NUMERICAL FUNCTIONS AND TRIPLETS

I. Bălăcenoiu, D. Bordea, V. Seleacu

We consider the functions: \( f_s, f_d, f_p, F : \mathbb{N}^* \to \mathbb{N} \), where
\( f_s(k) = n, f_d(k) = n, f_p(k) = n, F(k) = n \), \( n \) being, respectively, the least natural number such that \( k/n! - 1, k/n! + 1, k/n! \pm 1, k/n! \) or \( k/n! \pm 1 \).

This functions have the next properties:

1. Obviously, from definition of this function, it results:

\[ F(k) = \min\{S(k), f_p(k)\} = \min\{S(k), f_s(k), f_d(k)\} \]

where \( S \) is the Smarandache function (see [3]).

2. \( F(k) \leq S(k), F(k) \leq f_s(k), F(k) \leq f_d(k), F(k) \leq f_p(k) \)

3. \( F(k) = S(k) \) if \( k \) is even, \( k \geq 4 \).

**Proof.** For any \( n \in \mathbb{N}, n \geq 2, n! \) is even, \( n! \pm 1 \) are odd. If \( k \) is even, then \( k \) cannot divide \( n! \pm 1 \). So \( F(k) = S(k) = n \geq 2 \) if \( k \) is even, \( k \geq 4 \).

4. If \( p > 3 \) is prime number, then \( F(p) \leq p - 2 \).

**Proof.** According to Wilson's theorem \( (p - 1)! + 1 = M_p \). Because \( (p - 2)! - 1 + (p - 1)! + 1 = (p - 2)!p \) results for \( p > 3 \), \( (p - 2)! - 1 = M_p \) and so \( F(p) \leq p - 2 \).

5. \( F(m!) = F(m! \pm 1) = S(m!) = m \).

6. The equation \( F(k) = F(k + 1) \) has infinitely many solutions, because, according to the property 5), there is the solutions \( k = m! \), \( m \in \mathbb{N}^* \).
7. If \( F(k) = S(k) \) and \( n \) is the least natural number such that \( k/n! \), then 
\( k \) not divide \( s! \pm 1 \) for \( s < n \).
Let \( k = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_r^{a_r} \). According to \( S(k) = \max \{ S_{p_i}(\alpha_i) \} \), it results 
that \( S(k) \geq p_h \), where \( p_h = \min \{ p_1, p_2, \ldots, p_r \} \).
If \( k \) not divide \( s! \pm 1 \) for \( s \leq p_h \), then \( k \) not divide \( t! \pm 1 \) for \( t > p_h \).
Consequently, if \( k \) not divide \( (n - 1)! \), \( k/n! \) and \( k \) not divide \( s! \pm 1 \) for 
\( s \leq \min \{ n, p_h \} \), then \( F(k) = S(k) = n \).
Obviously, the numbers \( k = 3t \), \( t \) being odd, \( t \neq 1 \), have \( p_h = 3 \) and 
they satisfy the condition \( 3t \) not divide \( s! \pm 1 \) for \( s = 1, 2, 3 \).
Therefore, for \( k = 3t \), \( t \) odd, \( t \neq 1 \), \( F(3t) = S(3t) = n \), \( n \) being the 
least natural number such that \( 3t/n! \).

8. The partition “bai” of the odd numbers.

Let \( A = \{ k \in \mathbb{N} | k \) odd and \( F(k) = S(k) \} \)
\( B = \{ k \in \mathbb{N} | k \) odd and \( F(k) < S(k) \} \)

\((A, B)\) is the partition “bai” of the odd numbers.
Into \( A \) there are numbers \( k = 3t \), \( t \) odd, \( t \neq 1 \). Obviously, \( A \) has
infinitely many elements.
Into \( B \) there are numbers \( k = t! \pm 1 \) with \( t \geq 3 \), \( t \in \mathbb{N} \). Obviously, \( B \)
has infinitely many elements.

**Definition 1** Let \( n \in \mathbb{N}^* \). We called triplet \( \hat{n} \), the set:
\( n - 1, n, n + 1 \).

**Definition 2** Let \( k < n \). The triplets \( \hat{k}, \hat{n} \) are separated if
\( k + 1 < n - 1 \), i.e. \( n - k > 2 \).

**Definition 3** The triplets \( \hat{k}, \hat{n} \) are \( l_* \)-relatively prime if
\( (k - 1, n - 1) = 1 \), \( (k + 1, n + 1) \neq 1 \).

For example: \( 6 \) and \( 72 \) are \( l_* \)-relatively prime.

**Definition 4** The triplets \( \hat{k}, \hat{n} \) are \( l_4 \)-relatively prime if
\( (k - 1, n - 1) \neq 1 \), \( (k + 1, n + 1) = 1 \).

