ABOUT THE SMARANDACHE COMPLEMENTARY PRIME FUNCTION

by

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Let \( c : \mathbb{N} \to \mathbb{N} \) be the function defined by the condition that \( n + c(n) = p_n \), where \( p_n \) is the smallest prime number. \( p_n \geq n \).

Example

\[
\begin{align*}
    c(0) &= 2, \\
    c(1) &= 1, \\
    c(2) &= 0, \\
    c(3) &= 0, \\
    c(4) &= 1, \\
    c(5) &= 0, \\
    c(6) &= 1, \\
    c(7) &= 0 
\end{align*}
\]

and so on.

1) If \( p_k \) and \( p_{k-1} \) are two consecutive primes and \( p_k < n \leq p_{k-1} \), then:

\[
c(n) \in \{ p_{k-1} - 1, p_{k-1} - 2, \ldots, 1, 0 \},
\]

because:

\[
c(p_k - 1) = p_{k-1} - p_{k-1} - 1 \text{ and so on, } c(p_{k-1}) = 0.
\]

2) \( c(p) = c(p - 1) - 1 = 0 \) for every prime \( p \), because \( c(p) = 0 \) and \( c(p - 1) = 1 \).

We also can observe that \( c(n) \neq c(n + 1) \) for every \( n \in \mathbb{N} \).

1. Property

The equation \( c(n) = n \), \( n > 1 \) has no solutions.

Proof

If \( n \) is a prime it results \( c(n) = 0 < n \).

It is wellknown that between \( n \) and \( 2n \), \( n > 1 \) there exists at least a prime number. Let \( p_k \) be the smallest prime of them. Then if \( n \) is a composite number we have:

\[
c(n) = p_k - n < 2n - n = n, \text{ therefore } c(n) < n.
\]
It results that for every $n = p$, where $p$ is a prime, we have \( \frac{1}{n} \leq \frac{c(n)}{n} < 1 \). Therefore \( \sum_{n=1}^{\infty} \frac{c(n)}{n} \) diverges. Because for the primes \( c(p)/p = 0 \) we can say that \( \sum_{n=1}^{\infty} \frac{c(n)}{n} \) diverges.

2. Property

If $n$ is a composite number, then $c(n) = c(n-1) - 1$.

Proof

Obviously.

It results that for $n$ and $(n-1)$ composite numbers we have $\frac{c(n)}{c(n-1)} > 1$. Now, if $p_k < n < p_{k+1}$, where $p_k$ and $p_{k+1}$ are consecutive primes, then we have:

\[
c(n) c(n-1) \ldots c(p_{k-1} - 1) = (p_{k-1} - n)!
\]

and if $n \leq p_k < n$, then $c(n) c(n-1) \ldots c(p_{k-1} - 1) = 0$.

Of course, every $\prod_{n=k}^{p} c(n) = 0$ if there exists a prime number $p$, $k \leq p \leq r$.

If $n = p_k$ is any prime number, then $c(n) = 0$ and because $c(n+1) = p_{k-1} - n - 1$ it results that $|c(n) - c(n+1)| = 1$ if and only if $n$ and $(n+2)$ are primes (friend prime numbers).

3. Property

For every $k$-th prime number $p_k$ we have:

\[
c(p_k - 1) < (\log p_k)^2 - 1.
\]

Proof

Because $c(p_k - 1) = p_{k-1} - p_k - 1$ we have $p_{k-1} - p_k = c(p_k - 1) + 1$.

But, on the other hand we have $p_{k-1} - p_k < (\log p_k)^2$, then the assertion follows.

4. Property

$c(c(n)) < c(n)$ and $c^m(n) < c(n) < n$, for every $n > 1$ and $m \geq 2$.

Proof

If we denote $c(n) = r$ then we have:

\[
c(c(n)) = c(r) < r = c(n).
\]

Then we suppose that the assertion is true for $m: c^m(n) < c(n) < n$, and we prove it
5. Property
For every prime \( p \) we have \((c(p-1))^n \leq c((p-1)^n)\).

