Near Pseudo Smarandache Function

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Abstract.

The Pseudo Smarandache Functions $Z(n)$ are defined by David Gorski [1].
This new paper defines a new function $K(n)$ where $n \in \mathbb{N}$, which is a slight modification of $Z(n)$ by adding a smallest natural number $k$. Hence this function is “Near Pseudo Smarandache Function (NPSF)”. Some properties of $K(n)$ are presented here, separately, according to as $n$ is even or odd. A continued fraction consisting NPSF is shown to be convergent [3]. Finally some properties of $K^{'}(n)$ are also obtained.

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1.1 Definition

Near Pseudo Smarandache Function (NPSF) $K$ is defined as follows.

$K : \mathbb{N} \to \mathbb{N}$ defined by $K(n) = m$, where $m = \Sigma n + k$ and $k$ is the smallest natural number such that $n$ divides $m$.

1.2 Following are values of $K(n)$ for $n \leq 15$

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<table>
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<tr>
<th>$n$</th>
<th>$\Sigma n$</th>
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<th>$K(n)$</th>
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For more such values see appendix A

2.1 Properties

(i) $k = n$ if $n$ is odd and $n/2$ if $n$ is even.

(a) Let $n$ be odd.

Then $(n + 1)$ is even and hence $(n + 1)/2$ is an integer.

$\therefore \Sigma n = n(n + 1)/2$, being multiple of $n$, is divisible by $n$.

Hence $n$ divides $\Sigma n + k$ iff $n$ divides $k$ i.e. $k$ is a multiple of $n$. However, as $k$ is smallest $k = n$.

(b) Let $n$ be even.

Then $\Sigma n + k = n(n + 1)/2 + k = n^2/2 + n/2 + k$

As $n$ is even hence $n/2$ is an integer and $n^2/2$ is divisible by $n$.

Hence $n$ divides $\Sigma n + k$ iff $n$ divides $n/2 + k$

i.e. iff $n \leq n/2 + k$ or $k \geq n/2$.

However, as $k$ is smallest $k = n/2$. 
(ii) \[ K(n) = \frac{n(n+3)}{2} \] if \( n \) is odd and \[ K(n) = \frac{n(n+2)}{2} \] if \( n \) is even.

\[ K(n) = \sum n + k = \frac{n(n+1)}{2} + k \]

If \( n \) is odd then \( k = n \) and hence \( K(n) = \frac{n(n+3)}{2} \)
If \( n \) is even then \( k = \frac{n}{2} \) and hence \( K(n) = \frac{n(n+2)}{2} \).

(iii) For all \( n \in \mathbb{N} \); \( \frac{n(n+2)}{2} \leq K(n) \leq \frac{n(n+3)}{2} \)

We know \( K(n) \) is either \( \frac{n(n+2)}{2} \) or \( \frac{n(n+3)}{2} \)
depending upon whether \( n \) is even or odd.
Hence for all \( n \in \mathbb{N} \); \( \frac{n(n+2)}{2} \leq K(n) \leq \frac{n(n+3)}{2} \)

(iv) For all \( n \in \mathbb{N} \); \( K(n) > n \).

As \( K(n) \geq \frac{n(n+2)}{2} = n + \frac{n^2}{2} > n \)
Hence \( K(n) > n \) for all \( n \in \mathbb{N} \).

(v) \( K \) is strictly monotonic increasing function of \( n \).

Let \( m < n \implies m + 1 \leq n \) i.e. \( m + (3 - 2) \leq n \)
Or \( m + 3 \leq n + 2 \). So \( m < n \) and \( m + 3 \leq n + 2 \)
\[ \implies m(m+3) < n(n+2) \]
Or \( m(m+3)/2 < n(n+2)/2 \)
\[ \implies K(m) < K(n) \]
Hence \( K(n) \) is strictly monotonic increasing function of \( n \).

(vi) \( K(m+n) \neq K(m) + K(n) \)
and \( K(m,n) \neq K(m) \cdot K(n) \)

We know \( K(2) = 4, K(3) = 9, K(5) = 20, \) & \( K(6) = 24 \)
So \( K(2) + K(3) = 4 + 9 = 13 \) & \( K(2+3) = K(5) = 20 \)
Hence \( K(2+3) \neq K(2) + K(3) \)
Also \( K(2).K(3) = 4.9 = 36 \) & \( K(2.3) = K(6) = 24 \)
Hence \( K(2.3) \neq K(2).K(3) \)
2.2 (i) $K(2n + 1) - K(2n) = 3n + 2$