**Definition 5** The triplets \( \hat{k}, \hat{n} \) are \( l \)-relatively prime if
\( (k - 1, n - 1) = 1 \), \( (k + 1, n + 1) = 1 \).
Definition 6 The triplets $\hat{k}$, $\hat{n}$ are $d$-relatively prime if 

$$(k - 1, n + 1) = 1, (k + 1, n - 1) = 1.$$ 

For example: $\hat{2}$ and $\hat{6}$ are $d$-relatively prime.

Definition 7 Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are $d_s$-relatively prime if 

$$(k - 1, n + 1) = 1, (k + 1, n - 1) \neq 1.$$ 

For example: $\hat{6}$ and $\hat{120}$ are $d_s$-relatively prime.

Definition 8 Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are $d_d$-relatively prime if 

$$(k - 1, n + 1) = 1, (k + 1, n - 1) = 1.$$ 

Example: $\hat{6}$ and $\hat{24}$ are $d_d$-relatively prime.

Definition 9 The triplets $\hat{k}$, $\hat{n}$ are $p$-relatively prime if 

$$(k - 1, n - 1) = 1, (k - 1, n + 1) = 1, (k + 1, n - 1) = 1, (k + 1, n + 1) = 1.$$ 

Obviously, if $\hat{k}$, $\hat{n}$ are $p$-relatively prime, then they are $l$ and $d$-relatively prime.

For example: $\hat{24}$ and $\hat{120}$ are $p$-relatively prime.

Definition 10 Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are $F$-relatively prime if 

$$(k - 1, n - 1) = 1, (k + 1, n - 1) = 1,$$

$$(k - 1, n) = 1, (k + 1, n) = 1,$$

$$(k - 1, n + 1) = 1, (k + 1, n + 1) = 1.$$ 

Definition 11 The triplets $\hat{k}$, $\hat{n}$ are $t$-relatively prime if 

$$(k - 1, n - 1) \cdot (k - 1, n) \cdot (k - 1, n + 1) \cdot (k, n - 1) \cdot (k, n) \cdot (k, n + 1) \cdot (k + 1, n - 1) \cdot (k + 1, n + 1) = 6.$$ 

For example: $\hat{2}$ and $\hat{4}$ and $t$-relatively prime.

Definition 12 Let $H \subset \mathbb{N}^*$. The triplet $\hat{n}$, $n \in H$ is, respectively, $l_s$, $l_d$, $l$, $d$, $d_s$, $d_d$, $p$, $F$, $t$-prime concerned at $H$, if $\forall s \in H$, $s < n$, the triplets $\hat{s}$, $\hat{n}$ are, respectively, $l_s$, $l_d$, $l$, $d$, $d_s$, $d_d$, $p$, $F$, $t$-relatively prime.

Let $H = \{n!|n \in \mathbb{N}^*\}$. For the triplets $\hat{m}$, $m \in H$ there are particular properties.

Proposition 1 Let $k < n$. The triplets $\hat{(k!)}$, $\hat{(n!)}$ are separated if 

$n > \max\{2, k\}$. 

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Proof. Obviously, \( n! - k! > 2 \) if \( n > 2 \) and \( k < n \), i.e. \( n > \max\{2, k\} \).

Proposition 2 Let \( n > \max\{2, k\} \) and \( M_{kn} = \{m \in \mathbb{N}|k!+1 < m < n!-1\} \). If \( k_1 < k_2 \) and \( n_1 > \max\{2, k_1\} \), \( n_2 > \max\{2, k_2\} \), then \( n_1 - k_1 \leq n_2 - k_2 \Rightarrow \text{card}M_{k_1n_1} < \text{card}M_{k_2n_2} \).

Proof. For \( n > k \geq 2 \) it is true that
\[
n! - (n - 1)! > k! - (k - 1)! \tag{1}
\]
Let \( n > k \geq 2, 1 \leq s \leq k \). Using (1) we can write:
\[
n! - (n - 1)! > k! - (k - 1)!
(n - 1)! - (n - 2)! > (k - 1)! - (k - 2)!
\]
By summing this inequalities, it results:
\[
n! - (n - s)! > k! - (k - s)! \tag{2}
\]
Let \( 2 \leq k_1 < n_1, 2 \leq k_2 < n_2, k_1 < k_2, n_1 - k_1 \leq n_2 - k_2 \). Then \( n_2 - n_1 \geq k_2 - k_1 \geq 1 \) and there is \( n_3 \) such that \( n_2 > n_3 \geq n_1 \) and \( n_2 - n_3 = k_2 - k_1 \).
Using (2) we can write:
\[
n_2! - n_3! > k_2! - k_1!
\]
Since \( n_3! \geq n_1! \) we have:
\[
n_2! - n_1! > k_2! - k_1! \tag{3}
\]
According to \( \text{card}M_{k_1n_1} = n_1! - 1 - (k_1! + 1) \), \( \text{card}M_{k_2n_2} = n_2! - 1 - (k_2! + 1) \), it results that:
\[
\text{card}M_{k_2n_2} - \text{card}M_{k_1n_1} = n_2! - n_1! - (k_2! - k_1!)
\]
That is, taking into account (3), \( \text{card}M_{k_1n_1} < \text{card}M_{k_2n_2} \).