Proof
\[ c(p-1) = 1 \Rightarrow (c(p-1))^n = 1 \text{ while } (p-1)^n \text{ is a composite number. therefore} \]
\[ c((p-1)^n) \geq 1. \]

6. Property
The following kind of Fibonacci equation:
\[ c(n) - c(n-1) = c(n-2) \quad (1) \]
has solutions.

Proof
If \( n \) and \( (n-1) \) are both composite numbers, then \( c(n) > c(n-1) \geq 1 \). If \( (n+2) \) is a prime, then \( c(n+2) = 0 \) and we have no solutions in this case. If \( (n+2) \) is also a composite number, then:
\[ c(n) > c(n-1) > c(n-2) \geq 1, \text{ therefore } c(n) - c(n-1) > c(n-2) \]
and we have no solutions also in this case.

Therefore \( n \) and \( (n-1) \) are not both composite numbers in the equality \((1)\).

If \( n \) is a prime, then \( (n-1) \) is a composite number and we must have:
\[ 0 - c(n-1) = c(n-2), \text{ which is not possible (see \((2)\)).} \]

We have only the case when \( (n-1) \) is a prime: in this case we must have:
\[ 1 - 0 = c(n-2) \] but this implies that \( (n+3) \) is a prime number, so the only solutions are when \( (n-1) \) and \( (n-2) \) are friend prime numbers.

7. Property
The following equation:
\[ \frac{c(n) - c(n-2)}{2} = c(n-1) \quad (2) \]
has an infinite number of solutions.
Proof

Let \( p_k \) and \( p_{k+1} \) be two consecutive prime numbers, but not friend prime numbers.

Then, for every integer \( i \) between \( p_k - 1 \) and \( p_{k+1} - 1 \) we have:

\[
\frac{c(i-1) - c(i-1)}{2} = \frac{(p_{k+1} - i - 1) - (p_k - i - 1)}{2} = p_{k+1} - i = c(i).
\]

So, for the equation (2) all positive integer \( n \) between \( p_k - 1 \) and \( p_{k+1} - 1 \) is a solution.

If \( n \) is prime, the equation becomes \( \frac{c(n-2)}{2} = c(n + 1) \).

But \( (n+1) \) is a composite number, therefore \( c(n + 1) = 0 \) \( \Rightarrow \) \( c(n - 2) \) must be composite number. Because in this case \( c(n + 1) = c(n + 2) + 1 \) and the equation has the form \( \frac{c(n-2)}{2} = c(n + 2) + 1 \), so we have no solutions.

If \( (n-1) \) is prime, then we must have \( \frac{c(n) + c(n-2)}{2} = 0 \), where \( n \) and \( (n + 2) \) are composite numbers. So we have no solutions in this case, because \( c(n) \geq 1 \) and \( c(n + 2) \geq 1 \).

If \( (n + 2) \) is a prime, the equation has the form \( \frac{c(n)}{2} = c(n + 1) \), where \( (n - 1) \) is a composite number, therefore \( c(n - 1) = 0 \). From (2) it results that \( c(n) = 0 \), so \( n \) is also a composite number. This case is the same with the first considered case.

Therefore the only solutions are for \( p_k, p_{k+1} - 2 \), where \( p_k, p_{k+1} \) are consecutive primes, but not friend consecutive primes.

8. Property

The greatest common divisor of \( n \) and \( c(x) \) is 1:

\( (x, c(x)) = 1 \), for every composite number \( x \).

Proof

Taking into account of the definition of the function \( c \), we have \( x + c(x) = p \), where \( p \) is a prime number.

If there exists \( d \neq 1 \) so that \( d \div x \) and \( d \div c(x) \), then it implies that \( d \div p \). But \( p \) is a prime number, therefore \( d = p \).

This is not possible because \( c(x) < p \).

If \( p \) is a prime number, then \( (p, c(p)) = (p, 0) = p \).

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9. Property

The equation \([ x, y ] = [ c(x), c(y) ]\), where \([ x, y ]\) is the least common multiple of \(x\) and \(y\) has no solutions for \(x, y > 1\).

Proof

Let us suppose that \(x = dk_1\) and \(y = dk_2\), where \(d = (x, y)\). Then we must have

\([ x, y ] = dk_1k_2 = [ c(x), c(y) ]\).

But \((x, c(x)) = (dk_1, c(x)) = 1\), therefore \(dk_1\) is given in the least common multiple \([ c(x), c(y) ]\) by \(c(y)\).

