ABOUT THE SMARANDACHE SQUARE’S COMPLEMENTARY FUNCTION

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DEFINITION 1. Let \( a: \mathbb{N}^* \rightarrow \mathbb{N}^* \) be a numerical function defined by \( a(n) = k \) where \( k \) is the smallest natural number such that \( nk \) is a perfect square: \( nk = s^2, s \in \mathbb{N}^* \), which is called the Smarandache square’s complementary function.

PROPERTY 1. For every \( n \in \mathbb{N}^* \), \( a(n^2) = 1 \) and for every prime natural number \( a(p) = p \).

PROPERTY 2. Let \( n \) be a composite natural number and \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r} \), \( 0 < p_1 < p_2 < \ldots < p_r \), \( \alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{N} \) it’s prime factorization. Then

\[
a(n) = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot \ldots \cdot p_r^{\beta_r}
\]

where

\[
\beta_j = \begin{cases} 
1 & \text{if } a_j \text{ is an odd natural number} \\
0 & \text{if } a_j \text{ is an even natural number}
\end{cases}
\]

If we take into account of the above definition of the function \( a \), it is easy to prove both the properties.

PROPERTY 3. \( \frac{1}{n} \leq \frac{a(n)}{n} \leq 1 \), for every \( n \in \mathbb{N}^* \) where \( a \) is the above defined function.

Proof. It is easy to see that \( 1 \leq a(n) \leq n \) for every \( n \in \mathbb{N}^* \), so the property holds.

CONSEQUENCE. \( \sum_{n=1}^{\infty} \frac{a(n)}{n} \) diverges.

PROPERTY 4. The function \( a: \mathbb{N}^* \rightarrow \mathbb{N}^* \) is multiplicative:

\( a(x \cdot y) = a(x) \cdot a(y) \) for every \( x, y \in \mathbb{N}^* \) with \( (x, y) = 1 \).

Proof. For \( x = 1 = y \) we have \( (x, y) = 1 \) and \( a(1 \cdot 1) = a(1) \cdot a(1) \). Let \( x = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r} \) and \( y = q_1^{\gamma_1} \cdot q_2^{\gamma_2} \cdot \ldots \cdot q_s^{\gamma_s} \) be the prime factorization of \( x \) and \( y \), respectively, and \( x \cdot y \neq 1 \). Because \( (x, y) = 1 \) we have \( p_h \neq q_k \) for every \( h = 1, r \) and \( k = 1, s \). Then,
\[ a(x) = p_i^{\beta_i} \cdot p_j^{\beta_j} \cdots p_r^{\beta_r} \quad \text{where} \quad \beta_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is odd} \\ 0 & \text{if } \alpha_i \text{ is even} \end{cases} , j = 1, r, \]

\[ a(y) = q_j^{\delta_j} \cdot q_k^{\delta_k} \cdots q_s^{\delta_s} \quad \text{where} \quad \delta_k = \begin{cases} 1 & \text{if } \gamma_{jk} \text{ is odd} \\ 0 & \text{if } \gamma_{jk} \text{ is even} \end{cases} , k = 1, s \]

\[ a(xy) = p_i^{\beta_i} \cdot p_j^{\beta_j} \cdots p_r^{\beta_r} \cdot q_j^{\delta_j} \cdot q_k^{\delta_k} \cdots q_s^{\delta_s} = a(x) \cdot a(y) \]

**Property 5.** If \((x, y) = 1, x \text{ and } y \text{ are not perfect squares and } x, y > 1\) the equation \(a(x) = a(y)\) has not natural solutions.

**Proof.** It is easy to see that \(x = y\). Let \(x = \prod_{h=1}^{r} p_h^{\alpha_h} \) and \(y = \prod_{k=1}^{s} q_k^{\beta_k}\), \((\text{where } p_h \neq q_k, \forall h = 1, r, k = 1, s \text{ be their prime factorization.})\)

Then \(a(x) = \prod_{h=1}^{r} p_h^{\alpha_h} \) and \(a(y) = \prod_{k=1}^{s} q_k^{\beta_k}\), \((\text{where } \alpha_h \text{ for } h = 1, r \text{ and } \beta_k \text{ for } k = 1, s \text{ have the above significance, but there exist at least } \alpha_h \neq 0 \text{ and } \beta_k = 0. \text{(because } x \text{ and } y \text{ are not perfect squares). } \text{Then } a(x) \neq a(y).)\)

**Remark.** If \(x = 1\) from the above equation it results \(a(y) = 1\), so \(y\) must be a a perfect square (analogously for \(y = 1\)).

