Abstract. In the present paper we study some series concerning the following function of the Numbers Theory [1]: "S : N->N such that S(n) is the smallest k with property that k! is divisible by n".

1. Introduction. The following functions in Numbers Theory are well-known: the function μ(n) of Möbius, the function ζ(s) of Riemann (ζ(s) = ∑ 1/n^s, s = σ + it ∈ C), the function \( \Lambda(n) \) of Mangoldt \( \left( \Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \\ 0, & \text{if } n \neq p^m \end{cases} \right) \) etc.

The purpose of this paper is to study some series concerning the following function of the Numbers Theory "[1] S : N->N such that S(n) is the smallest integer k with the propriety that k! is divisible by n".

We first prove the divergence of some series involving the S function, using an unitary method, and then we prove that the series \( \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)...S(n)} \) is convergent to a number \( S \in (71/100, 101/100) \) and we study some applications of this series in the Numbers Theory.

Then we prove that series \( \sum_{n=2}^{\infty} \frac{1}{n!} \) is convergent to a real numbers \( s \in (0.717, 1.253) \) and that the sum of the remarkable series \( \sum_{n=2}^{\infty} \frac{S(n)}{n!} \) is an irrational number.

2. The main results

Proposition 1. If \( (x_n)_{n=1}^{\infty} \) is strict increasing sequence of natural numbers, then the series:

\[
\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)},
\]

is divergent.

Proof. We consider the function \( f[x_n, x_{n+1}] : R \to R \), defined by \( f(x) = \ln \ln x \) is meets the conditions of the Lagrange's theorem of finite increases. Therefore there is \( c_n \in (x_n, x_{n+1}) \) such that:

\[
\ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n} (x_n - x_{n+1}).
\]

Because \( x_n < c_n < x_{n+1} \), we have:

\[
\frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} - \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, (\forall) n \in N,
\]

if \( x_n \neq 1 \).

We know that for each \( n \in N^\times \backslash \{1\} \), \( \frac{S(n)}{n} \leq 1 \), i.e.

\[
0 < \frac{S(n)}{n \ln n} \leq \frac{1}{\ln n},
\]
from where it results that \( \lim_{n \to \infty} \frac{S(n)}{n \ln n} = 0 \). Hence there is \( k > 0 \) such that

\[
\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}.
\]

Introducing (5) in (3) we obtain:

\[
\ln \frac{x_{n+1}}{x_n} < k \frac{x_{n+1} - x_n}{S(x_n)}, (\forall)n \in \mathbb{N}^* \setminus \{1\}.
\]

Summing up after \( n \) it results:

\[
\sum_{n=1}^{m} \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k} (\ln \ln x_{m+1} - \ln \ln x_1).
\]

Because \( \lim \ln x_m = \infty \) we have \( \lim \ln \ln x_m = \infty \), i.e., the series:

\[
\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}
\]

is divergent. The Proposition 1 is proved.

**Proposition 2.** Series \( \sum_{n=2}^{\infty} \frac{1}{S(n)} \) is divergent.

**Proof.** We use Proposition 1 for \( x_n = n \).

**Remarks.**

1) If \( x_n \) is the \( n \)-th prime number, then the series \( \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)} \) is divergent.

2) If the sequence \( (x_n)_{n=1}^{\infty} \) forms an arithmetical progression of natural numbers, then the series \( \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)} \) is divergent.

3) The series \( \sum_{n=1}^{\infty} \frac{1}{S(2n+1)}, \sum_{n=1}^{\infty} \frac{1}{S(4n+1)} \) etc., are all divergent.

In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent forms are divergent.

**Proposition 3.** The series

\[
\sum_{n=2}^{\infty} \frac{1}{S(2)S(3) ... S(n)}
\]

is convergent to a number \( s \in (7/100, 101/100) \).

**Proof.** From the definition it results \( S(n) \leq n! \), \( (\forall)n \in \mathbb{N}^* \setminus \{1\} \), so \( \frac{1}{S(n)} \geq \frac{1}{n!} \).

Summing up, beginning with \( n=2 \) we obtain:

\[
\sum_{n=2}^{\infty} \frac{1}{S(2)S(3) ... S(n)} = \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.
\]

The product \( S(2) S(3) ... S(n) \) is greater than the product of prime numbers from the set \( \{1, 2, ..., n\} \), because \( S(p) = p \), for \( p \) prime number. Therefore:

\[
\prod_{i=2}^{n} \frac{1}{S(i)} < \prod_{i=2}^{n} \frac{1}{p_k},
\]

where \( p_k \) is the biggest number smaller or equal to \( n \).

There are the inequalities:

\[
\frac{1}{\prod_{i=2}^{n} S(i)} < \frac{1}{\prod_{i=2}^{n} p_k},
\]

\[
\lim_{n \to \infty} \frac{1}{\prod_{i=2}^{n} S(i)} = 0
\]

\[
\lim_{n \to \infty} \frac{1}{\prod_{i=2}^{n} p_k} = 0
\]
\[ S = \sum_{n=2}^\infty \frac{1}{S(2)S(3) \ldots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2)S(3)} + \frac{1}{S(2)S(3)S(4)} + \ldots + \]
\[ + \frac{1}{S(2)S(3) \ldots S(k)} + \ldots < \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 5 \cdot 7} + \]
\[ + \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}{p_{k+1} - p_k} + \ldots \]  
(8)

