A NEW SEQUENCE RELATED SMARANDACHE SEQUENCES AND ITS MEAN VALUE FORMULA*

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ABSTRACT. Let \( n \) be any positive integer, \( a(n) \) denotes the product of all non-zero digits in base 10. For natural \( x \geq 2 \) and arbitrary fixed exponent \( m \in \mathbb{N} \), let \( A_m(x) = \sum_{n<x} a^m(n) \). The main purpose of this paper is to give two exact calculating formulas for \( A_1(x) \) and \( A_2(x) \).

1. INTRODUCTION

For any positive integer \( n \), let \( b(n) \) denotes the product of base 10 digits of \( n \). For example, \( b(1) = 1, b(2) = 2,\ldots, b(10) = 0, b(11) = 1, \ldots \). In problem 22 of book [1], Professor F.Smaradache ask us to study the properties of sequence \( \{b(n)\} \). About this problem, it appears that no one had studied it yet, at least, we have not seen such a paper before. The problem is interesting because it can help us to find some new distribution properties of the base 10 digits. In this paper, we consider another sequence \( \{a(n)\} \), which related to Smarandache sequences. Let \( a(n) \) denotes the product of all non-zero digits in base 10 of \( n \). For example, \( a(1) = 1, a(2) = 2, a(12) = 2, \ldots, a(28) = 16, a(1023) = 6,\ldots\ldots \). For natural number \( x \geq 2 \) and arbitrary fixed exponent \( m \in \mathbb{N} \), let

\[
A_m(x) = \sum_{n<x} a^m(n).
\]

The main purpose of this paper is to study the calculating problem of \( A_m(x) \), and use elementary methods to deduce two exact calculating formulas for \( A_1(x) \) and \( A_2(x) \). That is, we shall prove the following:

**Theorem.** For any positive integer \( x \), let \( x = a_110^{k_1} + a_210^{k_2} + \cdots + a_s10^{k_s} \) with \( k_1 > k_2 > \cdots > k_s \geq 0 \) and \( 1 \leq a_i \leq 9, i = 2,3,\ldots,s \). Then we have the calculating formulas

\[
A_1(x) = \frac{a_1a_2\cdots a_s}{2} \sum_{i=1}^{s} \frac{a_i^2 - a_i + 2}{\prod_{j=i}^{s} a_j} \left( 45 + \left[ \frac{1}{k_i+1} \right] \right) \cdot 46^{k_i-1};
\]

\[
A_2(x) = \sum_{n<x} \frac{a^n(n)}{\prod_{i=1}^{s} a_i} \cdot \left( 45 + \left[ \frac{1}{k_i+1} \right] \right) \cdot 46^{k_i-1};
\]

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\[ A_2(x) = \frac{a_1^2 a_2^2 \cdots a_s^2}{6} \sum_{i=1}^{s} \frac{2a_i^3 - 3a_i^2 + a_i + 6}{\prod_{j=i}^{s} a_j^2} \left( 285 + \left[ \frac{1}{k_i + 1} \right] \right) \cdot 286^{k_i-1}, \]

where \([x]\) denotes the greatest integer not exceeding \(x\).

For general integer \(m \geq 3\), using our methods we can also give an exact calculating formula for \(A_m(x)\). That is, we have the calculating formula

\[ A_m(x) = a_1^m a_2^m \cdots a_s^m \sum_{i=1}^{s} \frac{1 + B_m(a_i)}{\prod_{j=i}^{s} a_j^m} \left( \left[ \frac{1}{k_i + 1} \right] + B_m(10) \right) \cdot (1 + B_m(10))^{k_i-1}, \]

where \(a_i\) as the definition as in the above Theorem, and \(B_m(N) = \sum_{1 \leq n < N} n^m\).

2. PROOF OF THE THEOREM

In this section, we complete the proof of the Theorem. First we need following two simple Lemmas.

Lemma 1. For any integer \(k \geq 1\) and \(1 \leq c \leq 9\), we have the identities

a) \(A_1(10^k) = 45 \cdot 46^{k-1}\);

b) \(A_1(c \cdot 10^k) = 45 \cdot \left( 1 + \frac{(c-1)c}{2} \right) \cdot 46^{k-1}\).

Proof. We first prove a) of Lemma 1 by induction. For \(k = 1\), we have \(A_1(10^1) = A_1(10) = 1 + 2 + \cdots + 9 = 45\). So that the identity

\[ A_1(10^k) = \sum_{n < 10^k} a(n) = 45 \cdot 46^{k-1} \quad \text{(2)} \]

holds for \(k = 1\). Assume (2) is true for \(k = m \geq 1\). Then by the inductive assumption we have

\[ A_1(10^{m+1}) = \sum_{n < 9 \cdot 10^m} a(n) + \sum_{9 \cdot 10^m \leq n < 10^{m+1}} a(n) \]

\[ = A_1(9 \cdot 10^m) + \sum_{0 \leq n < 10^m} a(n + 9 \cdot 10^m) \]

\[ = A_1(9 \cdot 10^m) + 9 \cdot \sum_{0 \leq n < 10^m} a(n) \]

\[ = A_1(9 \cdot 10^m) + 9 \cdot \sum_{n < 10^m} a(n) \]

\[ = A_1(9 \cdot 10^m) + 9 \cdot A_1(10^m) \]

\[ = A_1(8 \cdot 10^m) + 9 \cdot A_1(10^m) + 8 \cdot A_1(10^m) \]

\[ = \ldots \]

\[ = (1 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \cdot A_1(10^m) \]

\[ = 46 \cdot A_1(10^m) \]

\[ = 45 \cdot 46^m. \]
That is, (2) is true for \( k = m + 1 \). This proves the first part of Lemma 1.