Definition 13 Let \( k < n \). The triplets \((k!)\), \((n!)\) are linked if \( k! - 1 = n \) or \( k! + 1 = n \).

Proposition 3 For \( k \in \mathbb{N}^* \) there is \( p \) prime number, such that for any \( s \geq p \) the triplets \((k!)\), \((s!)\) are not \( F \)-relatively prime.
Proof. Obviously, for \( k = 1 \) and \( k = 2 \), the proposition is true.

If \( n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \) divide \( k! - 1 \) or \( k! + 1 \), then \( p_j > k \geq 3 \), for \( j \in \{1, 2, \ldots, i\} \).

Let \( \bar{n} = p_1 \cdot p_2 \cdots p_l \) and \( p = \max\{p_j\} \).

Obviously, \( \bar{n} \geq 3 \) because \( p > k \geq 3 \), \( \bar{n}/k! - 1 \) or \( \bar{n}/k! + 1 \).

For any \( s \geq p \), \( \bar{n}/s! \) and so, the triplets \( (k!),(\bar{n}) \) are not \( F \)-relatively prime.

Remark 1 i) Let \( k < n \). If \((k!), (\bar{n})\) are linked, then \( n - k = k! - k \pm 1 \).

If \( 2 < k_1 < n_1 \), \((k_1!)(\bar{n}_1!\) are linked and \( k_2 < n_2 \), \((k_2!)(\bar{n}_2!\) are linked, then \( k_1 < k_2 \Rightarrow n_1 - k_1 < n_2 - k_2 \) and in view of the proposition 2, results \( \text{card}M_{n_1,n_1} < \text{card}M_{n_2,n_2} \).

ii) There are twin prime numbers with the triplet \((\bar{n})\). For example 5 with 7 are from \((3!)\).

Definition 14 Considering the canonical decomposition of natural numbers \( n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \), we define \( \bar{n} = \{p_1^{a_1}, p_2^{a_2}, \ldots, p_r^{a_r}\} \),

\( M = \{\bar{n}|n \in \mathbb{N}^*\} \).

Definition 15 On \( M \) we consider the relation of order \( \sqsubseteq \) defined by:

\[ \{p_1^{a_1}, p_2^{a_2}, \ldots, p_r^{a_r}\} \sqsubseteq \{q_1^{b_1}, q_2^{b_2}, \ldots, q_t^{b_t}\} \]

if and only if \( \{p_1, p_2, \ldots, p_r\} \sqsubseteq \{q_1, q_2, \ldots, q_t\} \) and if \( p_i = q_j \), then \( \alpha_i \leq \beta_j \).

Remark 2 For any triplet \((\bar{n})\), \( n \in \mathbb{N}^* \), we consider the sets:

\( A_n = \{k \in \mathbb{N}^*|k \sqsubseteq \bar{n}\} \), \( A^*_n = \{k \in A_n|k \not\sqsubseteq A_h \text{ for } h < n\} \)

\( B_n = \{k \in \mathbb{N}^*|k \subseteq n! - 1\} \), \( B^*_n = \{k \in B_n|k \not\subseteq B_h \text{ for } h < n\} \)

\( C_n = \{k \in \mathbb{N}^*|k \subseteq n! + 1\} \), \( C^*_n = \{k \in C_n|k \not\subseteq C_h \text{ for } h < n\} \)

\( M_n = \{k \in \mathbb{N}^*|k \subseteq \bar{n}! \text{ or } k \subseteq n! - 1 \text{ or } k \subseteq n! + 1\} \)

\( M^*_n = \{k \in M_n|k \not\subseteq M_h \text{ for } h < n\} \).

It is obvious that:

\( A^*_n = S^{-1}(n) \), \( B^*_n = f_s^{-1}(n) \), \( C^*_n = f_d^{-1}(n) \), \( M^*_n = F^{-1}(n) \).

If \( k \in A^*_n \), it is said that \( k \) has a factorial signature which is equivalent with the factorial signature of \( n! \) (see [1])

Let \( k \in B^*_n \), \( k = t_1^{i_1} \cdot t_2^{i_2} \cdots t_i^{i_i} \). Then \( \{t_r\} \not\subseteq \bar{n}! \) for \( r = 1, i \) and for any \( h < n \), there are \( t_j^i \), \( 1 \leq j \leq i \), such that \( \{t_j^i\} \not\subseteq h! - 1 \).

Similarly, for \( k \in C^*_n : \{t_r\} \not\subseteq \bar{n}! \) for \( r = 1, i \) and for any \( h < n \), there are \( t_j^i \), \( 1 \leq j \leq i \), such that \( \{t_j^i\} \not\subseteq h! + 1 \).
References


Current address:
Department of Mathematics, University of Craiova
13, Al. I. Cuza st., Craiova 1100, Romania