s} (s)^{n+1}]$$

replacing $s$ by $k$ we get

$$= \frac{1}{(r+1)!} \sum_{k=1}^{r+1} [(-1)^{(r+1)-k} C_{k} (k)^{n+1}]$$

in this if we include $k = 0$ case we get

$$T_1 = \frac{1}{(r+1)!} \sum_{k=0}^{r+1} [(-1)^{(r+1)-k} C_{k} (k)^{n+1}] \quad \text{-----(3.5)}$$

$T_1$ is nothing but the right hand side member of (3.3).

To prove (3.3) we have to prove $a_{(n+1,r+1)} = T_1$

In view of (3.4) our next step is to prove that $T_2 = 0$
\[ T_2 = \frac{1}{r!} \sum_{k=0}^{r} ((-1)^{r-k} \cdot C_k \{ \sum_{q=0}^{r-1} \binom{n}{q} k^q \}) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \{ nC_0 k^0 + nC_1 k^1 + nC_2 k^2 + \ldots + nC_{r-1} k^{r-1} \} \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \{ nC_1 \left[ (1/r!) \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k k \right] \} + \]

\[ nC_2 \left[ (1/r!) \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k k^2 \right] + \ldots + nC_{r-1} \left[ (1/r!) \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k k^{r-1} \right] \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \{ nC_1 \left[ (1/r!) \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k k \right] \} + \]

\[ \left[ nC_2 \cdot a_{(2,r)} + nC_3 \cdot a_{(3,r)} + \ldots + nC_{r-1} \cdot a_{(r-1,r)} \right] \]

\[ = X + Y + Z \text{ say where} \]

\[ X = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \{ nC_1 \left[ (1/r!) \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k k \right] \} \]

\[ Y = nC_1 \left[ (1/r!) \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k k \right] \]

\[ Z = \left[ nC_2 \cdot a_{(2,r)} + nC_3 \cdot a_{(3,r)} + \ldots + nC_{r-1} \cdot a_{(r-1,r)} \right] \]

We shall prove that \( X = 0 \), \( Y = 0 \), \( Z = 0 \) seperately.

(1) \( X = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_{r-k} \]

let \( r - k = w \) then we get at \( k = 0 \) \( w = r \) and at \( k = r \) \( w = 0 \).

\[ = \frac{1}{r!} \sum_{w=r}^{0} (-1)^w \cdot C_w \]

249
\[= \frac{(1 - 1)^r}{r!}\]
\[= 0\]

We have proved that \(X = 0\)

\((2)\)

\[Y = \binom{n}{1} \sum_{k=0}^{r} \left\{ (-1)^{r-k} \cdot \binom{r}{k} \right\}\]
\[= \binom{n}{1} \sum_{k=1}^{r-1} \left\{ (-1)^{r-1-(k-1)} \cdot \binom{r-1}{k-1} \right\}\]
\[= \binom{n}{1} \left( \frac{1}{(r-1)!} \right) (1 - 1)^{r-1}\]
\[= 0\]

We have proved that \(Y = 0\)

\(3\) To prove

\[Z = \left[ \binom{n}{2} \cdot a_{(2,r)} + \binom{n}{3} \cdot a_{(3,r)} + \ldots + \binom{n}{r-1} \cdot a_{(r-1,r)} \right] = 0 \quad ----(3.6)\]

**Proof:**

Refer the matrix

\[
\begin{array}{ccccccc}
\mathbf{a}_{(1,1)} & \mathbf{a}_{(1,2)} & \mathbf{a}_{(1,3)} & \mathbf{a}_{(1,4)} & \ldots & \mathbf{a}_{(1,r)} \\
\mathbf{a}_{(2,1)} & \mathbf{a}_{(2,2)} & \mathbf{a}_{(2,3)} & \mathbf{a}_{(2,4)} & \ldots & \mathbf{a}_{(2,r)} \\
\mathbf{a}_{(3,1)} & \mathbf{a}_{(3,2)} & \mathbf{a}_{(3,3)} & \mathbf{a}_{(3,4)} & \ldots & \mathbf{a}_{(3,r)} \\
\mathbf{a}_{(4,1)} & \mathbf{a}_{(4,2)} & \mathbf{a}_{(4,3)} & \mathbf{a}_{(4,4)} & \ldots & \mathbf{a}_{(4,r)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{array}
\]

250
The Diagonal elements are underlined. And the the elements above the leading diagonal are shown with bold face.

We have

\[
a_{(1,r)} = \left[ (1/r!) \sum_{k=0}^{r} \left\{ (-1)^{r-k} \cdot \binom{r}{k} k \right\} \right] = \frac{Y/nC_1}{Y/n} = 0 \text{ for } r > 1
\]

All the elements of the first row except \(a_{(1,1)}\) (the one on the leading diagonal) are zero.

Also

\[
a_{(n+1,r)} = a_{(n,r-1)} + r \cdot a_{(n,r)} \quad \text{(3.7)}
\]

(7) gives us

\[
a_{(2,r)} = a_{(1,r-1)} + r \cdot a_{(1,r)} = 0 \text{ for } r > 2
\]

i.e. \(a_{(2,r)}\) can be expressed as a linear combination of two elements of the first row (except the one on the leading diagonal).

\[\Rightarrow a_{(2,r)} = 0 \quad r > 2\]

Similarly, \(a_{(3,r)}\) can be expressed as a linear combination of two elements of the second row of the type \(a_{(2,r)}\) with \(r > 3\)

\[\Rightarrow a_{(2,r)} = 0 \quad r > 3\]

and so on \(a_{(r-1,r)} = 0\)

substituting

\[a_{(2,r)} = a_{(3,r)} = \ldots = a_{(r-1,r)} = 0 \text{ in (6)}\]

we get \(Z = 0\)
With \( X = Y = Z = 0 \) we get \( T_2 = 0 \)

or \( a_{(n+1, r+1)} = T_1 - T_2 = T_1 \)

from (5) we have

\[
T_1 = \left(\frac{1}{(r+1)!}\right) \sum_{k=0}^{r+1} \left[ (-1)^{(r+1)} \cdot k \cdot r^1 C_k (k)^{n+1} \right]
\]

which gives

\[
a_{(n+1, r+1)} = \left(\frac{1}{(r+1)!}\right) \sum_{k=0}^{r+1} \left[ (-1)^{(r+1)} \cdot k \cdot r^1 C_k (k)^{n+1} \right]
\]

We have proved, if the proposition (3.3) is true for \( r \) it is true for \( r+1 \) as well. We have already verified it for 1, 2, 3 etc. Hence by induction (3.3) is true for all \( n \).

This completes the proof of theorem (3.1).

Remarks: This proof is quite lengthy, clumsy and heavy in algebra. The readers can try some analytic, combinatorial approach.

REFERENCES:


[3] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.