EXPLORING SOME NEW IDEAS ON SMARANDACHE TYPE SETS, FUNCTIONS AND SEQUENCES

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ABSTRACT: In this article I have defined a number of SMARANDACHE type sets, sequences which I found very interesting. The problems and conjectures proposed would give food for thought and would pave ways for more work in this field.

(1) SMARANDACHE PATTERNED PERFECT SQUARE SEQUENCES.
Consider following sequence of numbers
13, 133, 1333, 13333, ... --------(1)

The sequence formed by the square of the numbers is
169, 17689, 1776889, 177768889, ... ----------(2)

We define (1) as the root sequence.
It is evident that the above sequence (2) follows a pattern.
i.e. The square of one followed by n three's is, one followed by (n-1) seven's, followed by a six, followed by (n-1) eight's followed by a nine.

There are a finite number of such patterned perfect square sequences. Here we list the root sequences.

(I) 13, 133, 1333, 13333, ... 
(2) 16, 166, 1666, 16666, ... 
(3) 19, 199, 1999, 19999, ... 
(4) 23, 233, 2333, 23333, ... 
(5) 26, 266, 2666, 26666, ... 
(6) 29, 299, 2999, 29999, ...

on similar lines we have the root sequences with the first terms as
(7) 33, (8) 36, (9) 39, (10) 43, (11) 46, (12) 49, (13) 53, (14) 66, 
(15) 73, (16) 93, (17) 96, (18) 99.

There are some root sequences which start with a three digit number, like
799, 7999, 79999, ...
The patterned perfect square sequence is
638401, 63984001, 6399840001, 639998400001, ...

( the nine's and zero's inserted are shown in darker print to identify the pattern.)
Open Problem: (1) Are there any patterned perfect cube sequences? 
(2) Are there any patterned higher perfect power sequences?

(2) SMARANDACHE BREAKUP SQUARE SEQUENCES

4, 9, 284, 61209, . . .

the terms are such that we have
4 = 2^2
49 = 7^2
49284 = 22^2
4928461209 = 70203^2

T_n = the smallest number whose digits when placed adjacent to other terms of the sequence in the following manner

\[ T_1 T_2 \ldots T_{n-1} T_n \] yields a perfect square.

\[ Lt \left( \frac{T_1 T_2 \ldots T_{n-1} T_n}{n} \right)^{1/2} \] where \( k \) is the number of digits in the numerator for this kind of sequence can be analyzed. As it is evident that for large values of \( n \) the value of \( \left( \frac{T_1 T_2 \ldots T_{n-1} T_n}{10^k} \right)^{1/2} \) is close to either 2.22... or to 7.0203...

(3) SMARANDACHE BREAKUP CUBE SEQUENCES

On similar lines SMARANDACHE BREAKUP CUBE SEQUENCES can be defined. The same idea can be extended to define SMARANDACHE BREAKUP PERFECT POWER SEQUENCES

(4) SMARANDACHE BREAKUP INCREMENTED PERFECT POWER SEQUENCES

1, 6, 6375,

1 = 1^1 , 16 = 4^2 , 166375 = 55^3 , etc.

T_n = the smallest number whose digits when placed adjacent to other terms of the sequence in the following manner

\[ T_1 T_2 \ldots T_{n-1} T_n \] yields a perfect \( n^{th} \) power.
(5) SMARANDACHE BREAKUP PRIME SEQUENCE
2, 3, 3, ...
2, 23, 233 etc. are primes.
\( T_1T_2...T_{n-1}T_n \) is a prime

(6) SMARANDACHE SYMMETRIC PERFECT SQUARE SEQUENCE
1, 4, 9, 121, 484, 14641, ...

(7) SMARANDACHE SYMMETRIC PERFECT CUBE SEQUENCE
1, 8, 343, 1331 ...

This can be extended to define

(8) SMARANDACHE SYMMETRIC PERFECT POWER SEQUENCE

(9) SMARANDACHE DIVISIBLE BY \( n \) SEQUENCE
1, 2, 3, 2, 5, 2, 5, 6, 1, 0, 8, 4 ...

the terms are the smallest numbers such that \( n \) divides \( T_1T_2...T_{n-1}T_n \) the terms placed adjacent digit wise.

e.g. 1 divides 1, 2 divides 12, 3 divides 123, 4 divides 1232, 5 divides 12325, 6 divides 123252, 7 divides 1232535, 8 divides 12325256, 9 divides 123252561, 10 divides 1232525610, 11 divides 12325256108, 12 divides 123252561084, etc.

