Carlitz considered the numbers $\eta_k(q)$ which are determined by

$$\eta_0(q) = 0, \quad (q\eta(q) + 1)^k - \eta_k(q) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}.$$ 

These numbers $\eta_k(q)$ induce Carlitz's $k$th $q$-Bernoulli numbers $\beta_k(q) = \beta_k$ as

$$\beta_0 = 1, \quad q(\beta + 1)^k - \beta_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

where we use the usual convention about replacing $\beta^i$ by $\beta_i$ $(i \geq 0)$.

Now, we modify the above number $\eta_m(q)$, that is,

$$B_0(q) = \frac{q-1}{\log q}, \quad (qB(q) + 1)^k - B_k(q) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

where we use the usual convention about replacing $B^i(q)$ by $B_i(q)$ $(i \geq 0)$.

In [1], I have constructed a complex $q$-series which is a $q$-analogue of Hurwitz's $\zeta$-function. In this a short note, I will compute the values of zeta by using the $q$-series.

Let $F_q(t)$ be the generating function of $B_i(q)$:

$$F_q(t) = \sum_{k=0}^{\infty} B_k(q) \frac{t^k}{k!} \quad \text{for } q \in \mathbb{C} \text{ with } |q| < 1.$$ 

This is the unique solution of the following $q$-difference equation:

$$F_q(t) = e^t F_q(qt) - t.$$ 

It is easy to see that

$$F_q(t) = -t \sum_{n=0}^{\infty} q^n c[n] \frac{t^n}{n!} + \frac{q-1}{\log q} e^{t/q} - t.$$ 

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Thus we have:

\[ B_k(q) = \frac{d^k}{dt^k} F_q(t) \bigg|_{t=0} = -k \sum_{n=0}^{\infty} \frac{q^n}{(n)_q^{k-1}} + \frac{(-1)^k}{(q-1)^{k-1}} \log q. \]

Hence, we can define a $q$-analogue of the $\zeta$-function as follows:

For $s \in \mathbb{C}$, define (see[1])

\[ \zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^n}{(n)_q^s} - \frac{1}{s-1} \frac{(1-q)^s}{\log q}. \]

Note that, $\zeta_q(s)$ is analytic continuation in $\mathbb{C}$ with only one simple pole at $s = 1$ and

\[ \zeta_q(1-k) = -\frac{B_1(q)}{k} \quad \text{where } k \text{ is any positive integer.} \]

Now, we define $q$-Bernoulli polynomial $B_n(x; q)$ as

\[ B_n(x; q) = (q^x B(q) + [x]) = \sum_{k=0}^{n} \binom{n}{k} q^k B_k(q)[x]^{n-k}. \]

Let $T_q(x, t)$ be generating function of $q$-Bernoulli polynomials.

Note that

\[ T_q(x, t) = F_q(q^x t e^{x t}). \]

Thus

\[ B_{k+1}(x; q) = \frac{d^{k+1}}{dt^{k+1}} T_q(x, t) \bigg|_{t=0} = -(k+1) \sum_{n=0}^{\infty} \binom{[n]_q x + [x]}{n+1} q^{n+1} + \frac{q-1}{\log q} \left( \frac{1}{q-1} \right)^{k+1}. \]

So, we can also define a $q$-analogue of the Hurwitz $\zeta$-function as follows:

For $s \in \mathbb{C}$ (see [1])

\[ \zeta_q(s, x) = \sum_{n=0}^{\infty} \frac{q^{n+x}}{([n]_q^{x} + [x])^{s}} - \frac{(1-q)^s}{\log q} \frac{1}{s-1} \]

\[ = \sum_{n=0}^{\infty} \frac{q^{n+x}}{([n+x]_q^{x})^{s}} - \frac{(1-q)^s}{\log q} \frac{1}{s-1}, \quad 0 < x \leq 1. \]

Note that, $\zeta_q(s, x)$ has an analytic continuation in $\mathbb{C}$ with only one simple pole at $s = 1$.