$K(2n + 1) = (2n + 1)(2n + 4)/2 = 2n^2 + 5n + 2$

$K(2n) = 2n(2n + 2)/2 = 2n^2 + 2n$

Hence $K(2n + 1) - K(2n) = 3n + 2$

(ii) $K(2n) - K(2m) = 2(n - m)(n + m + 1)$

$K(2n) = 2n(2n + 2)/2 = 2n^2 + 2n$

$\therefore K(2n) - K(2m) = 2(n^2 - m^2) + 2(n - m)$

Hence $K(2n) - K(2m) = 2(n - m)(n + m + 1)$

(iii) $K(2n + 1) - K(2n - 1) = 4n + 3$

$K(2n + 1) = (2n + 1)(2n + 4)/2 = 2n^2 + 5n + 2$

$K(2n - 1) = (2n - 1)(2n + 2)/2 = 2n^2 + n - 1$

Hence $K(2n + 1) - K(2n - 1) = 4n + 3$

(iv) $K(n) - K(m) = \frac{n - m}{n + m} K(n + m)$ where $m, n$ are even and $n > m$.

$K(n) - K(m) = \frac{n}{2} (n + 2) - \frac{m}{2} (m + 2)$

$= \frac{1}{2} (n^2 + 2n - m^2 - 2m)$

$= \frac{1}{2} \{(n^2 - m^2) + 2(n - m)\}$

$= \left(\frac{n - m}{2}\right) (n + m + 2)$

$= (n - m) \frac{1}{n + m} \frac{n + m}{2} (n + m + 2)$

$= \frac{n - m}{n + m} K(n + m)$
(v) Let $K(n) = m$ and

(a) Let $n$ be even then $n \cdot m$ is a perfect square iff $(n + 2)/2$ is a perfect square.

(b) Let $n$ be odd then $n \cdot m$ is a perfect square iff $(n + 3)/2$ is a perfect square.

(c) $n \cdot m$ is a perfect cube iff $n = 2$ or 3.

(a) If $n$ is even then $K(n) = m = n(n + 2)/2$

$\therefore n \cdot m = n^2(n + 2)/2$ Hence if $n$ is even then $n \cdot m$ is a perfect square iff $(n + 2)/2$ is a perfect square.

(b) If $n$ is odd then $K(n) = m = n(n + 3)/2$

$\therefore n \cdot m = n^2(n + 3)/2$ Hence if $n$ is odd then $n \cdot m$ is a perfect square iff $(n + 3)/2$ is a perfect square.

(c) Let $n$ be even and let $n = 2p$

Then $m = K(n) = K(2p) = 2p/2(2p + 2)$

$\therefore n \cdot m = (2p) \cdot p \cdot 2(p + 1) = (2p) \cdot (2p) \cdot (p + 1)$

$\therefore n \cdot m$ is a perfect cube iff $p + 1 = 2p$

i.e. iff $p = 1$ i.e. iff $n = 2$

Let $n$ be odd and let $n = 2p - 1$

Then $m = K(n) = K(2p - 1) = (2p - 1)(2p - 1 + 3)/2$

$= (2p - 1)(p + 1)$

$\therefore n \cdot m = (2p - 1) \cdot (2p - 1) \cdot (p + 1)$

$\therefore n \cdot m$ is a perfect cube iff $p + 1 = 2p - 1$

i.e. iff $p = 2$ i.e. iff $n = 3$

$\therefore n = 2$ and $n = 3$ are the only two cases where $n \cdot m$ is a perfect cube.

Verification :- $K(2) = 4 \& 2 \cdot 4 = 8 = 2^3$

$K(3) = 9 \& 3 \cdot 9 = 27 = 3^3$
2.3 Ratios

(i) \[ \frac{K(n)}{K(n+1)} = \frac{n}{n+1} \]
if \( n \) is odd.

As \( n \) is odd \( \because n + 1 \) is even. Hence \( K(n) = \frac{n(n + 3)}{2} \)

and \( K(n + 1) = \frac{(n + 1)(n + 1 + 2)}{2} = \frac{(n + 1)(n + 3)}{2} \)

Hence \( \frac{K(n)}{K(n+1)} = \frac{n}{n+1} \) if \( n \) is odd.

(ii) \[ \frac{K(n)}{K(n+1)} = \frac{n(n+2)}{(n+1)(n+4)} \]
if \( n \) is even.