But \((y, c(y)) = (dk_1, c(y)) = 1 \Rightarrow d = 1 \Rightarrow (x, y) = 1 \Rightarrow\)

\([ x, y ] = xy > c(x)c(y) \geq [ c(x), c(y) ]\), therefore the above equation has no solutions, for \(x, y > 1\).

For \(x = 1 = y\) we have \([ x, y ] = [ c(x), c(y) ] = 1\).

10. Property

The equation:

\((x, y) = (c(x), c(y))\) \hspace{1cm} (3)

has an infinite number of solutions.

Proof

If \(x = 1\) and \(y = p - 1\) then \((x, y) = 1\) and \((c(x), c(y)) = (1, 1) = 1\), for an arbitrary prime \(p\).

Easily we observe that every pair \((n, n - 1)\) of numbers is a solutions for the equation \((3)\), if \(n\) is not a prime.

11. Property

The equation:

\(c(x) - x = c(y) - y\) \hspace{1cm} (4)

has an infinite number of solutions.

Proof

From the definition of the function \(c\) it results that for every \(x\) and \(y\) satisfying...
\(p_k < x \leq y \leq p_{k-1}\) we have \(c(x) - x = c(y) - y = p_{k-1}\). Therefore we have \((p_{k-1} - p_k)^2\) couples \((x, y)\) as different solutions. Then, until the \(n\)-th prime \(p_n\), we have \(\sum_{k=1}^{n} (p_{k-1} - p_k)^2\) different solutions.

**Remark**

It seems that the equation \(c(x) + y = c(y) + x\) has no solutions \(x \neq y\), but it is not true.

Indeed, let \(p_k\) and \(p_{k-1}\) be consecutive primes such that \(p_{k-1} - p_k = 6\) (is possible: for example \(29 - 23 = 6, 37 - 31 = 6, 53 - 47 = 6\) and so on) and \(p_k - 2\) is not a prime.

Then \(c(p_k - 2) = 2, c(p_k - 1) = 1, c(p_k) = 0, c(p_k + 1) = 5, c(p_k + 2) = 4, c(p_k + 3) = 3\) and we have:

1. \(c(p_k + 1) - c(p_k - 2) = 5 - 2 = 3 = (p_k - 1) - (p_k - 2)\)
2. \(c(p_k - 2) - c(p_k - 1) = 3 = (p_k + 2) - (p_k - 1)\)
3. \(c(p_k + 3) - c(p_k) = 3 = (p_k + 3) - p_k\), thus

\(c(x) - c(y) = x - y (\Leftrightarrow c(x) + y = c(y) + x)\) has the above solutions if \(p_k - p_{k-1} > 3\).

If \(p_k - p_{k-1} = 2\) we have only the two last solutions.

In the general case, when \(p_{k-1} - p_k = 2h, h \in \mathbb{N}^*\), let \(x = p_k - u\) and \(y = p_k + v, u, v \in \mathbb{N}\) be the solutions of the above equation.

Then \(c(x) = c(p_k - u) = u\) and \(c(y) = c(p_k + v) = 2h - v\).

The equation becomes:

\(u + (p_k + v) = (2h - v) + (p_k - u)\), thus \(u + v = h\).

Therefore, the solutions are \(x = p_k - u\) and \(y = p_k + h - u\), for every \(u = 0, h\) if \(p_k - p_{k-1} > h\) and \(x = p_k - u, y = p_k + h - u\), for every \(u = 0, h\) if \(p_k - p_{k-1} = h + 1 \leq h\).

**Remark**

\(c(p_k + 1)\) is an odd number, because if \(p_k\) and \(p_{k-1}\) are consecutive primes, \(p_k > 2\), then \(p_k\) and \(p_{k-1}\) are, of course, odd numbers, then \(p_{k-1} - p_k - 1 = c(p_k + 1)\) are always odd.

**12. Property**

The sumatory function of \(c, F_c(n) \overset{\text{def}}{=} \sum_{d \mid n} c(d)\) has the properties:
a) \( F_\pi(2p) = 1 + c(2p) \)

b) \( F_\pi(pq) = 1 + c(pq) \), where \( p \) and \( q \) are prime numbers.

**Proof**
a) \( F_\pi(2p) = c(1) + c(2) + c(p) + c(2p) = 1 + c(2p) \).
b) \( F_\pi(pq) = c(1) + c(p) + c(q) + c(pq) = 1 + c(pq) \).

