An inequality between prime powers dividing \( n! \)

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For any positive integer \( n \geq 1 \) and for any prime number \( p \) let \( e_p(n) \) be the exponent at which the prime \( p \) appears in the prime factor decomposition of \( n! \). In this note we prove the following:

**Theorem.**

Let \( p < q \) be two prime numbers, and let \( n > 1 \) be a positive integer such that \( pq \mid n \). Then,

\[ p^{e_p(n)} > q^{e_q(n)}. \quad (1) \]

Inequality (1) was suggested by Balacenoii at the First International Conference on Smarandache Notions in Number Theory (see [1]). In fact, in [1], Balacenoii showed that (1) holds for \( p = 2 \). In what follows we assume that \( p \geq 3 \).

We begin with the following lemmas:

**Lemma 1.**

(i) The function

\[ f(x) = \frac{x - 1}{\log x} \quad (2) \]

is increasing for \( x \geq e \).

(ii) Let \( p \geq 3 \) be a real number. Then,

\[ x > (p - 1)\log_p(x) \quad \text{for} \ x \geq p. \quad (3) \]

(iii) Let \( p \geq 3 \) be a real number. The function

\[ g_p(x) = \frac{x - 2}{x - (p - 1)\log_p(x)} \quad (4) \]

is positive and decreasing for \( x \geq p(p + 2) \).

(iv) \[ \frac{p + 2}{p} > \frac{\log(p + 4)}{\log p} \quad \text{for} \ p > e^2. \quad (5) \]

(v) \[ \frac{p + 1}{p} > \frac{\log(p + 2)}{\log p} \quad \text{for} \ p > e. \quad (6) \]

**Proof.** (i) Notice that

\[ \frac{df}{dx} = \frac{1}{\log^2 x} \cdot \left( \log \left( \frac{x}{e} \right) + \left( \frac{1}{x} \right) \right) > 0 \quad \text{for} \ x > e. \]

(ii) Suppose that \( x \geq p \geq 3 \). From (i) it follows that

\[ \frac{x}{\log x} > \frac{z - 1}{\log z} > \frac{p - 1}{\log p}. \quad (7) \]

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Inequality (7) is clearly equivalent to
\[ x > (p-1) \frac{\log x}{\log p} = (p-1) \log_p (x). \]

(iii) The fact that \( g_p(x) > 0 \) for \( x \geq p \geq 3 \) follows from (ii). Suppose that \( x \geq p(p+2) \), and that \( p \geq 3 \). Then,
\[
\frac{dg_p}{dx} = \frac{-\log(p)((p-1)x \log x - (2 \log p + p - 1)x + 2(p-1))}{x((p-1) \log x - x \log p)^2}.
\]  

From (8), it follows that in order to check that \( \frac{dg_p}{dx} < 0 \) it suffices to show that
\[
(p-1)x \log x - (2 \log p + p - 1)x \geq 0,
\]

or that
\[
\log x > \frac{2 \log p}{p-1} + 1 = \left( \frac{2}{f(p)} + 1 \right).
\]  

The left hand side of (9) is increasing in \( x \). By (i), the right hand side of (9) is decreasing in \( p \). Thus, since \( p \geq 3 \), and \( x \geq p(p+2) \geq 15 \), it suffices to show that inequality (9) holds for \( x = 15 \) and \( p = 3 \). But this is straightforward.

(iv) Inequality (5) is equivalent to
\[
p^{p+2} > (p+4)^p,
\]

or
\[
p^2 > \left(1 + \frac{4}{p}\right)^4 = \left(\frac{1 + \frac{4}{p}}{p-1}\right)^4.
\]  

Since
\[
e > (1+x)^{1/2} \text{ for all } x > 0,
\]

it follows, from inequality (11) with \( x = 4/p \), that
\[
e > \left(1 + \frac{4}{p}\right)^{p/4}.
\]  

From inequality (12) one can immediately see that (10) holds whenever \( p > e^2 \).

(v) Follows from arguments similar to the ones used at (iv).