As \( n \) is even \( \because n + 1 \) is odd. Also \( K(n) = \frac{n(n + 2)}{2} \) and

\[ K(n + 1) = \frac{(n + 1)(n + 1 + 3)}{2} = \frac{(n + 1)(n + 4)}{2} \]

Hence \( \frac{K(n)}{K(n+1)} = \frac{n(n+2)}{(n+1)(n+4)} \) if \( n \) is even.

(iii) \[ \frac{K(2n)}{K(2n+2)} = \frac{n}{n+2} \]

\[ K(2n) = 2n(2n + 2)/2 = 2n(n + 1) \]

\[ K(2n + 2) = (2n + 2)(2n + 4)/2 = 2(n + 1)(n + 2) \]

Hence \( \frac{K(2n)}{K(2n+2)} = \frac{n}{n+2} \)

2.4 Equations

(i) \( \) Equation \( K(n) = n \) has no solution.

We know \( K(n) = \frac{n(n + 2)}{2} \) OR \( n(n + 3)/2 \)

\( \therefore K(n) = n \) iff \( n(n + 2)/2 = n \) OR \( n(n + 3)/2 = n \)

i.e. iff \( n = 0 \) OR \( n = -1 \) which is not possible as \( n \in \mathbb{N} \).

Hence Equation \( K(n) = n \) has no solution.

(ii) \( \) Equation \( K(n) = K(n + 1) \) has no solution.

If \( n \) is even (or odd) then \( n + 1 \) is odd (or even)

Hence \( K(n) = K(n + 1) \)

iff \( n(n + 2)/2 = (n + 1)(n + 4)/2 \)
OR \[ n(n+3)/2 = (n+1)(n+3)/2 \]
i.e. iff \[ n(n+2) = (n+1)(n+4) \]

OR \[ n(n+3) = (n+1)(n+3) \]
i.e. iff \[ n^2 + 2n = n^2 + 5n + 4 \] OR \[ n^2 + 3n = n^2 + 4n + 3 \]
i.e iff \[ 3n + 4 = 0 \] OR \[ n + 3 = 0 \]
i.e iff \[ n = -4/3 \] OR \[ n = -3 \] which is not possible as \( n \in N \).

Hence Equation \( K(n) = K(n+1) \) has no solution.

(iii) \( Equation \ K(n) = K(n+2) \) has no solution.

If \( n \) is even (or odd) then \( n+2 \) is even (or odd).

Hence \( K(n) = K(n+2) \)
iff \[ n(n+2)/2 = (n+2)(n+4)/2 \]
OR \[ n(n+3)/2 = (n+2)(n+5)/2 \]
i.e. iff \[ n(n+2) = (n+2)(n+4) \]

OR \[ n(n+3) = (n+2)(n+5) \]
i.e. iff \[ n^2 + 2n = n^2 + 6n + 8 \] OR \[ n^2 + 3n = n^2 + 7n + 10 \]
i.e iff \[ 4n + 8 = 0 \] OR \[ 4n + 10 = 0 \]
i.e iff \[ n = -2 \] OR \[ n = -5/2 \] which is not possible as \( n \in N \).

Hence \( Equation \ K(n) = K(n+2) \) has no solution.

(iv) To find \( n \) for which \( K(n) = n^2 \)

(a) Let \( n \) be even.
Then \( K(n) = n^2 \) iff \[ n(n+2)/2 = n^2 \]
i.e. iff \[ n^2 + 2n = 2n^2 \] Or \( n(n-2) = 0 \)
i.e. iff \( n = 0 \) or \( n = 2 \). Hence \( n = 2 \) is the only even value of \( n \) for which \( K(n) = n^2 \)

(b) Let \( n \) be odd.
Then \( K(n) = n^2 \) iff \[ n(n+3)/2 = n^2 \]
i.e. iff \[ n^2 + 3n = 2n^2 \] Or \( n(n-3) = 0 \)
i.e. iff \( n = 0 \) or \( n = 3 \). Hence \( n = 3 \) is the only odd value of \( n \) for which \( K(n) = n^2 \)

So 2 and 3 are the only solutions of \( K(n) = n^2 \)

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2.5 Summation and product

(i) For \( n \) odd \( \Sigma K(2n) - \Sigma K(2n-1) = K(n) \)

\[
\Sigma K(2n) = \Sigma n(2n+2) = 2\Sigma n(n+1) = 2\Sigma (n^2 + n)
\]

\[
\Sigma K(2n-1) = \Sigma (2n-1)(2n+2)/2n
\]

\[
= \Sigma (2n-1)(n+1) = \Sigma (2n^2 + n - 1)
\]

\[
\therefore \Sigma K(2n) - \Sigma K(2n-1) = \Sigma(n+1) = n(n+1)/2 + n
\]

\[
= n(n+3)/2 = K(n)
\]