**Consequence.** The equation \(a(x) = a(x + 1)\) has not natural solutions, because for \(x > 1\) \(x\) and \(x + 1\) are not both perfect squares and \((x, x + 1) = 1\).

**Property 6.** We have \(a(x \cdot y) = a(x), \text{ for every } x, y \in \mathbb{N}^\ast\).

**Proof.** If \((x, y) = 1\), then \((x, y^2) = 1\) and using property 4 and property 1 we have \(a(x \cdot y^2) = a(x) \cdot a(y^2) = a(x)\). If \((x, y) \neq 1\) we can write: \(x = \prod_{h=1}^{r} p_h^{\alpha_h} \cdot \prod_{t=1}^{n} d_t^{\nu_h} \) and \(y = \prod_{k=1}^{s} q_k^{\delta_k} \cdot \prod_{t=1}^{n} d_t^{\eta_k}\) where \(p_h \neq q_k, q_k \neq d_t, p_h = q_k, \forall h = 1, r, k = 1, s, t = 1, n\), but this implies \(\left(\prod_{h=1}^{r} p_h^{\alpha_h} \cdot \prod_{k=1}^{s} q_k^{\delta_k} \cdot \prod_{t=1}^{n} d_t^{\nu_h + \nu_h + \nu_h} \right) = 1\) and

\[ \left(\prod_{h=1}^{r} p_h^{\alpha_h} \cdot \prod_{k=1}^{s} q_k^{\delta_k} \right) = 1 \implies a(xy^2) = a \left(\prod_{h=1}^{r} p_h^{\alpha_h} \cdot \prod_{k=1}^{s} q_k^{\delta_k} \cdot \prod_{t=1}^{n} d_t^{\nu_h + \nu_h + \nu_h} \right) = a \left(\prod_{h=1}^{r} p_h^{\alpha_h} \right) \cdot a \left(\prod_{k=1}^{s} q_k^{\delta_k} \right) \cdot a \left(\prod_{t=1}^{n} d_t^{\nu_h + \nu_h + \nu_h} \right) \]
\[
\begin{aligned}
&= \begin{cases} 
1 & \text{if } \alpha \_i \text{ is odd} \\
0 & \text{if } \alpha \_i \text{ is even}
\end{cases}
\end{aligned}
\]

**Consequence 1.** For every \( x \in \mathbb{N} \) and \( n \in \mathbb{N} \), \( a(x^n) = \left\{ \begin{array}{ll} 1 & \text{if } n \text{ is even} \\ a(x) & \text{if } n \text{ is odd} \end{array} \right. \)

**Consequence 2.** If \( x = \frac{m^2}{n} \) where \( \frac{m}{n} \) is a simplified fraction, then \( a(x) = a(y) \). It is easy to prove this, because \( x = km^2 \) and \( y = kn^2 \) and using the above property we have:

\[
a(x) = a(km^2) = a(k) = a(kn^2) = a(y).
\]

**Property 7.** The sumatory numerical function of the function \( a \) is

\[
F(n) = \prod_{j=1}^{k} (H(\alpha_\_j)(p_{\_j} + 1) + \frac{1+(-1)^{\alpha_{\_j}}}{2})
\]

where the prime factorization of \( n \) is \( n = p_{\_n}^{a_{\_n}} \cdot p_{\_n}^{a_{\_n}} \cdots p_{\_n}^{a_{\_n}} \) and \( H(\alpha) \) is the number of the odd numbers which are smaller than \( \alpha \).

**Proof.** The sumatory numerical function of \( a \) is defined as \( F(n) = \sum_{d|n} a(d) \), because \((p_{\_n}^{a_{\_n}}, \prod_{j=1}^{k} p_{\_j}^{a_{\_j}}) = 1\) we can use the property 4 and we obtain:

\[
F(n) = \left( \sum_{d_i \mid p_{\_n}^{a_{\_n}}} a(d_i) \right) \left( \sum_{d_j \mid p_{\_j}^{a_{\_j}}} a(d_j) \right)
\]

and so on, making a finite number of steps we obtain

\[
F(n) = \prod_{j=1}^{k} F(p_{\_j}^{a_{\_j}}).
\]

But we observe that

\[
F(p^\alpha) = \begin{cases} 
\frac{\alpha}{2} (p+1) + 1 & \text{if } \alpha \text{ is an even number} \\
\left[ \frac{\alpha}{2} \right] + 1 & \text{if } \alpha \text{ is an odd number}
\end{cases}
\]

where \( p \) is a prime number.