The second part b) follows from a) of Lemma 1 and the recurrence formula

\[
A_1(c \cdot 10^k) = \sum_{n < (c-1) \cdot 10^k} a(n) + \sum_{(c-1) \cdot 10^k 
\leq n < c \cdot 10^k} a(n) 
= \sum_{n < (c-1) \cdot 10^k} a(n) + \sum_{0 \leq n < 10^k} a(n + (c - 1) \cdot 10^k) 
= \sum_{n < (c-1) \cdot 10^k} a(n) + (c - 1) \cdot \sum_{n < 10^k} a(n) 
= A_1((c - 1) \cdot 10^k) + (c - 1) \cdot A_1(10^k).
\]

This completes the proof of Lemma 1.

**Lemma 2.** For any integer \( k \geq 1 \) and \( 1 \leq c \leq 9 \), we have the identities

\[
c) \quad A_2(10^k) = 285 \cdot 286^{k-1}; \\
d) \quad A_2(a \cdot 10^k) = 285 \cdot \left[ 1 + \frac{(a-1)a(2a-1)}{6} \right] \cdot 286^{k-1}.
\]

**Proof.** Note that \( A_2(10) = 285 \). The Lemma 2 can be deduced by Lemma 1, induction and the recurrence formula

\[
A_2(10^{k+1}) = \sum_{n < 10 \cdot 10^k} a^2(n) + \sum_{9 \cdot 10^k \leq n < 10^{k+1}} a^2(n) 
= \sum_{n < 9 \cdot 10^k} a^2(n) + \sum_{0 \leq n < 10^k} a^2(n + 9 \cdot 10^k) 
= \sum_{n < 9 \cdot 10^k} a^2(n) + 9^2 \cdot \sum_{0 \leq n < 10^k} a^2(n) 
= A_2(9 \cdot 10^k) + 9^2 \cdot A_2(10^k) 
= \ldots \ldots 
= (1 + 1^2 + 2^2 + \ldots + 9^2) \cdot A_2(10^k) 
= 286 \cdot A_2(10^k).
\]

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any positive integer \( x \), let \( x = a_1 \cdot 10^{k_1} + a_2 \cdot 10^{k_2} + \ldots + a_s \cdot 10^{k_s} \) with \( k_1 > k_2 > \ldots > k_s \geq 0 \) under the base 10. Then applying Lemma 1 repeatedly we have

\[
A_1(x) = \sum_{n < a_1 \cdot 10^{k_1}} a(n) + \sum_{a_1 \cdot 10^{k_1} \leq n < x} a(n) 
= A_1(a_1 \cdot 10^{k_1}) + \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a(n + a_1 \cdot 10^{k_1}) 
= A_1(a_1 \cdot 10^{k_1}) + a_1 \cdot \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a(n)
\]
This proves the first part of the Theorem.

Applying Lemma 2 and the first part of the Theorem repeatedly we have

\[ A_2(x) = \sum_{n < a_1 \cdot 10^{k_1}} a^2(n) + \sum_{a_1 \cdot 10^{k_1} \leq n < x} a^2(n) \]

\[ = A_2(a_1 \cdot 10^{k_1}) + \sum_{n = a_1 \cdot 10^{k_1}}^{n < x} a^2(n + a_1 \cdot 10^{k_1}) \]

\[ = A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a^2(n) \]

\[ = A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot A_2(x - a_1 \cdot 10^{k_1}) \]

\[ = A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot A_2(a_2 \cdot 10^{k_2}) + a_1^2 a_2^2 \cdot A_2(x - a_1 \cdot 10^{k_1} - a_2 \cdot 10^{k_2}) \]

\[ = \ldots \ldots \]

\[ = \sum_{i=1}^{s} \frac{a_1^2 a_2^2 \cdots a_s^2}{\prod_{j=i}^{s} a_j^2} A_2(a_i \cdot 10^{k_i}) \]

\[ = \frac{a_1^2 a_2^2 \cdots a_s^2}{6} \sum_{i=1}^{s} \frac{2a_i^3 - 3a_i^2 + a_i + 6}{\prod_{j=i}^{s} a_j^2} \left( 285 + \left[ \frac{1}{k_i + 1} \right] \right) \cdot 286^{k_i - 1}. \]

This completes the proof of the second part of the Theorem.

**REFERENCES**