Hence for \( n \) odd \( \Sigma K(2n) - \Sigma K(2n-1) = K(n) \)

(ii) \[ \sum_{m=1}^{m=n} K(a^m) = K(a) + K(a^2) + K(a^3) + \ldots + K(a^n) \]

\[
= \frac{a(a^n-1)}{2(a^2-1)}(a^{n+1} + 3a + 2) \text{ if } a \text{ is even}
\]

\[
= \frac{a(a^n-1)}{2(a^2-1)}(a^{n+1} + 4a + 3) \text{ if } a \text{ is odd}
\]

(a) Let \( a \) be even. Then

\[
\sum_{m=1}^{m=n} K(a^m) = K(a) + K(a^2) + K(a^3) + \ldots + K(a^n)
\]

\[
= a(a+2)/2 + a^2(a^2+2)/2 + a^3(a^3+2)/2 + \ldots + a^n(a^n+2)/2
\]

\[
= \left( a^2/2 + a \right) + \left( a^4/2 + a^2 \right) + \ldots + \left( a^n/2 + a^n \right)
\]

\[
= (1/2) \left\{ a^2 + a^4 + a^6 + \ldots + a^{2n} \right\}
\]

\[
+ \left\{ a + a^2 + a^3 + \ldots + a^n \right\}
\]

\[
= (1/2) \left\{ a^2 + (a^2)^2 + (a^2)^3 + \ldots + (a^2)^n \right\}
\]

\[
+ \left\{ a + a^2 + a^3 + \ldots + a^n \right\}
\]

\[
= \frac{1}{2} a^2 \left( \frac{a^{2n}-1}{a^2-1} \right) + \frac{a(a^n-1)}{a-1}
\]

\[
= \frac{a^2}{2} \left( \frac{a^n-1}{a-1} (a+1) \right) + \frac{a(a^n-1)}{a-1}
\]

\[
= \frac{a(a^n-1)}{2(a-1)} \left\{ \frac{a(a^n+1)}{(a+1)} + 2 \right\}
\]

\[
= \frac{a(a^n-1)}{2(a-1)} \left\{ \frac{a^{n+1} + a + 2a + 2}{(a+1)} \right\}
\]
\[\begin{align*}
\text{Hence } K(a) + K(a^2) + K(a^3) + \ldots + K(a^n) & = \frac{a(a^n-1)}{2(a^2-1)} (a^{n+1} + 3a + 2) \text{ if } a \text{ is even} \\
\end{align*}\]

(b) Let \(a\) is odd. Then

\[\begin{align*}
\sum_{m=1}^{n} K(a^m) & = K(a) + K(a^2) + K(a^3) + \ldots + K(a^n) \\
& = a(a+3)/2 + a^2(a^2+3)/2 + a^3(a^3+3)/2 \\
& \quad + \ldots + a^n(a^n+3)/2 \\
& = (1/2) \left\{ a^2 + 3a + a^4 + 3a^2 + a^6 \\
& \quad + 3a^3 + \ldots + a^{2n} + 3a^n \right\} \\
& = (1/2) \left\{ a^2 + a^4 + a^6 + \ldots + a^{2n} \right\} \\
& \quad + \left\{ a + a^2 + a^3 + \ldots + a^n \right\} \\
& = (1/2) \left\{ (a^2 + (a^2)^2 + \ldots + (a^2)^n \right\} \\
& \quad + 3 \left\{ (a + a^2 + a^3 + \ldots + a^n) \right\} \\
& = \frac{1}{2} \left\{ a^2 \frac{(a^n-1)}{a-1} + 3a \frac{(a^n-1)}{a-1} \right\} \\
& = a(a^n-1) \left\{ \frac{a(a^n+1)}{(a+1)} + 3 \right\} \\
& = a(a^n-1) \left\{ \frac{a^{n+1}+a+3a+3}{(a+1)} \right\} \\
& = a(a^n-1) \left( \frac{a^{n+1}+4a+3}{2(a^2-1)} \right) \text{ if } a \text{ is odd} \\
\end{align*}\]

(iii) \(\Pi K(2n) = 2^n \cdot n! \cdot (n+1)!\)

\[\begin{align*}
\Pi K(2n) & = \Pi 2n(2n+2)/2 = \Pi 2n(n+1) \\
& = \Pi 2 \cdot \Pi n \cdot \Pi (n+1) \\
& = 2n \cdot n! \cdot (n+1)! \\
\end{align*}\]