(9) SMARANDACHE SEQUENCE OF NUMBERS WITH SUM OF THE DIGIT'S = PRIME
2, 3, 5, 7, 11, 12, 14, 16, 20, 21, 23, 25, 29, ...

(10) SMARANDACHE SEQUENCE OF PRIMES WITH SUM OF THE DIGIT'S = PRIME
2, 3, 5, 7, 11, 23, 29, 41, 43, 47, 61, 67, 83, 89, ...

(11) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT \( 2p + 1 \) IS ALSO A PRIME
2, 3, 5, 11, 23, 29, 41, 53, ...
(11) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT $2P - 1$ IS ALSO A PRIME

3, 7, 19, 31, . . .

(13) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT $P^2 + 2$ IS ALSO A PRIME

3, 17, . . .

(14) SMARANDACHE SEQUENCE OF SMALLEST PRIME WHICH DIFFER BY $2n$ FROM ITS PREDECESSOR

5, 17, 29, 97, . . .

( $T_1 = 5 = 3 + 2$, $T_2 = 17 = 13 + 4$, $T_3 = 29 = 23 + 6$, $T_4 = 97 = 89 + 8$ etc.)

(15) SMARANDACHE SEQUENCE OF SMALLEST PRIME $p$ FOR WHICH $p + 2r$ IS A PRIME

3, 13, 23, 89, . . .

$3 + 2 \times 1 = 5$ is a prime, $13 + 2 \times 2 = 17$ is a prime, $23 + 2 \times 3 = 29$, $89 + 2 \times 4 = 97$ is a prime etc.

(16) SMARANDACHE SEQUENCE OF THE SMALLEST NUMBER WHOSE SUM OF DIGITS IS $n$.

1, 2, 3, 4, 5, 6, 7, 8, 9, 19, 29, 39, 49, 59, 69, 79, 89, 99, 199, 299, 399, 499, 599, 699, . . .

It is a sequence of the only numbers which have the following property.

$$N + 1 = \prod_{r=1}^{k} (a_r + 1)$$

PROOF:

Let $N$ be a $k$-digit number with $a_r$ the $r^{th}$ digit ($a_1 = \text{LSB}$) such that

$$N + 1 = \prod_{r=1}^{k} (a_r + 1) \quad \text{------------------------(1)}$$

to find all such $k$-digit numbers.
The largest \( k \)-digit number is \( N = 10^k - 1 \), with all the digits as 9. It can be verified that this is a solution. Are there other solutions?

Let the \( m^{th} \) digit be changed from 9 to \( a_m \) (\( a_m < 9 \)). Then the right member of (1) becomes \( 10^{(k-1)} (a_m + 1) \). This amounts to the reduction in value by \( 10^{(k-1)} (9-a_m) \). The value of the \( k \)-digit number \( N \) goes down by \( 10^{(m-1)}(9-a_m) \). For the new number to be a solution these two values have to be equal which occurs only at \( m = k \). This gives 8 more solutions. In all there are 9 solutions given by \( a_1 \) \( 10^k - 1 \), for \( a = 1 \) to 9.

\[ 199, 299, 399, 499, 599, 699, 799, 899, 999. \]

Are there infinitely many primes in this sequence.

(17) SMARANDACHE SEQUENCE OF NUMBERS SUCH THAT THE SUM OF THE DIGITS DIVIDES \( n \)

\[ 1,3,6,9,10,12,18,20,21,24,27,30,36,40,42,45,48,50,54,60,63,72,80,81,84,90,100,102,108,110,112,114,120,126,132,133,135,140,144,150, \ldots \]

(18) SMARANDACHE SEQUENCE OF NUMBERS SUCH THAT EACH DIGIT DIVIDES \( n \)

\[ 1,2,3,4,5,6,7,8,9,10,11,12,15,20,22,24,30,33,36,40,44,50,55,60,66, \ldots \]

(19) SMARANDACHE POWER STACK SEQUENCE FOR \( n \)

\[ \text{SPSS}(2) \]

1, 12, 124, 1248, 124816, 12481632, \ldots

The \( n^{th} \) term is obtained by placing the digits of the powers of 2 starting from \( 2^0 \) to \( 2^n \) from left to right.

\[ \text{SPSS}(3) \]

1, 13, 139, 13927, 1392781, 1392781243, \ldots

Problem: If \( n \) is an odd number not divisible by 5 how many of the above sequence \( \text{SPSS}(n) \) are prime? (It is evident that \( n \) divides \( T_n \) iff \( n \equiv 0 \mod (5) \)).