If we take into account of the definition of \( H(\alpha) \) we find

\[
H(\alpha) = \begin{cases} 
\frac{\alpha}{2} & \text{if } \alpha \text{ is even} \\
\left[ \frac{\alpha}{2} \right] + 1 & \text{if } \alpha \text{ is odd}
\end{cases}
\]

so we can write \( F(p^\alpha) = H(\alpha) \cdot (p+1) + \frac{1+(-1)^{\alpha}}{2} \),

therefore:

\[
F(n) = \prod_{j=1}^{k} (H(\alpha_\_j)(p_{\_j} + 1) + \frac{1+(-1)^{\alpha_{\_j}}}{2}).
\]

In the sequel we study some equations which involve the function \( a \).
1) Find the solutions of the equation: \( xa(x) = m \), where \( x, m \in \mathbb{N}^* \).

If \( m \) is not a perfect square then the above equation has not solutions.

If \( m \) is a perfect square, \( m = z^2 \), \( z \in \mathbb{N}^* \), then we have to give the solutions of the equation \( xa(x) = z^2 \).

Let \( z = p_{i_1}^{a_{i_1}} \cdot p_{i_2}^{a_{i_2}} \cdots p_{i_k}^{a_{i_k}} \) be the prime factorization of \( z \). Then \( xa(x) = p_{i_1}^{2a_{i_1}} \cdot p_{i_2}^{2a_{i_2}} \cdots p_{i_k}^{2a_{i_k}} \), so taking account of the definition of the function \( a \), the equation has the following solutions:

\[
\begin{align*}
&x_1^{(0)} = p_{i_1}^{a_{i_1}} \cdot p_{i_2}^{a_{i_2}} \cdots p_{i_k}^{a_{i_k}} \quad \text{(because \( a(x_1^{(0)}) = 1 \))}, \\
&x_2^{(1)} = p_{i_1}^{2a_{i_1}} \cdot p_{i_2}^{2a_{i_2}} \cdots p_{i_k}^{2a_{i_k}} \quad \text{(because \( a(x_2^{(1)}) = p_{i_1} \)),} \\
&x_3^{(2)} = p_{i_1}^{4a_{i_1}} \cdot p_{i_2}^{4a_{i_2}} \cdots p_{i_k}^{4a_{i_k}} \\
&\vdots \\
&x_t^{(t)} = p_{i_1}^{2^{t-1}a_{i_1}} \cdot p_{i_2}^{2^{t-1}a_{i_2}} \cdots p_{i_k}^{2^{t-1}a_{i_k}} \quad \text{(because \( a(x_t^{(t)}) = p_{i_1} \)),} \\
&j_1 \neq j_2, j_1, j_2 \in \{i_1, i_2, \ldots, i_k\}, t = 1, 2, \ldots, C_k^2, \\
&x_t^{(t)} = \frac{z^2}{p_{i_1} \cdot p_{i_2} \cdots p_{i_k}} \quad \text{where \( j_1, j_2, j_3 \in \{i_1, \ldots, i_k\} \)}.
\end{align*}
\]

So the above equation has \( 1 + C_k^2 + C_k^4 + \cdots = 2^k \) different solutions where \( k \) is the number of the prime divisors of \( m \).

2) Find the solutions of the equation: \( xa(x) + ya(y) = za(z) \), \( x, y, z \in \mathbb{N}^* \).

\textbf{Proof.} We note \( xa(x) = m^2 \), \( ya(y) = n^2 \) and \( za(z) = s^2 \), \( x, y, z \in \mathbb{N}^* \) and the equation \( m^2 + n^2 = s^2 \) has the following solutions: \( m = u^2 - v^2 \), \( n = 2uv \), \( s = u^2 + v^2 \), \( u > v > 0 \), \( (u,v)=1 \) and \( u \) and \( v \) have different evenes.

If \((m,n,s)\) as above is a solution, then \((am,an,as), \alpha \in \mathbb{N}^*\) is also a solution of the equation (*)

If \((m,n,s)\) is a solution of the equation (*), then the problem is to find the solutions of the equation \( xa(x) = m^2 \) and we see from the above problem that there are \( 2^k \) solutions (where \( k_1 \) is the number of the prime divisors of \( m \), then the solutions of the equations \( ya(y) = n^2 \) and respectively \( za(z) = s^2 \), so the number of the different solutions of the given equations, is \( 2^{k_2} \cdot 2^{k_3} = 2^{k_2 + k_3} \) (where \( k_2 \) and \( k_3 \) have the same signifience as \( k_1 \), but concerning \( n \) and \( s \), respectively).