Hence \(\Pi K(2n) = 2^n \cdot n! \cdot (n+1)!\)
(iv) \[ \Pi K(2n-1) = \left( \frac{1}{2^n} \right) \cdot 2n! \cdot n! \cdot (n+1) \]

\[ \Pi K(2n-1) = \Pi (2n-1) (2n+2)/2 \]
\[ = \Pi (2n-1) (n+1) \]
\[ = \Pi (2n-1) \Pi (n+1) \]
\[ = (2n-1)! \cdot (n+1)! \]
\[ = (1/2n) \cdot 2n! \cdot n! \cdot (n+1) \]

2.6 Inequalities

(i) (a) For even numbers \( a \) and \( b > 4 \); \( K(a, b) > K(a) \cdot K(b) \)

Assume that \( K(a, b) \leq K(a) \cdot K(b) \)

i.e. \( ab \cdot (ab + 2)/2 \leq a(a + 2)/2 \cdot b(b + 2)/2 \)

\[ \therefore ab + 2 \leq (a + 2) \cdot (b + 2)/2 \]

i.e. \( ab \leq 2(a + b) \) . . . . . . (A)

Now as \( a \) and \( b > 4 \) so let \( a = 4 + h, b = 4 + k \) for some \( h, k \in N \).

\[ (A) \Rightarrow (4 + h)(4 + k) \leq (8 + 2h) + (8 + 2k) \]

i.e. \( 16 + 4h + 4k + hk \leq 16 + 2h + 2k \)

i.e. \( 2h + 2k + hk \leq 0 \) . . . . . . (I)

But as \( h, k \in N \), hence \( 2h + 2k + hk > 0 \)

This contradicts (I). Hence if both \( a \) and \( b \) are even and \( a, b > 4 \) then \( K(a, b) > K(a) \cdot K(b) \)

(b) For odd numbers \( a, b \geq 7 \); \( K(a, b) > K(a) \cdot K(b) \)

Let \( K(a, b) \leq K(a) \cdot K(b) \)

i.e. \( ab \cdot (ab + 3)/2 \leq a(a + 3)/2 \cdot b(b + 3)/2 \)

\[ \therefore ab + 3 \leq (a + 3) \cdot (b + 3)/2 \]

i.e. \( 2ab + 6 \leq ab + 3a + 3b + 9 \)

or \( ab \leq 3a + 3b + 3 \) . . . . . . (B)

Now as \( a, b \geq 7 \) so let \( a = 7 + h, b = 7 + k \) for some \( h, k \in W \)

\[ (B) \Rightarrow (7 + h)(7 + k) \leq 3(7 + h) + 3(7 + k) + 3 \]

i.e. \( 49 + 7h + 7k + hk \leq 45 + 3h + 3k \)

i.e. \( 4 + 4h + 4k + hk \leq 0 \) . . . . . . (II)
But $h, k \in W$ hence $4 + 4h + 4k + hk > 0$

This contradicts (II) Hence $K(a, b) > K(a) \cdot K(b)$

(c) For $a$ odd, $b$ even and $a, b > 5$; $K(a, b) > K(a) \cdot K(b)$

Let $K(a, b) \leq K(a) \cdot K(b)$

i.e. $ab(a + b + 2)/2 \leq a(a + 3)/2 \cdot b(b + 2)/2$

$\therefore ab + 2 \leq (a + 3) \cdot (b + 2)/2$

i.e. $ab \leq 2a + 3b + 2$ . . . . . . (C)

Now $a, b > 5$ so let $a = 6 + h$ and $b = 6 + k$

for some $h, k \in W$

$\therefore (C) \Rightarrow (6 + h)(6 + k) \leq 2(6 + h) + 3(6 + k) + 2$

i.e. $36 + 6h + 6k + hk \leq 12 + 2h + 18 + 3k + 2$

i.e. $4h + 3k + hk + 4 \leq 0$ . . . . . . (III)

But $h, k \in W$ . . . . . . $4h + 3k + hk + 4 > 0$

This contradicts (III) Hence $K(a, b) > K(a) \cdot K(b)$

Note :- It follows from (xii) (a), (b) and (c) that in general if $a, b > 5$ then $K(a, b) > K(a) \cdot K(b)$

(ii) If $a > 5$ then for all $n \in N$; $K(a^n) > n K(a)$

As $a > 5$ . . . . . . . . $K(a^n) = K(a, a, a, \ldots, n \text{ times})$

$> K(a) \cdot K(a) \cdot K(a) \text{ up to } n \text{ times}$

$> \{K(a)\}^n \geq n K(a)$

Hence if $a > 5$ then for all $n \in N; K(a^n) > n K(a)$

2.7 Summation of reciprocals.

(i) $\sum_{n=1}^{\infty} \frac{1}{K(2n)}$ is convergent.