**Remark**
The function \( c \) is not multiplicative: \( 0 = c(2) \cdot c(p) < c(2p) \).

13. **Property**
\[ c^k(p) = \begin{cases} 
0 & \text{for } k \text{ odd number} \\
2 & \text{for } k \text{ even number}, \ k \geq 1 
\end{cases} \]

**Proof**
We have:
\[
c^1(p) = 0;
c^2(p) = c(c(p)) = c(0) = 2;
c^3(p) = c(2) = 0;
c^4(p) = c(0) = 2.
\]
Using the complete mathematical induction, the property holds.

**Consequences**
1) We have \( \frac{c^k(p)}{2} + \frac{c^{k+1}(p)}{2} = 1 \) for every \( k \geq 1 \) and \( p \) prime number.

2) \( \sum_{k=1}^{\infty} c^k(p) = \left[ \frac{p}{2} \right] \cdot 2, \) where \( \left[ x \right] \) is the integer part of \( x \), and
\[
\sum_{k=2}^{\infty} \frac{1}{c^k(p)} = \left[ \frac{p}{2} \right] \cdot \frac{1}{2}, \] thus \( \sum_{k=1}^{\infty} c^k(p) \) and \( \sum_{k=2}^{\infty} \frac{1}{c^k(p)} \) are divergent series.

**Remark**
\( c^k(p - 1) = c^{k-1}(c(p - 1)) = c^{k-1}(1) = 1 \), for every prime \( p > 3 \) and \( k \in \mathbb{N}^* \), therefore \( c^k(p - 1) = c^k(p_2 - 1) \) for every primes \( p_1, p_2 > 3 \) and \( k_1, k_2 \in \mathbb{N}^* \).

14. **Property**
The equation:
\[ c(x) + c(y) + c(z) = c(x)c(y)c(z) \] (5)
has an infinite number of solutions.

**Proof**

The only non-negative solutions for the diophantine equation \(a + b - c = abc\) are \(a = 1, b = 2\) and \(c = 3\) and all circular permutations of \(\{1, 2, 3\}\).

Then:

\[c(x) = 1 \Rightarrow x = p_k - 1, \quad \text{prime number, } p_k > 3\]

\[c(y) = 2 \Rightarrow y = p_k - 2, \quad \text{where } p_{k-1} \text{ and } p_k \text{ are consecutive prime numbers such that } p_k - p_{k-1} \geq 3\]

\[c(z) = 3 \Rightarrow z = p_k - 3, \quad \text{where } p_{k-1} \text{ and } p_k \text{ are consecutive prime numbers such that } p_k - p_{k-1} \geq 4\]

and all circular permutations of the above values of \(x, y\) and \(z\).

Of course, the equation \(c(x) = c(y)\) has an infinite number of solutions.

**Remark**

We can consider \(c^{-1}(y)\), for every \(y \in \mathbb{N}^*\), defined as \(c^{-1}(y) = \{ x \in \mathbb{N} | c(x) = y \}\).

For example \(c^{-1}(0)\) is the set of all primes, and \(c^{-1}(1)\) is the set \(\{1, p_k\} \quad \text{prime and so on.}\)

A study of these sets may be interesting.

**Remark**

If we have the equation:

\[c^k(x) = c(y), k \geq 2\]  \hfill (6)

then, using property 13, we have two cases.

If \(x\) is prime and \(k\) is odd, then \(c^k(x) = 0\) and (5) implies that \(y\) is prime.

In the case when \(x\) is prime and \(k\) is even it results \(c^k(x) = 2 = c(y)\), which implies that \(y\) is a prime, such that \(y - 2\) is not prime.

If \(x = p, y = q, p\) and \(q\) primes, \(p, q > 3\), then \((p - 1, q - 1)\) are also solutions, because \(c^k(p - 1) = 1 = c(q - 1)\), so the above equation has an infinite number of couples as solutions.

Also a study of \((c^k(x))^+\) seems to be interesting.
Remark

The equation:
\[ c(n) - c(n-1) - c(n-2) = c(n-1) \] (7)

has solutions when \( c(n-1) = 3, c(n) = 2, c(n-1) = 1, c(n-2) = 0 \), so the solutions are \( n = p - 2 \) for every \( p \) prime number such that between \( p - 4 \) and \( p \) there is not another prime.