For \( \alpha > 1 \) we have \( xa(x) = \alpha^2 m^2 \), \( ya(y) = \alpha^2 n^2 \), \( za(z) = \alpha^2 s^2 \) and, using an analogue way as above, we find \( 2^{k_1-2} \cdot 2^{k_2} \cdot 2^{k_3} = 2^{k_1 + k_2 + k_3} \) different solutions, where \( k_1, k_2, k_3 \) are the number of the prime divisors of \( \alpha m, \alpha n \) and \( \alpha s \), respectively.

\textbf{Remark.} In the particular case \( u=2, v=1 \) we find the solution \((3,4,5)\) for (*) So we must find the solutions of the equations \( xa(x) = 3^2 \alpha^2 \), \( ya(y) = 2^4 \alpha^2 \) and \( za(z) = 5^4 \alpha^2 \), for \( \alpha \in \mathbb{N}^* \). Suppose that \( \alpha \) has not 2, 3 and 5 as prime factors in this prime factorization \( \alpha = p_{i_1}^{a_{i_1}} \cdot p_{i_2}^{a_{i_2}} \cdots p_{i_k}^{a_{i_k}} \). Then we have:
So any triplet \((x_0, y_0, z_0)\) with \(x_0, y_0\) and \(z_0\) arbitrary of above corresponding values, is a solution for the equation (for example \((9, 16, 25)\), is a solution).

**Definition.** The triplets which are the solutions of the equation 

\[ x_0(x) + y_0(y) = z_0(z), \quad x, y, z \in \mathbb{Z}. \]

we call MIV numbers.

3) Find the natural numbers \(x\) such that \(a(x)\) is a three-cornered, a squared and a pentagonal number.

**Proof.** Because 1 is the only number which is at the same time a three-cornered, a squared and a pentagonal number, then we must find the solutions of the equation \(a(x)=1\), therefore \(x\) is any perfect square.

4) Find the solutions of the equation:

\[ \frac{1}{xa(x)} + \frac{1}{ya(y)} = \frac{1}{za(z)}, \quad x, y, z \in \mathbb{N}. \]

**Proof.** We have \(xa(x) = m^2, ya(y) = n^2, za(z) = s^2\), \(m, n, s \in \mathbb{N}\).

The equation \(\frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{s^2}\) has the solutions:

\[ m = t(u^2 + v^2)2uv, \]

\[ n = t(u^2 + v^2)(u^2 - v^2), \]

\[ s = t(u^2 - v^2)2uv, \]

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u>v, (u, v)=1, u and v have different eveness and t ∈ N*, so we have
\[ xa(x) = t^2(u^2 + v^2)^2 4u^2v^2 \]
\[ ya(y) = t^2(u^2 + v^2)^2(u^2 - v^2)^2 \]
\[ za(z) = t^2(u^2 - v^2)^2 4u^2v^2 \] and we find x, y and z in the same way which is indicated in the first problem.

For example, if u=2, v=1, t=1 we have
\[ m=20, n=15, s=12, \] so we must find the solutions of the following equations:
\[ xa(x) = 20^2 = 2^4 \cdot 5^2 \Rightarrow x \in \{2^3 \cdot 5^2 = 400, 2^3 \cdot 7 = 80, 2^3 \cdot 5 = 40, 2^4 \cdot 5^2 = 400\} \]
\[ ya(y) = 15^2 = 3^2 \cdot 5^2 \Rightarrow y \in \{15, 45, 75, 225\} \]
\[ za(z) = 12^2 = 2^2 \cdot 3^2 \Rightarrow z \in \{24, 48, 72, 144\} \]
Therefore for this particular values of u, v and t we find 4 \cdot 4 \cdot 4 = 64 solutions. (because \( k_1 = k_2 = k_3 = 2 \))

5) Find the solutions of the equation: \( a(x) + a(y) + a(z) = a(x)a(y)a(z), x, y, z \in N^* \).

Proof. If \( a(x) = m, a(y) = n \) and \( a(z) = s \), the equation \( m + n + s = m \cdot n \cdot s \), \( m, n, s \in N^* \) has a solutions the permutations of the set \{1, 2, 3\} so we have:
\[ a(x) = 1 \Rightarrow x \text{ must be a perfect square, therefore } x = u^2, u \in N^* \]
\[ a(y) = 2 \Rightarrow y = 2v^2, v \in N^* \]
\[ a(z) = 3 \Rightarrow z = 3t^2, t \in N^* \]
Therefore the solutions are the permutation of the sets \{u^2, 2v^2, 3t^2\} where \( u, v, t \in N^* \).

6) Find the solutions of the equation \( Aa(x) + Ba(y) + Ca(z) = 0, A, B, C \in Z^* \).

