DECOMPOSITION OF THE DIVISORS OF A NATURAL NUMBER INTO PAIRWISE CO-PRIME SETS

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Given \( n \) a natural number. Let \( d_1, d_2, d_3, \ldots \) be the divisors of \( N \). A query comes to my mind, as to, in how many ways, we could choose a divisor pair which are co-prime to each other? Similarly, in how many ways one could choose a triplet, or a set of four divisors etc. such that, in each chosen set, the divisors are pairwise co-prime?

We start with an example. Let \( N = 48 = 2^4 \times 3 \). The ten divisors are
\[ 1, 2, 3, 4, 6, 8, 12, 16, 24, 48 \]

We denote the set of co-prime pairs by \( D_2(48) \), co-prime triplets by \( D_3(48) \) etc.

We get \( D_2(48) = \{ (1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (1,16), (1,24), (1,48), (2,3), (4,3), (8,3), (16,3) \} \)

Order of \( D_2(48) = 13 \).

\( D_3(48) = \{ (1,2,3), (1,3,4), (1,3,8), (1,3,16) \} \), Order of \( D_3(48) = 4 \).

\( D_4(48) = \{ \} = D_5(48) = \ldots = D_9(48) = D_{10}(48) \).

Another example \( N = 30 = 2 \times 3 \times 5 \) (a square free number). The 8 divisors are
\[ 1, 2, 3, 5, 6, 10, 15, 30 \]

\( D_2(30) = \{ (1,2), (1,3), (1,5), (1,6), (1,10), (1,15), (1,30), (2,3), (2,5), (2,15), (3,5), (3,10), (5,6) \} \).

Order of \( D_2(30) = 13 \). = \( O[D_2( p_1p_2p_3)] \) \hspace{1cm} (A)

\( D_3(30) = \{ (1,2,3), (1,2,5), (1,3,5), (2,3,5), (1,3,10), (1,5,6), (1,2,15) \} \)

Order of \( D_3(30) = 7 \).

\( D_4(30) = \{ (1,2,3,5) \} \), Order of \( D_4(30) = 1 \).

OPEN PROBLEM: To determine the order of \( D_r(N) \).

In this note, we consider the simple case of \( n \) being a square-free number for \( r = 2, 3 \) etc.

(A) \( r = 2 \)
We rather derive a reduction formula for $r = 2$. And finally a direct formula.

Let $N = p_1 p_2 \ldots p_n$ where $p_k$ is a prime for $k = 1$ to $n$.

We denote $D_2(N) = D_2(1 \# n)$ for convinience. We shall derive a reductio formula for $D_2(1 \# (n+1))$.

Let $q$ be a prime such that $(q, N) = 1$, (HCF = 1).

Then $D_2(Nq) = D_2(1 \# (n+1))$

1. We have by definition $D_2(1 \# n) \subset D_2(1 \# (n+1))$

   This provides us with $O[D_2(1 \# n)]$ elements of $D_2(1 \# (n+1))$.

   (2) Consider an arbitrarily chosen element $(d_k, d_s)$ of $D_2(1 \# n)$. This element when
combined with $q$ yields exactly two elements of $D_2(1 \# (n+1))$, i.e. $(qd_k, d_s)$ and $(d_k, qd_s)$.

   Hence the set $D_2(1 \# n)$ contributes twice the order of itself.

2. The element $(1, q)$ has not been considered in the above mentioned cases hence the the
   total number of elements of $D_2(1 \# (n+1))$ are $3 \times O[D_2(1 \# n)] + 1$.

   $O[D_2(1 \# (n+1))] = 3 \times O[D_2(1 \# n)] + 1. \quad (B)$

Applying Reduction Formula (B) for evaluating $O[D_2(1 \# 4)]$

From (A) we have $O[D_2(p_1 p_2 p_3)] = O[D_2(1 \# 3)] = 13$ hence

   $O[D_2(1 \# 4)] = 3 \times 13 + 1 = 40$.