$K(2n) = 2n(2n+2)/2 = 2n(n+1)$

$\therefore \frac{1}{K(2n)} = \frac{1}{2n(n+1)} = \frac{1}{2n^2(1+1/n)} \leq \frac{1}{n^2}$

So series is dominated by convergent series and hence it is convergent.
(ii) \[ \sum_{n=1}^{\infty} \frac{1}{K(2n-1)} \text{ is convergent.} \]

\[ K(2n-1) = (2n-1)(2n+2)/2 = (2n-1)(n+1) \]

\[ \therefore \frac{1}{K(2n-1)} = \frac{1}{(2n-1)(n+1)} = \frac{1}{n^2 \left( 2 - \frac{1}{n} \right) \left( 1 + \frac{1}{n} \right)} \leq 1/n^2 \]

Hence by comparison test series is convergent.

(iii) \[ \sum_{n=1}^{\infty} \frac{1}{K(n)} \text{ is convergent.} \]

\[ K(n) \geq n(n+2)/2 \]

\[ \therefore \frac{1}{K(n)} \leq \frac{2}{n^2 \left( 1 + 2/n \right)} \leq 1/n^2 \]

Hence series is convergent.

(iv) \[ \sum_{n=1}^{\infty} \frac{K(n)}{n} \text{ is divergent.} \]

\[ \frac{K(n)}{n} \geq \frac{n+2}{2} \geq \frac{n}{2} \]

Hence series is divergent.

2.8 \textbf{Limits.}

(i) \[ \lim_{n \to \infty} \frac{K(2n)}{\sum 2n} = 2 \]

\[ K(2n) = 2n(2n+2)/2 = 2n(n+1) \]

\[ \Sigma 2n = 2 \Sigma n = n(n+1) \]

\[ \frac{K(2n)}{\sum 2n} = \frac{2n(n+1)}{n(n+1)} = 2 \]

\[ \therefore \lim_{n \to \infty} \frac{K(2n)}{\sum 2n} = 2 \]
\[(ii) \quad \lim_{n \to \infty} \frac{K(2n-1)}{\sum (2n-1)} = 2\]

\[K(2n-1) = (2n-1)(2n-1+3)/2 = (2n-1)(2n+2)/2 = (2n-1)(n+1)\]

\[\Sigma 2n-1 = 2n(n+1)/2 - n = n^2\]

\[\therefore \quad \frac{K(2n-1)}{\sum (2n-1)} = \frac{(2n-1)(n+1)}{n^2} = (2 - \frac{1}{n})(1 + \frac{1}{n})\]

\[\therefore \quad \lim_{n \to \infty} \frac{K(2n-1)}{\sum (2n-1)} = 2\]

\[(iii) \quad \lim_{n \to \infty} \frac{K(2n+1)}{K(2n-1)} = 1\]

\[K(2n+1) = (2n+1)(2n+1+3)/2 = (2n+1)(n+2)\]

\[K(2n-1) = (2n-1)(2n-1+3)/2 = (2n-1)(2n+2)/2 = (2n-1)(n+1)\]

\[\therefore \quad \frac{K(2n+1)}{K(2n-1)} = \frac{(2n+1)(n+2)}{(2n-1)(n+1)}\]

\[\text{OR} \quad \frac{K(2n+1)}{K(2n-1)} = \frac{(2 + \frac{1}{n})(1 + \frac{2}{n})}{(2 - \frac{1}{n})(1 + \frac{1}{n})}\]

\[\therefore \quad \lim_{n \to \infty} \frac{K(2n+1)}{K(2n-1)} = 1\]

\[(iv) \quad \lim_{n \to \infty} \frac{K(2n+2)}{K(2n)} = 1\]

\[K(2n+2) = (2n+2)(2n+2+2)/2 = 2(n+1)(n+2)\]

\[K(2n) = 2n(2n+2)/2 = 2n(n+1)\]

\[\therefore \quad \frac{K(2n+2)}{K(2n)} = \frac{2(n+1)(n+2)}{2n(n+1)}\]

\[\text{OR} \quad \frac{K(2n+2)}{K(2n)} = (1 + \frac{2}{n})\]

\[\therefore \quad \lim_{n \to \infty} \frac{K(2n+2)}{K(2n)} = 1\]
2.9 Additional Properties.