(20) SMARANDACHE SELF POWER STACK SEQUENCE

\[ \text{SSPSS} \]
1, 14, 1427, 1427256, 14272563125, 142725631257776, ...

\[ T_r = T_{r-1}a_1a_2a_3...a_k \text{ where } r^f = a_1a_2a_3...a_k \]
( the digits are placed adjacent).

How many terms of the above sequence, SSPSS are prime?

(21) SMARANDACHE PERFECT SQUARE COUNT PARTITION SEQUENCE

the \( r \)th term of \( \text{SPSCPS}(n) \) is defined as

\[ T_r = \{ x \mid x \text{ is a perfect square, } nr + 1 \leq x \leq nr + n \} \]

\( O \) stands for the order of the set

e.g. for \( n = 12 \) \( \text{SPSCPS}(12) \) is

3, 1, 2, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1

( number of perfect squares \( \leq 12 \) is 3 (1, 4, and 9), number of perfect squares between 13 to 24 is 1 (only 16) etc.)

(21) SMARANDACHE PERFECT POWER COUNT PARTITION SEQUENCE

The \( r^{th} \) term of \( \text{SPPCPS}(n,k) \) is defined as

\[ T_r = \{ x \mid x \text{ is a } k^{th} \text{ perfect power, } nr + 1 \leq x \leq nr + n \} \]

where \( O \) stands for the order of the set

By this definition we get

\( \text{SPSCPS}(12) = \text{SPPCPS}(12,2) \)

Another example, \( \text{SPPCPS}(100,3) \) is

4,1,1,1,0,1,0,1,0,0,0,1,0,0,1,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,1, ...

Problem: Does \( \sum (T_r/(nr)) \) converge as \( n \to \infty \)?

(22) SMARANDACHE BERTRAND PRIME SEQUENCE

According to Bertrand’s postulate there exists a prime between \( n \) and \( 2n \). Starting from 2 let us form a sequence by taking the largest prime less than double of the previous prime in the sequence. We get
2, 3, 5, 7, 13, 23, 43, 83, 163, . . .

(23) SMARANDACHE SEMI- PERFECT NUMBER SEQUENCE

6, 12, 18, 20, 24, 30, 36, 40, . . .

A semi perfect number is defined as one which can be expressed as the sum of its (all or fewer) distinct divisors.

e.g. \(12 = 2 + 4 + 6 = 1 + 2 + 3 + 6\)

\(20 = 1 + 4 + 5 + 10\)

\(30 = 2 + 3 + 10 + 15 = 1 + 3 + 5 + 6 + 15\) etc.

It is evident that every perfect number is also a semi perfect number.

THEOREM: There are infinitely many semi perfect numbers.

Proof: We shall prove that \(N = 2^n p\) where \(p\) is a prime less than \(2^{n+1} - 1\), is a semi-perfect number.

The divisors of \(N\) are

row 1------ \(1, 2, 2^2, 2^3, 2^4, \ldots 2^n\)

row 2------ \(p, 2p, 2^2p, 2^3p, 2^4p, \ldots 2^np\)

we have \(\sum_{r=0}^{n-1} 2^r p = p (1 + 2 + 2^2 + 2^3 + \ldots 2^{n-1}) = p(2^n - 1) = M\)

\(M\) is short of \(N\) by \(p\). The task ahead is to express \(p\) as the sum of divisors from the first row. It is an established fact that every number can be expressed as the sum of powers of 2, i.e.

\(p = \sum_{r=0}^{n} a_r 2^r\), where \(a_r = 0\) or \(a_r = 1\). iff \(p \leq 2^{n+1} - 1\), the equality giving a perfect number.

(note: \(a_1a_2a_3\ldots a_n\) is the binary representation of \(p\)).

\(N = M + p\) is expressible as the sum of its divisors.

Remark: This of-course is not exhaustive. There are many more such examples possible giving infinitely many semi perfect numbers. One can explore the possibility of more such expressions.
SMARANDACHE CO-PRIME BUT NO PRIME SEQUENCE

4, 9, 10, 21, 22, 25, 26, 27, 28, 33, 34, 35, 36, 49, 50, 51, 52, ...