Proof. If we note \( a(x) = u, a(y) = v, a(z) = t \) we must find the solutions of the equation \( Au + Bv + Ct = 0 \).
Using the method of determinants we have:
\[
\begin{vmatrix}
A & B & C \\
A & B & C \\
m & n & s \\
\end{vmatrix} = 0, \quad \forall m, n, s \in Z \Rightarrow A(Bs - Cn) + B(Cm - As) + C(An - Bm) = 0, \quad \text{and it is known that the only solutions are}\]
\[
\begin{align*}
u &= Bs - Cn \\
v &= Cm - As \\
t &= An - Bm, \quad \forall m, n, s \in Z \\
\end{align*}
\]
so, we have \( a(x) = Bs - Cn \)
\( a(y) = Cm - As \)
\( a(z) = An - Bm \) and now we know to find x, y and z.

Example. If we have the following equation: \( 2a(x) - 3a(y) - a(z) = 0 \), using the above result we must find (with the above mentioned method) the solutions of the equations:
\[ a(x) = -3s + n \]
\[ a(y) = -m - 2s \]
\[ a(z) = 2n + 3m, \quad m, n \text{ and } s \in \mathbb{Z}. \]

For \( m = -1, \ n = 2, \ s = 0 \), the solution in this case is \((2\alpha^2, \beta^3, \gamma^1)\), \(\alpha, \beta, \gamma \in \mathbb{Z}^+\). For the other values of \( m, n, s \) we find the corresponding solutions.

7) The same problem for the equation \( Aa(x) + Ba(y) = C, \quad A, B, C \in \mathbb{Z} \).

Proof. \( Aa(x) + Ba(y) - C = 0 \Rightarrow Aa(x) + Ba(y) + (-C)a(z) = 0 \) with \( a(z) = 1 \) so we must have \( An - Bm = 1 \). If \( n_0 \) and \( m_0 \) are solutions of this equation \((An_0 - Bm_0 = 1)\) it remains us to find the solutions of the following equations:
\[ a(x) = Bs + Cn_0 \]
\[ a(y) = -Cm_0 + As, \quad s \in \mathbb{Z} \], but we know how to find them.

Example. If we have the equation \( 2a(x) - 3a(y) = 5 \), \( x, y \in \mathbb{N}^+ \) using the above results, we get: \( A = 2, \ B = -3, \ C = -5 \) and \( a(z) = 1 = 2n + 3m \). The solutions are \( m = 2k + 1 \) and \( n = -1 - 3k, \ k \in \mathbb{Z} \). For the particular value \( k = 1 \) we have \( m_0 = -1 \) and \( n_0 = 2 \) so we find \( a(x) = -3 + 5 \cdot 2 = 10 - 3s \) and \( a(y) = 5(-1) - 2s = 5 - 2s \).

If \( s_0 = 0 \) we find \( a(x) = 10 \Rightarrow x = 10u^2, \ u \in \mathbb{Z}^+ \)
\[ a(y) = 5 \Rightarrow y = 5v^2, \ v \in \mathbb{Z}^+ \] and so on.

8) Find the solutions of the equation: \( a(x) = ka(y) \quad k \in \mathbb{N}^+, \ k > 1 \).

Proof. If \( k \) has in his prime factorization a factor which has an exponent \( \geq 2 \), then the problem has not solutions.

If \( k = p_1 \cdot p_2 \cdots p_r \) and the prime factorization of \( a(y) \) is \( a(y) = q_1 \cdot q_2 \cdots q_s \), then we have solutions only in the case \( p_i, p_1, \ldots, p_r \in \{ q_1, q_2, \ldots, q_s \} \).

This implies that \( a(x) = p_1 \cdot p_2 \cdots p_r \cdot q_1 \cdot q_2 \cdots q_m \cdot \alpha^2 \)
\[ y = q_1 \cdot q_2 \cdots q_m \cdot \beta^2, \quad \alpha, \beta \in \mathbb{Z}^+. \]

9) Find the solutions of the equation \( a(x) = x \) (the fixed points of the function \( a \)).

Proof. Obviously, \( a(1) = 1 \). Let \( x > 1 \) and let \( x = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \), \( a_j \geq 1, \) for \( j = 1, r \) be the prime factorization of \( x \). Then \( a(x) = p_1^{b_1} \cdot p_2^{b_2} \cdots p_r^{b_r} \) and \( b_j \leq 1 \) for \( j = 1, r \). Because \( a(x) = x \) this implies that \( \alpha_j = \beta_j = 1, \) \( \forall j = 1, r \), therefore \( x = p_1 \cdot p_2 \cdots p_r \), where \( p_j, \ j = 1, r \) are prime numbers.

REFERENCES

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