(i) Let $C$ be the continued fraction of the sequence \{\(K(n)\)\}

\[
C = K(1) + \frac{K(2)}{K(3) + \frac{K(4)}{K(5) + \frac{K(6)}{K(7) + \ldots}}}
\]

\[
= K(1) + \frac{K(2)}{K(3) + \frac{K(4)}{K(5) + \frac{K(6)}{K(7) + \ldots}}}\]

The \(n\)th term \(T_n = \frac{K(2n)}{K(2n+1)} = \frac{2n^2 + 2n}{2n^2 + 5n + 2}\)

Hence \(T_n < 1\) for all \(n\) and \(\therefore\) with respect to [3], \(C\) is convergent and \(2 < C < 3\).

(ii) \(K(2^n - 1) + 1\) is a triangular number.

Let \(x = 2n\) then

\[
K(2n-1) + 1 = K(x-1) + 1
\]

\[
= \left\{ (x-1) \frac{(x+2)}{2} \right\} + 1
\]

\[
= \left\{ \frac{x^2 + x}{2} \right\}
\]

\[
= \frac{x(x+1)}{2}\quad \text{which is a triangular number.}
\]

(iii) Fibonacci sequence does not exist in the sequence \{\(K(n)\)\}

(a) If possible then let \(K(n) + K(n+1) = K(n+2)\) for some \(n\) where \(n\) is even.

\[
\therefore n(n+2)/2 + (n+1)(n+4)/2 = (n+2)(n+4)/2
\]

\[
\therefore (n^2 + 2n) + (n^2 + 5n + 4) = n^2 + 6n + 8
\]

\[
\therefore n^2 + n - 4 = 0\quad \text{OR}\quad n = \frac{-1 \pm \sqrt{17}}{2}\quad \text{which is not possible as}\quad n \in \mathbb{N}\]

(b) Let \(K(n) + K(n+1) = K(n+2)\) for some \(n\) where \(n\) is odd.

\[
\therefore n(n+3)/2 + (n+1)(n+3)/2 = (n+2)(n+5)/2
\]

\[
\therefore (n+3)(2n+1) = n^2 + 7n + 10
\]

\[
\therefore n^2 = 7\quad \text{OR}\quad n = \sqrt{7}\quad \text{which is not possible as}\quad n \in \mathbb{N}.
\]

Hence there is no Fibonacci sequence in \{\(K(n)\)\}

Similarly there is no Lucas sequence in \{\(K(n)\)\}
(iv) \( K(n) > \max \{ K(d) \} \): Where \( d \) is a proper divisor of \( n \) and \( n \) is composite.

As \( d \) is a proper divisor of \( n \) \( \therefore d < n \) and as function \( K \) is strictly monotonic increasing hence \( K(d) < K(n) \).

So for each proper divisor \( d \) we have \( K(n) > K(d) \) and hence \( K(n) > \max \{ K(n) \} \)

(v) Palindromes in \( \{ K(n) \} \)

\[ K(11) = 77, \quad K(21) = 252, \quad K(29) = 464, \]
\[ K(43) = 989, \quad K(64) = 212 \]

are only Palindromes for \( n \leq 100 \).

(vi) Pythagorean Triplet

We know that \((5, 12, 13)\) is a Pythagorean Triplet.

Similarly \((K(5), K(12), K(13))\) is a Linear Triplet because \(K(5) + K(12) = K(13)\).

(vii) \( K(2^n) = 2^n (2^n + 2) / 2 = 2^{2^n - 1} + 2^n \)

\( \therefore K(2^3) = 2^3 + 2^3 = 32 + 8 = 40 \) and \(40 + 1 = 41\) is prime.

Similarly \( K(2^4) = 2^7 + 2^4 = 128 + 16 = 144 \) and \(140 + 1 = 139\) is prime.

Hence it is conjectured that \( K(2^n) - 1 \) or \( K(2^n) + 1 \) is prime.

3.1 To find \( K^{-1} \) when \( n \) is odd

\( \therefore K(n) = n(n + 3) / 2 = t \) (say)

\( \therefore n = K^{-1}(t) \) Also as \( n(n + 3) / 2 = t \)

\( \therefore n = \frac{-3 + \sqrt{9 + 8t}}{2} \) \( \text{OR } K^{-1}(t) = n = \frac{-3 + \sqrt{9 + 8t}}{2} \)

\( \text{OR } K^{-1}(t_r) = \frac{-3 + \sqrt{9 + 8t}}{2} = n_r \)
Note:

(I) In the above expression plus sign is taken to ensure that

\[ K^{-1}(t_r) \in N. \]

(II) Also \( K^{-1}(t_r) \in N \) \text{ iff } \sqrt{9 + 8t_r} \text{ is an odd integer.}

and for this \( 9 + 8t_r \) should be a perfect square.