For every prime number \( p \) and every positive integer \( n \) let \( \tau_p(n) \) be the sum of the digits of \( n \) written in the base \( p \).

**Lemma 2.**

Let \( p < q \) be two prime numbers and let \( n \) be a positive integer. Assume that \( pq \mid n \). Then,

(i) \( \tau_q(n) \geq 2 \).

(ii) \( \tau_p(n) < (p-1) \log_p (n) \).

**Proof.** (i) Since \( n > 0 \) it follows that \( \tau_q(n) \geq 1 \). If \( \tau_q(n) = 1 \), it follows that \( n \) is a power of \( q \) which contradicts the fact that \( p \mid n \). Hence, \( \tau_q(n) \geq 2 \).

(ii) Let \( n = pq \) for some integer \( l \geq 1 \). Let
\[
ql = a_0 + a_1 p + \ldots + a_s p^s,
\]
where $0 \leq a_i \leq p - 1$ for $1 \leq i \leq s$, and $a_s \neq 0$, be the representation of $qf$ in the base $p$. Clearly, 

$$s = \lfloor \log_p(qf) \rfloor < \log_p(qf).$$

Since 

$$n = pqf = a_0p + a_1p^2 + \ldots + a_sp^{s+1},$$

it follows that 

$$\tau_p(n) = \sum_{i=0}^{s} a_i \leq (p-1)(s+1) < (p-1)(\log_p(qf) + 1) = (p-1)\log_p(n).$$

The Proof of the Theorem. Suppose that $q > p \geq 3$ are prime numbers, and that $n > 1$ is such that $pq | n$. By applying logarithms in (1) it suffices to prove that 

$$e_p(n)\log p > e_q(n)\log q.$$ 

Since 

$$e_p(n) = \frac{n - \tau_p(n)}{p-1} \quad \text{and} \quad e_q(n) = \frac{q - \tau_q(n)}{q-1},$$

it follows that (13) can be rewritten as 

$$\frac{n - \tau_p(n)}{p-1} > \frac{n - \tau_q(n)}{q-1}.$$ 

or 

$$\frac{(q - 1)\log p}{(p - 1)\log q} > \frac{n - \tau_q(n)}{n - \tau_p(n)}.$$ 

We distinguish two cases:

CASE 1. $q = p + 2$. We distinguish two subcases:

CASE 1.1. $n = pq$. In this case, since $q = p + 2$, and $p \geq 3$, it follows that 

$$\tau_p(n) = \tau_p(p^2 + 2p) = 3, \quad \text{and} \quad \tau_q(n) = \tau_q(pq) = p.$$ 

Therefore inequality (14) becomes 

$$\frac{(p + 1)\log p}{p(p - 1)} > \frac{p^2 + 2p - p}{p^2 + 2p - 3} = \frac{p(p + 1)}{p^2 + 2p - 3}.$$ 

Inequality (15) is equivalent to 

$$\frac{p^2 + 2p - 3}{p(p - 1)} \geq \frac{p + 1}{p}.$$ 

By lemma 1 (v) we conclude that in order to prove inequality (16) it suffices to show that 

$$\frac{p^2 + 2p - 3}{p(p - 1)} \geq \frac{p + 1}{p}.$$ 

But (17) is equivalent to 

$$\frac{p^2 + 2p - 3}{p - 1} \geq p + 1,$$ 

or $p^2 + 2p - 3 \geq p^2 - 1$, or $p \geq 1$ which is certainly true. This disposes of Case 1.1.
CASE 1.2. $n = pqI$ where $l \geq 2$. In this case $n \geq 2p(p + 2) > 2p^2$. By lemma 2 (i) and (ii), it follows that

$$\frac{n - 2}{n - (p - 1) \log_p(n)} > \frac{n - \tau_q(n)}{n - \tau_p(n)}.$$  \hfill (19)

Thus, in order to prove (14) it suffices to show that

$$\frac{(p + 1) \log p}{(p - 1) \log(p + 2)} \geq \frac{n - 2}{n - (p - 1) \log_p(n)} = g_p(n).$$  \hfill (20)