The $n^{th}$ term $T_n$ is defined as follows

$$T_n = \{ x \mid (T_{n-1}, x) = 1, \text{x is not a prime and } (T_{n-1}, y) \neq 1 \text{ for } T_{n-1} < y < x \}$$

The smallest number which is not a prime but is relatively prime to the previous term in the sequence.

Open problem: Is it possible to as large as we want but finite increasing sequence $k, k+1, k+2, k+3, \ldots$ included in the above sequence?

DEFINITION: We define a prime to be week, strong or balanced prime accordingly as $p_r < = or > (p_{r-1} + p_{r+1}) / 2$. where $p_r$ is the $r^{th}$ prime.

e.g. $3 < (2+5)/2$ 3 is week prime. $5= (3 + 7)/2$ is a balanced prime. $71 > (67 + 73)/2$ is a strong prime.

SMARANDACHE WEEK PRIME SEQUENCE:

3, 7, 13, 19, 23, 29, 31, 37, ...

SMARANDACHE STRONG PRIME SEQUENCE:

11, 17, 41, ...

SMARANDACHE BALANCED PRIME SEQUENCE:

5, 157, 173, 257, 263, 373, ...

It is evident that for a balanced prime $> 5$, $p_r = p_{r-1} + 6k$.

OPEN PROBLEM: Are there infinitely many terms in the SMARANDACHE BALANCED PRIME SEQUENCE?
How big is $N$? One of the first estimates of its size was approximately [6]:

$$10^{6846168}$$

But this is a rather large number; to test all odd numbers up to this limit would take more time and computer power than we have. Recent work has improved the estimate of $N$. In 1989 J.R. Chen and T. Wang computed $N$ to be approximately [7]:

$$10^{43000}$$

This new value for $N$ is much smaller than the previous one, and suggests that some day soon we will be able to test all odd numbers up to this limit to see if they can be written as the sum of three primes.

Anyway assuming the truth of the generalized Riemann hypothesis [5], the number $N$ has been reduced to $10^{20}$ by Zinoviev [9], Saouter [10] and Deshouillers. Effinger, te Riele and Zinoviev[11] have now successfully reduced $N$ to 5.

Therefore the weak Goldbach conjecture is true, subject to the truth of the generalized Riemann hypothesis.

Let's now analyse the generalizations of Goldbach conjectures reported in [3] and [4]; six different conjectures for odd numbers and four conjectures for even numbers have been formulated. We will consider only the conjectures 1, 4 and 5 for the odd numbers and the conjectures 1, 2 and 3 for the even ones.

4.1 First Smarandache Goldbach conjecture on even numbers.

_Every even integer $n$ can be written as the difference of two odd primes, that is $n = p - q$ with $p$ and $q$ two primes._

This conjecture is equivalent to:

_For each even integer $n$, we can find a prime $q$ such that the sum of $n$ and $q$ is itself a prime $p$. _

A program in Ubasic language to check this conjecture has been written.
(2) SMARANDACHE DIVISOR SEQUENCES:

Define $A_n = \{ x | d(x) = n \}$

Then

- $A_1 = \{ 1 \}$
- $A_2 = \{ p | p \text{ is a prime} \}$
- $A_3 = \{ x | x = p^2, p \text{ is a prime} \}$
- $A_4 = \{ x | x = p^3 \text{ or } x = p_1 p_2, p, p_1, p_2 \text{ are primes} \}$

$A_4 \rightarrow 6, 8, 10, 14, 15, 21, 22, 26, 27, \ldots$

We have

$$\sum 1/T_n = 1 \text{ for } A_1$$

This limit does not exist for $A_2$

$$\lim_{n \rightarrow \infty} \sum 1/T_n \text{ exists and is less than } \pi^2/6 \text{ for } A_3 \text{ as } \lim_{n \rightarrow \infty} \sum 1/n^2 = \pi^2/6.$$  

The above limit does exist for $A_p$ where $p$ is a prime.

* Whether these limits exist for $A_4, A_5$ etc is to be explored.

DIVISOR SUB SEQUENCES

The sub sequences for $A_4, A_5$ etc can be defined as follows:

$B(r_1, r_2, r_3, \ldots r_k)$ is the sequence of numbers $\pi_1^{r_1} \pi_2^{r_2} \pi_3^{r_3} \ldots \pi_k^{r_k}$ in increasing order, where $p_1, p_2, p_3, \ldots p_k$ are primes. All the numbers having the same unique factorization structure.