The equation:
\[ c(n-2) - c(n-1) - c(n-1) - c(n-2) = 4c(n) \] (8)

has as solutions \( n = p - 3 \), where \( p \) is a prime such that between \( p - 6 \) and \( p \) there is not another prime, because \( 4c(n) = 12 \) and \( c(n-2) - c(n-1) - c(n-1) - c(n-2) = 12 \).

For example \( n = 29 - 3 = 26 \) is a solution of the equation (7).

The equation:
\[ c(n) - c(n-1) - c(n-2) - c(n-3) + c(n-4) = 2c(n-5) \] (9)
(see property 7) has as solution \( n = p - 5 \), where \( p \) is a prime, such that between \( p - 6 \) and \( p \) there is not another prime. Indeed we have \( 0 + 1 + 2 + 3 + 4 = 2 \cdot 5 \).

Thus, using the properties of the function \( c \) we can decide if an equation, which has a similar form with the above equations, has or has not solutions.

But a difficult problem is: "For any even number \( a \), can we find consecutive primes such that \( p_{a+1} - p_a = a \) ?"

The answer is useful to find the solutions of the above kind of equations, but is also important to give the answer in order to solve another open problem:

"Can we get, as large as we want, but finite decreasing sequence \( k, k - 1, \ldots, 2, 1, 0 \) (odd \( k \)), included in the sequence of the values of \( c \) ?"

If someone gives an answer to this problem, then it is easy to give the answer (it will be the same) at the similar following problem:

"Can we get, as large as we want, but finite decreasing sequence \( k, k - 1, \ldots, 2, 1, 0 \) (even \( k \)), included in the sequence of the values of \( c \) ?"
We suppose the answer is negative.

In the same order of ideas, it is interesting to find \( \max_n \frac{c(n)}{n} \).

It is well known (see [4], page 147) that \( p_{n+1} - p_n < (\ln p_n)^2 \), where \( p_n \) and \( p_{n+1} \) are two consecutive primes.

Moreover, \( \frac{c(n)}{n} \) reaches its maximum value for \( n = p_k - 1 \), where \( p_k \) is a prime.

So, in this case:

\[
\frac{c(n)}{n} = \frac{p_{k+1} - p_k - 1}{p_k + 1} < \frac{(\ln p_k)^2 - 1}{p_k + 1} \xrightarrow{k \to \infty} 0
\]

Using this result, we can find the maximum value of \( \frac{c(n)}{n} \).

For \( p > 100 \) we have

\[
\frac{(\ln p)^2 - 1}{p + 1} < \frac{(\ln 100)^2 - 1}{101} < \frac{1}{4}
\]

Using the computer, by a straightforward computation, it is easy to prove that

\[
\max_{22 < n < 100} \frac{c(n)}{n} = \frac{3}{8}, \text{ which is reached for } n = 8.
\]

Because \( \frac{c(n)}{n} < \frac{1}{4} \) for every \( n > 100 \) it results that \( \max_{n \geq 2} \frac{c(n)}{n} = \frac{3}{8} \) reached for \( n = 8 \).

**Remark**

There exists an infinite number of finite sequences \( \{c(k_1), c(k_1 + 1), \ldots, c(k_2)\} \) such that \( \sum_{k=k_1}^{k_2} c(k) \) is a three-cornered number for \( k_1, k_2 \in \mathbb{N}^* \) (the \( n \)-th three-cornered number is \( T_n = \frac{n(n + 1)}{2}, n \in \mathbb{N}^* \)).

For example, in the case \( k_1 = p_k \) and \( k_2 = p_{k+1}, \) two consecutive primes, we have the finite sequence \( \{c(p_k), c(p_k + 1), \ldots, c(p_{k+1} - 1), c(p_{k+1})\} \) and

\[
\sum_{k=p_k}^{p_{k+1}} c(k) = 0 + (p_{k+1} - p_k - 1) + \ldots + 2 + 1 = 0 = \frac{(p_{k+1} - p_k - 1)(p_{k+1} - p_k)}{2} = T_{p_{k+1} - p_k - 1}
\]

Of course, we can define the function \( c' : \mathbb{N} \setminus \{0, 1\} \to \mathbb{N}, c'(n) = n - k \), where \( k \) is the smallest natural number such that \( n - k \) is a prime number, but we shall give some properties of this function in another paper.
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