From above two observations we get possible values of \( t_r \) as \( 2, 9, 20, 35 \) etc...

3.2 Following are some examples of \( K^{-1}(t_r) \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( t_r )</th>
<th>( K^{-1}(t_r) = n_r )</th>
<th>( q_r = t_r / n_r )</th>
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</tbody>
</table>

3.3 Following results are obvious.

(i) \( K^{-1}(t_r) = n_r = 2r - 1 \)

(ii) \( t_r = t_{r-1} + (4r - 1) \)

(iii) \( t_r = n_r q_r = (2r - 1) q_r \)

(iv) \( n_r = q_r + (r - 2) \)

(v) \( \Sigma t_r = \Sigma t_{r-1} + r \cdot n_r \)

(vi) Every \( t_{r+1} \) is a triangular number.

(vii) As \( t_r - t_{r-1} = 4r - 1 \)

\( \therefore \) Second difference \( D^2(t_r) = 4r - 1 - [4(r-1) - 1] = 4 \)
3.4 To find $K^{-1}$ when $n$ is even

\[ K(n) = \frac{n(n + 2)}{2} = t \quad \text{(say)} \]

\[ n = K^{-1}(t) \quad \text{Also as} \quad n(n + 2) / 2 = t \]

\[ n = \frac{-2 + \sqrt{4 + 8t}}{2} \quad \text{OR} \quad K^{-1}(t) = n = -1 + \sqrt{1 + 2t} \]

\[ \text{OR} \quad K^{-1}(t_r) = -1 + \sqrt{1 + 2t_r} = n_r \]

**Note:**

(I) In the above expression plus sign is taken to ensure that $K^{-1}(t_r) \in \mathbb{N}$.

(II) Also $K^{-1}(t_r) \in \mathbb{N}$ iff $\sqrt{1 + 2t_r}$ is an odd integer.

And for this first of all $1 + 2t_r$ should be a perfect square of some odd integer.

From above two observations we get possible values of $t_r$ as 4, 12, 24, 40 etc...

3.5 Following are some examples of $K^{-1}(t_r)$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$t_r$</th>
<th>$K^{-1}(t_r) = n_r$</th>
<th>$q_r = t_r / n_r$</th>
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</tbody>
</table>

3.6 Following results are obvious.

(i) $K^1(t_r) = n_r = 2r$

(ii) $t_r = t_{r-1} + 4r$

(iii) $t_r = n_r q_r = 2r \cdot q_r$

(iv) $n_r = q_r + (r - 1)$

(v) $\Sigma t_r = \Sigma t_{r-1} + (r + 1) \cdot n_r$

(vi) $t_r = n_r [n_r - r + 1]$

(vi) Every $t_r$ is a multiple of 4

(vii) $t_r = 4p$ where $p$ is a triangular number.

(viii) For $r = 8$, $t_r = 144$, $n_r = 16$ and $q_r = 9$. So for $r = 8$; $t_r$, $n_r$, and $q_r$
are all perfect square.

(ix) As \( t_r - t_{r-1} = 4r \)

\[ : \text{Second difference } D^2 (t_r) = 4r - [4(r-1)] = 4 \]

3.7 Monoid

Let \( M = \{ K^l(2), K^l(4), K^l(9), K^l(12) \ldots \} \) be the collection of images of \( K^l \) including both even and odd \( n \).

Let \( \cdot \) stands for multiplication. Then \((M, \cdot)\) is a Monoid.

For it satisfies (I) Closure (II) Associativity (III) Identity

Here identity is \( K^{-l}(2) \).

In fact \((M, \cdot)\) is a Commutative Monoid.

As inverse of an element does not exist in \( M \) hence it is not a group.

Coincidently, \( M \) happens to be a cyclic monoid with operation +.

Because \( K^l(9) = [K^l(2)]^3 \)

References :-

   (Journal of Recreational Mathematics 1996, P. 249)

   (Smarandache Notion Journal Vol. 12, 2000, P. 140)

   (Smarandache Notion Journal Vol. 9, 1998, P. 40)
Appendix – [A]

Values of $K(\ n\ )$ for $n = 1$ To 100

<table>
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<th>$k$</th>
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