Abstract: In this article, a formula is given to obtain the next prime in an arithmetic progression.

Theorem: We consider the arithmetic progression \( a + di \ i \geq 0 \) of positive integers with \( GCD(a, d) = 1 \) and considering that the final term is \( a + dM \) is to say \( 0 \leq i \leq M \).

Let \( p \) a term in the arithmetic progression (it doesn’t have to be prime).

Then the next prime in the arithmetic progression is:

\[
\text{nxt}(a, d)(p) = p + d + d \cdot \sum_{k=1}^{M} \prod_{j=1}^{k} \frac{2 - \sum_{s=1}^{\lfloor \frac{a + jd}{s} \rfloor} \left( \frac{a + jd - 1}{s} \right)}{a + jd}
\]

and the improved formula:

\[
\text{nxt}(a, d)(p) = p + d + d \cdot \sum_{k=1}^{M} \prod_{j=1}^{k} \left[ - \left( 2 + 2 \sum_{s=1}^{\lfloor \frac{a + jd - 1}{s} \rfloor} \left( \frac{a + jd}{s} - \frac{a + jd - 1}{s} \right) \right) \left( a + jd \right) \right]
\]

Where \( \lfloor x \rfloor \) is the floor function. And where \( x / y \) is the integer division in the improved formula.

Proof:

By a past article [1] we have that the next prime function is:

\[
\text{nxt}(p) = p + 1 + \sum_{k=p+1}^{2p} \prod_{i=1}^{k} \left[ - \left( \frac{2 - \sum_{s=1}^{\lfloor \frac{i - 1}{s} \rfloor} \left( \frac{i - 1}{s} \right)}{i} \right) \right]
\]

Where the expression of the product is the Smarandache Prime Function:

\[
G(i) = \begin{cases} 1 & \text{if } i \text{ is composite} \\ 0 & \text{if } i \text{ is prime} \end{cases}
\]
We consider that \( a + j_0d \) is the next prime of a number \( p \) in an arithmetic progression \( a + jd \). We have that \( G(a + j_0d) = 0 \).

And for all \( j \) such that \( p < a + jd < a + j_0d \) we have that \( G(a + jd) = 1 \).

It is deduced that:

\[
\prod_{j=1\,(p-a)/d+1}^{k} G(a + jd) = \prod_{j=1\,(p-a)/d}^{k-1} G(a + jd) \cdot \prod_{j=k+1}^{k} G(a + jd) = \begin{cases} 0 & k > j_0 - 1 \\ 1 & k \leq j_0 - 1 \end{cases}
\]

since the first product has the value of 1, and the second product is zero since it has the factor \( G(a + j_0d) = 0 \).

As a result in the formula \( \text{nxt}(a, d) \) the non zero terms are summed until \( j_0 - 1 \) and has the value of 1.

\[
\text{nxt}(a, d)(p) = p + d + d \cdot \sum_{k=(p-a)/d+1}^{k-1} = p + d + d \cdot (j_0 - 1 + 1 - 1 - (p-a)/d) =
\]

\[
= p + d + j_0d - d - p + a = a + j_0d
\]

And the result is proven.

The improved formula [2] is obtained by considering that the sum, in the Smarandache prime function, until the integer part of the square root and multiplied by 2 the result. Also the floor function is changed \( \lfloor x \rfloor \) for the integer division operator \( x/y \) that it faster for the computation.

Let us see an example made in MATHEMATICA:

\[
\text{a}=5
\]

\[
\text{dd}=4
\]

\[
\text{M}=20
\]

\[
\text{p}=5
\]

\[
\text{DD}[i_] := \text{Sum}[\text{Quotient}[(a+i*dd),j]-\text{Quotient}[a+i*dd-1,j], \{j,1,\text{Sqrt}[a+i*dd]\}]
\]

\[
\text{G}[i_] := \text{Quotient}[(2-2*\text{DD}[i]),(a+i*dd)]
\]

\[
\text{F}[m_] := \text{Product}[\text{G}[i],\{i, (p-a)/dd+1,m\}]
\]

\[
\text{S}[n_] := \text{Sum}[\text{F}[m],\{m, (p-a)/dd+1,M\}]
\]

\[
\text{While}[p < a + (M-1)*dd+1, \text{Print}["\text{nxt}(" , p, ")=", p+dd+dd*\text{S}[p]]; \]

\[
p = p+dd+dd*\text{S}[p]
\]

\[
\text{nxt}(5) = 13
\]

\[
\text{nxt}(13) = 17
\]

\[
\text{nxt}(17) = 29
\]

\[
\text{nxt}(29) = 37
\]

\[
\text{nxt}(37) = 41
\]

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nxt(41) = 53  
nxt(53) = 61  
nxt(61) = 73  
nxt(73) = 89 

The question is that if these formulas can be applied to prove the Dirichlet’s Theorem [3] for arithmetic progressions.

That is to say: does any arithmetic progression $a + jd$ such that $GCD(a, d) = 1$ have infinite primes?

REFERENCES:

http://personal.telefonica.terra.es/web/smruiz/ 

www.primepuzzles.net 