DIVISOR MULTIPLE SEQUENCE

$SDMS = \{ n | n = k \cdot d(n) \}$

$SDMS \rightarrow 1, 2, 8, 9, 12, \ldots$

(3) SMARANDACHE QUAD PRIME SEQUENCE GENERATOR:

$SQPSG = \{ r | 90r+11, 90r+13, 90r+17, 90r+19 \text{ are all primes} \}$
Are there infinitely many terms in the above sequence?

(4) SMARANDACHE PRIME LOCATION SEQUENCES

Define
\[ P_0 = \text{sequence of primes}. \]
\[ P_1 = \text{sequence of primeth primes} \]
\[ P_1 \to 3, 5, 11, 17, \ldots \]
\[ P_2 = \text{sequence of primeth, primeth prime} \]
\[ P_r = \text{sequence of primeth, primeth, … } r \text{ times primes} \]

* If \( T_n \) is the \( n \)th term of \( P_r \), then what is the minimum value of \( r \) for which

\[ \lim_{n \to \infty} \sum 1/T_n \text{ exists?} \]

(5) SMARANDACHE PARTITION SEQUENCES

(i) PRIME PARTITION

Number of partitions into prime parts
\[ S_{pp}(n) \to 0, 1, 1, 1, 2, 2, 3, \ldots \]

(ii) COMPOSITE PARTITION

Number of partitions into composite parts
\[ S_{pc}(n) \to 1, 1, 2, 1, 3, \ldots \]

(iii) DIVISOR PARTITIONS

Number of partitions into parts which are the divisors of \( n \).
\[ S_{pd}(n) \to 1, 1, 2, 1, \]
On similar lines following two partition sequences can be defined.

(iv) CO-PRIME PARTITIONS : $SP_{cp}(n)$
Number of partitions into co-prime parts.

(v) NON-CO-PRIME PARTITIONS $SP_{ncp}(n)$
Number of partitions into non coprime parts.

(vi) PRIME SQUARE PARTITIONS
Partitions into prime square parts.

This idea could be generalized to define more such functions.

(6) SMARANDACHE COMBINATORIAL SEQUENCES.

(I) Let the first two terms of a sequence be 1 & 2. The $(n+1)^{th}$
term is defined as

$$T_{n+1} = \text{sum of all the products of the previous terms of the sequence taking two at a time}.$$  

$T_1 = 1$, $T_2 = 2$, $\Rightarrow T_3 = 2$, and $T_4 = 8$,

$SCS(2) = 1, 2, 2, 8, 48, \ldots$

The above definition can be generalized as follows:

Let $T_k = k$ for $k = 1$ to $n$.

$T_{n+1} = \text{sum of all the products of the previous terms of the sequence taking } r \text{ at a time. This defines } SCS(r)$.

Another generalization could be:

Let $T_k = k$ for $k = 1$ to $n$.

$T_r = \text{sum of all products of (r-1) terms of the sequence taking (r-2) at a time (r > n). This defines } SC_pS.$

for $n = 2$ $T_1 = 1$, $T_2 = 2$, $T_3 = 3$, $T_4 = 17$ etc

$SC_pS \rightarrow 1, 2, 3, 17, \ldots$
PROBLEM: (1) How many of the consecutive terms of SCS(r) are pairwise coprime?

(2) How many of the terms of SCvS are primes?

(ii) SMARANDACHE PRIME PRODUCT SEQUENCES

SPPS(n)

\[ T_n = \text{sum of all the products of primes chosen from first } n \text{ primes taking } (n-1) \text{ primes at a time.} \]

SPPS(n) → 1, 5, 31, 247, 2927...

\[ T_1 = 1, \quad T_2 = 2 + 3, \quad T_3 = 2 \times 3 + 2 \times 5 + 3 \times 5 = 31. \]

\[ T_4 = 2 \times 3 \times 5 + 2 \times 3 \times 7 + 2 \times 5 \times 7 + 3 \times 5 \times 7 = 247 \text{ etc.} \]

How many of these are primes?

(7) SMARANDACHE \( \phi \)-SEQUENCE

(S\( \phi \)S) = \( \{ n \ | \ n = k \phi(n) \} \)

S\( \phi \)S → 1, 2, 4, 6, 8, 12...

(8) SMARANDACHE PRIME DIVISIBILITY SEQUENCE

SPDS = \( \{ n \ | \ n \text{ divides } p_n + 1, \ p_n \text{ is the } n^{\text{th}} \text{ prime.} \} \)

SPDS → 1, 2, 3, 4, 10, ...