Since $n > 2p^2 > p(p + 2)$, and since $g_p(n)$ is decreasing for $n > p(p + 2)$ (thanks to lemma 1 (iii)), it follows that in order to prove (20) it suffices to show that

$$\frac{(p + 1) \log p}{(p - 1) \log(p + 2)} > g_p(2p^2) = \frac{2p^2 - 2}{2p^2 - \log_p(2p^2)}.$$  \hfill (21)

Since $p \geq 3 > 2^{3/2}$, it follows that $p^{3/3} > 2$. Hence,

$$\log_p(2p^2) < \log_p(p^{2/3}) = \frac{8}{3}.$$

We conclude that in order to prove (21) it suffices to show that

$$\frac{(p + 1) \log p}{(p - 1) \log(p + 2)} > \frac{2p^2 - 2}{2p^2 - \frac{8}{3}} = \frac{3(p - 1)(p + 1)}{3p^2 - 4}.$$  \hfill (22)

Inequality (22) is equivalent to

$$\frac{3p^2 - 4}{3(p - 1)^2} > \frac{\log(p + 2)}{\log p}.$$  \hfill (23)

Using inequality (6), it follows that in order to prove (23) it suffices to show that

$$\frac{3p^2 - 4}{3(p - 1)^2} > \frac{p + 1}{p}.$$  \hfill (24)

Notice now that (24) is equivalent to

$$3p^3 - 4p > 3(p - 1)^2(p + 1) = 3p^3 - 3p^2 - 3p + 3,$$

or $3p^2 > p + 3$ which is certainly true for $p \geq 3$. This disposes of Case 1.2.

CASE 2. $q \geq p + 4$. Using inequality (19) it follows that in order to prove inequality (14) it suffices to show that

$$f(q) \cdot \frac{\log p}{p - 1} = \frac{(q - 1) \log q}{(p - 1) \log p} > \frac{n - 2}{n - (p - 1) \log_p(n)} = g_p(n).$$  \hfill (25)

Since $f(q)$ is increasing for $q \geq 3$ (thanks to lemma 1 (i)), and since $g_p(n)$ is decreasing for $n \geq pq \geq p(p + 4) > p(p + 2)$, it follows that in order to prove (25)
it suffices to show that inequality (25) holds for \( q = p + 4 \), and \( n = pq = p(p + 4) \).

Hence, we have to show that

\[
\frac{(p + 3) \log p}{(p - 1) \log(p + 4)} > \frac{p^2 + 4p - 2}{p^2 + 4p - (p - 1) \log p(p(p + 4))}.
\]  

Inequality (26) is equivalent to

\[
\frac{(p + 3)}{(p - 1) \log(p + 4)} > \frac{p^2 + 4p - 2}{(p^2 + 3p + 1) \log p - (p - 1) \log(p + 4)},
\]

or

\[
\frac{(p + 3)(p^2 + 3p + 1)}{(p - 1)(p^2 + 4p - 2) + (p - 1)(p + 3)} > \frac{\log(p + 4)}{\log p},
\]

or

\[
\frac{p^3 + 6p^2 + 10p + 3}{p^2 + 4p^2 - 4p - 1} > \frac{\log(p + 4)}{\log p}.
\]  

One can easily check that (27) is true for \( p = 3, 5, 7 \). Suppose now that \( p \geq 11 > e^2 \).

By lemma 1 (iv), it follows that in order to prove (27) it suffices to show that

\[
p^3 + 6p^2 + 10p + 3 > p^2 + 4p^2 - 4p - 1\log p.
\]

Notice that (28) is equivalent to

\[
p^4 + 6p^3 + 10p^2 + 3p > (p + 2)(p^2 + 4p^2 - 4p - 1) = p^4 + 6p^3 + 4p^2 - 9p - 2,
\]

or \( 6p^2 + 11p + 2 > 0 \), which is obvious. This disposes of the last case.

Reference


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123