Remark. $\zeta_q(s, x)$ is called $q$-analogue of Hurwitz $\zeta$-function.
For \( u \in \mathbb{C} \) with \( u \neq 0, 1 \), let \( H_k(u : q) \) be \( q \)-Euler numbers (See [4]). It is known in [4] that \( H_k(u : 1) = H_k(u) \) is the ordinary Euler number which is defined by

\[
\frac{1 - u}{e^t - u} = \sum_{k=0}^{\infty} H_k(u) \frac{t^k}{k!}.
\]

In the case \( u = -1, H_k(-1) = E_k \) is the classical Euler number is defined by

\[
\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.
\]

Note that \( E_{2k} = 0 \) \((k \geq 1)\). In [4], \( \ell_q(s, u) \) is defined by \( \ell_q(s, u) = \sum_{n=1}^{\infty} \frac{u^n}{n!} \) and \( \ell_q(-k, u) = u^{-k} H_k(u : q) \) for \( k > 1 \).

**Theorem 1.** For \( s \in \mathbb{C}, f \in \mathbb{N}\setminus\{1\} \), we have

1. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} q^n = -\zeta_q^*(s) + \frac{2}{(2^f)!} \zeta_q^*(s) \).
2. \( \zeta_q^*(s) = \left[ \frac{1}{2} \right] \sum_{n=1}^{\infty} \zeta_q^*(s, \frac{n}{2}) \), where \( \zeta_q^*(s) = \sum_{n=1}^{\infty} \frac{q^n}{n^s} \).

It is easy to see that

\[
\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n!^{2k+1}} \sum_{j=0}^{\infty} \frac{g^{2j+1} [n]^{2j+1}}{(2j + 1)!} + \frac{1}{\log q} \sum_{j=0}^{k-1} \frac{(1-q)^{2k-2j}}{2k-2j-1} \frac{g^{2j+1}}{(2j + 1)!}
\]

\[= \sum_{j=0}^{k-1} \frac{(-1)^j g^{2j+1}}{(2j + 1)!} \left( -\frac{\zeta_q(2k-2j) + \zeta_q(2k-2j) \frac{2}{[2]^{2k-2j}}} {2^{2k-2j}} \right) - \frac{q}{1 + q (2k + 1)!} (-1)^k + \sum_{j=k+1}^{\infty} \frac{g^{2j+1} (-1)^j}{(2j + 1)!} \frac{q^{-1}}{1 + q^{-1} H_{2j-2k} (-q^{-1}, q)}.\]

If \( q \to 1 \), then we have

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k+1}} \sin(n \theta) = \sum_{j=0}^{k-1} \frac{(-1)^j g^{2j+1}}{(2j + 1)!} \frac{2}{2^{2k-2j}} \left( \frac{2}{2^{2k-2j}} - 1 \right)
\]

\[= (-1)^{k-j+1} \frac{(2\pi)^{2k-2j}}{2^{2k-2j} (2k-2j)!} B_{2k-2j} - \frac{1}{2} \frac{g^{2k+1}}{(2k + 1)!} (-1)^k.\]

Let \( k = 2 \) and \( \theta = \frac{\pi}{2} \). Then we have

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)^5} = \pi^5 \left( \frac{1}{2^5} - \frac{B_2}{2^4} \cdot \frac{5}{3!} + \frac{7}{12} B_4 \right).
\]
It is easy to see that
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^5} = 2 \sum_{n=1}^{\infty} \frac{1}{(4n+3)^5} + \sum_{n=1}^{\infty} \frac{1}{(2n)^5} - \sum_{n=1}^{\infty} \frac{1}{n^5} - 1 = \frac{1}{2^5} \zeta(5, \frac{1}{4}) - \frac{2^5 - 1}{2^5} \zeta(5) - 1.
\]
Thus we have
\[
\zeta(5) - \frac{1}{24} \frac{1}{2^5 - 1} \zeta \left(5, \frac{1}{4} \right) - 1 = -2^5 \pi^5 \left( \frac{1}{2^5 \cdot 5!} - \frac{B_2}{2^4 \cdot 3!} + \frac{7}{12} B_4 \right).
\]
Therefore we obtain the following:

Proposition 2. \( \zeta(5) - \frac{1}{24} \frac{1}{2^5 - 1} \zeta(5, \frac{1}{4}) - 1 \) is irrational.

REFERENCES


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