This can be verified by considering $N = 2 \times 3 \times 5 \times 7 = 210$. The divisors are

$1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210,$

$D_2(210) = \{(1, 2), (1, 3), (1, 5), (1, 6), (1, 7), (1, 10), (1, 14), (1, 15), (1, 21), (1, 30),$

$ (1, 35), (1, 42), (1, 70), (1, 105), (1, 210), (2, 3), (2, 5), (2, 7), (2, 15), (2, 21),$

$(2, 35), (2, 105), (3, 5), (3, 7), (3, 10), (3, 14), (3, 35), (3, 70), (5, 6), (5, 7),$

$(5, 14), (5, 21), (5, 42), (7, 6), (7, 10), (7, 15), (7, 30), (6, 35), (10, 21), (14, 15) \}$

$O[D_2(210)] = 40.$

The reduction formula (B) can be reduced to a direct formula by applying simple induction and we get

$O[D_2(1 \# n)] = (3^n - 1) / 2 \quad (C)$

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For $r = 3$ we derive a reduction formula.

(1) We have $D_3 (1#n) \subseteq D_3 (1#(n+1))$ hence this contributes $O[D_3 (1#n)]$ elements to $D_3 (1#(n+1))$.

(2) Let us choose an arbitrary element of $D_3 (1#n)$ say $(a, b, c)$. The additional prime $q$ yields $(qa, b, c)$, $(a, qb, c)$, $(a, b, qc)$ i.e. three elements. In this way we get $3 \times O[D_3 (1#n)]$ elements.

3. Let the product of the $n$ primes $= N$. Let $(d_1, d_2, d_3, \ldots, d_{\varphi(N)})$ be all the divisors of $N$. Consider $D_2 (1#n)$ which contains $d(N) - 1$ elements in which one member is unity $= d_1$, i.e., $(1, d_2), (1, d_3), \ldots, (1, d_{\varphi(N)})$.

If $q$ is placed as the third element with these as the third element we get $d(N) - 1$ elements of $D_3 (1#(n+1))$. The remaining elements of $D_2 (1#n)$ yield elements repetitive elements already covered under (2).

Considering the exhaustive contributions from all the three above we get

$$O[D_3 (1#(n+1))] = 4 \times O[D_3 (1#n)] + d(N) - 1$$

$$O[D_3 (1#(n+1))] = 4 \times O[D_3 (1#n)] + 2^r - 1 \quad \text{(D)}$$

$$O[D_3 (210)] = 4 \times O[D_3 (30)] - 1$$

$$O[D_3 (210)] = 4 \times 7 + 8 - 1 = 35$$

To verify the elements are listed below.

$$D_3 (210) = \{ (1, 2, 3), (1, 2, 5), (1, 3, 5), (1, 2, 7), (1, 3, 7), (1, 5, 7), (1, 2, 15), (1, 2, 35), (1, 2, 21), (1, 2, 105), (1, 3, 10), (1, 3, 14), (1, 3, 35), (1, 3, 70), (1, 5, 6), (1, 5, 14), (1, 5, 21),$$

$$ (1, 5, 42), (1, 7, 6), (1, 7, 10), (1, 7, 15), (1, 7, 30), (2, 3, 5), (2, 3, 7), (2, 5, 7), (2, 3, 35), (2, 5, 21),$$

$$ (2, 7, 15), (3, 5, 7), (3, 5, 14), (3, 7, 10), (5, 7, 6), (1, 6, 35), (1, 10, 21), (1, 14, 15) \}$$

Open Problem: To obtain a direct formula from the reduction formula (D).

Regarding the general case i.e. $O[D_r (1#n)]$ we derive an inequality.

Let $(d_1, d_2, d_3, \ldots, d_r)$ be an element of $O[D_r (1#n)]$. 

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Introducing a new prime q other than the prime factors of N we see that this element in conjunction with q gives r elements of $D_r(1\#(n+1))$ i.e. $(qd_1, d_2, d_3, \ldots d_r), (d_1, qd_2, d_3, \ldots d_r), \ldots$

$(d_1, d_2, d_3, \ldots qd_r)$. Also $D_r(1\#n) \subset D_r(1\#(n+1))$. Hence we get

$O[D_r(1\#(n+1))] > (r+1)$. $O[D_r(1\#n)]$

To find an accurate formula is a tough task ahead for the readers.

Considering the general case is a further challenging job.