CONVERGENCE OF THE SMARANDACHE GENERAL CONTINUED FRACTION

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ABSTRACT. We give a positive answer to the Smarandache General Continued Fraction convergence (see [2]).

The Smarandache General Continued Fraction associated with the Smarandache reverse sequence
1, 21, 321, 4321, 54321, ..., 121110987654321, ..., is given by

$$\frac{1}{1 + \frac{1}{12 + \frac{1}{21 + \frac{1}{123 + \frac{1}{321 + \frac{1}{1234 + \frac{1}{4321 + \frac{1}{12345 + \ldots}}}}}}}}$$

(See for more details [2]).

Define the sequences \( \{a_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \) by:

- \( a_0 = 1, a_1 = 12, a_2 = 123, ... \)
- \( b_0 = 1, b_1 = 21, b_2 = 321, ... \)

We verify easily that

$$a_{n+1} = 10a_n + (n + 1) \quad \text{and} \quad b_{n+1} = 10b_n + \frac{10^{(n+1)} - 1}{9}, \text{ for any } n \geq 0.$$

With notations of [1], the continued fraction (1) can be written as follows:

$$a_0 + \frac{b_0}{|a_1|} + \frac{b_1}{|a_2|} + \frac{b_2}{|a_3|} + \ldots + \frac{b_n}{|a_n|} + \ldots$$

Let \( \frac{A_k}{B_k} \) be the result of the \( k^{th} \) reduce of continued fraction:

$$a_0 + \frac{b_0}{|a_1|} + \frac{b_1}{|a_2|} + \frac{b_2}{|a_3|} + \ldots + \frac{b_k-1}{|a_k|}\)$$

Thus we define two sequences \( \{A_n\}_{n \geq 0} \) and \( \{B_n\}_{n \geq 0} \) of real numbers. Using the elementary algebraic theory of continued fraction given by Euler (see [1]) we have the following,

Lemma 0.1. The sequences \( \{A_n\}_{n \geq 0} \) and \( \{B_n\}_{n \geq 0} \) satisfy the following statements:

- \( A_n = a_nA_{n-1} + b_nA_{n-2}, \text{ for } n \geq 2, \ A_{-1} = 0 \text{ and } A_0 = a_1. \)
- \( B_n = a_nB_{n-1} + b_nB_{n-2}, \text{ for } n \geq 2, \ B_{-1} = 0 \text{ and } B_0 = 1. \)

In consequence, we have:

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}b_0b_1\ldots b_{n-1}, \text{ for any } n \geq 1$$

And if \( B_n \neq 0, \text{ for any } n \geq 0, \text{ we have,}$$
An easy computation gives,
\[
\frac{A_n}{B_n} = \frac{A_{n-1}}{B_{n-1}} = (-1)^{(n-1)} b_1 b_2 \ldots b_{n-1}, \quad \text{for any } n \geq 1
\]

Hence, we have the following result.

**Lemma 0.2.** The Smarandache General Continued Fraction (1) is convergent if and only if the alternate series \( \sum_{k=1}^{n} (-1)^{(k-1)} \frac{b_1 \ldots b_{k-1}}{B_{k+1} B_k} \) is also convergent.

Let \( \{u_n\}_{n \geq 0} \) be the sequence of positive real numbers defined by
\[
u_n := \frac{b_0 b_1 \ldots b_{n-1}}{B_{n-1} B_n}
\]
for \( n \geq 1 \).

We have,
\[
u_{n+1} - \nu_n = \frac{b_0 b_1 \ldots b_n}{B_{n-1} B_n} - \frac{b_0 b_1 \ldots b_{n-1}}{B_{n-1} B_n} - \frac{1}{B_{n+1}} = \frac{b_0 b_1 \ldots b_{n-1} \cdot b_n - 1}{B_{n-1} B_n}.
\]

And using the lemma 0.1, we get
\[
u_{n+1} - \nu_n = \frac{b_0 b_1 \ldots b_n - 1}{B_{n-1} B_n} \left( \frac{b_n - b_{n+1} (B_{n-1} - a_{n+1} B_n)}{B_{n-1} B_{n+1}} \right)
\]

And by (2) we have
\[
u_{n+1} - \nu_n = \frac{b_0 b_1 \ldots b_n - 1}{B_{n-1} B_n} \left( \frac{b_n - b_{n+1} (B_{n-1} - a_{n+1} B_n)}{B_{n-1} B_{n+1}} \right)
\]

Because the \( B_n \)'s are positive, we deduce that the sequence \( \{u_n\}_{n \geq 0} \) is decreasing. On the other hand, we have, \( B_n B_{n-1} = (a_n B_{n-2} + b_n B_{n-1}) \), which implies that
\[
u_n \leq \frac{b_0 b_1 \ldots b_n - 1}{b_1 b_2 \ldots b_n B_0} = \frac{1}{b_n B_1} - \frac{1}{b_0}
\]

The last inequality assert that \( \lim_{n \to \infty} u_n = 0 \). Finally, we have the result

**Theorem 0.1.** The Smarandache General Continued Fraction (1) is convergent.

**References**