Using the inequality \[ p_1 p_2 \ldots p_k > p_{k+1}^\alpha, (\forall) k \geq 5[5], \]
we obtain:
\[ S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{1}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \ldots - \frac{1}{12^2}. \]
(9)

We symbolise by \[ P = \frac{1}{2^2} + \frac{1}{3^2} + \ldots \] and observe that \[ P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \ldots \]
It results:
\[ P < \frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{12^2}\right), \]
where
\[ \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \text{ (EULER)}. \]

Introducing in (9) we obtain:
\[ S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{1}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \ldots - \frac{1}{12^2}. \]

Estimating with an approximation of an order not more than \[ \frac{1}{10^2}, \]
we find:
\[ 0.71 < \frac{1}{S(2)S(3) \ldots S(n)} < 0.79. \]
(10)

The proposition 3 is proved.

**Remark.** Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right ranging:
\[ \sum_{n=2}^\infty \frac{1}{S(2)S(3) \ldots S(n)} < 0.97. \]
(11)

**Proposition 4.** Let \( \alpha \) be a fixed real number, \( \alpha \geq 1. \) Then the series \[ \sum_{n=2}^\infty \frac{n^\alpha}{S(2)S(3) \ldots S(n)} \]
is convergent.

**Proof.** Be \((x_k)_{k=1}^\infty\) the sequence of prime numbers. We can write:
\[ \frac{2^\alpha}{S(2)} = \frac{2^\alpha}{2} = 2^{\alpha-1} \]
\[ \frac{3^\alpha}{S(2)S(3)} = \frac{3^\alpha}{p_1 p_2} \]
\[ \frac{4^\alpha}{S(2)S(3)S(4)} < \frac{4^\alpha}{p_1 p_2} < \frac{p_3^\alpha}{p_1 p_2} \]
\[ \frac{5^\alpha}{S(2)S(3)S(4)S(5)} < \frac{5^\alpha}{p_1 p_2 p_3} < \frac{p_4^\alpha}{p_1 p_2 p_3} \]
\[ \frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^\alpha}{p_1 p_2 p_3 p_4} < \frac{p_5^\alpha}{p_1 p_2 p_3 p_4} \]
\[ \vdots \]
\[ \frac{n^\alpha}{S(2)S(3) \ldots S(n)} < \frac{n^\alpha}{p_1 p_2 \ldots p_k} < \frac{p_{k+1}^\alpha}{p_1 p_2 \ldots p_k} \]
where \( p_i \leq n, i \in \{1, \ldots, k\}, p_{k+1} > n. \)
Therefore
\[ \sum_{n=2}^{\infty} \frac{n^a}{S(2)S(3)\ldots S(n)} < 2^{a-1} + \sum_{n=2}^{\infty} \frac{(p_{k+1} - p_k) \cdot p_k^a}{S(2)S(3)\ldots S(n)} < 2^{a-1} + \sum_{n=2}^{\infty} \frac{p_k^a}{p_1p_2\ldots p_k} . \]

Then it exists \( k_0 \in \mathbb{N} \) such that for any \( k \geq k_0 \) we have:
\[ p_1p_2\ldots p_k > p_{k+1}^{a+3} . \]

Therefore
\[ \sum_{n=2}^{\infty} \frac{n^a}{S(2)S(3)\ldots S(n)} < 2^{a-1} + \sum_{k=1}^{k_0} \frac{p_{k+1}^a}{p_1p_2\ldots p_k} + \sum_{k=k_0}^{\infty} \frac{1}{p_k^2} . \]

Because the series \( \sum_{k=k_0}^{\infty} \frac{1}{p_k^2} \) is convergent it results that the given series is convergent too.

**Consequence 1.** It exists \( n_0 \in \mathbb{N} \) so that for each \( n \geq n_0 \) we have \( S(2)S(3)\ldots S(n) > n^a \).

**Proof.** Because \( \lim_{n \to \infty} \frac{n^a}{S(2)S(3)\ldots S(n)} = 0 \), there is \( n_0 \in \mathbb{N} \) so that \( \frac{n^a}{S(2)S(3)\ldots S(n)} < 1 \) for each \( n \geq n_0 \).

**Consequence 2.** It exists \( n_0 \in \mathbb{N} \) so that:
\[ S(2) + S(3) + \ldots + S(n) > (n-1)n^{a-1} \] for each \( n \geq n_0 \).

**Proof.** We apply the inequality of averages to the numbers \( S(2), S(3), \ldots, S(n) \):
\[ S(2) + S(3) + \ldots + S(n) > (n-1)\sqrt[n-1]{S(2)S(3)\ldots S(n)} > (n-1)n^{a-1}, \forall n \geq n_0 . \]

We can write it as it follows:
\[ \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \ldots = \sum_{i=2}^{\infty} \frac{a(n)}{n!} , \]
where \( a(n) \) is the number of solutions for the equation \( S(x) = n \).

It results from the equality \( S(x) = n \) that \( x \) is a divisor of \( n! \), so \( a(n) \) is smaller than \( d(n!) \).

So, \( a(n) < d(n!) \).

**Lemma 1.** We have the inequality:
\[ d(n) \leq n-2, \quad \text{for each } n \in \mathbb{N}, n \geq 7 . \]

**Proof.** Be \( n = p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k} \) with \( p_1, p_2, \ldots, p_k \) prime numbers, and \( a_i \geq 1 \) for each \( i \in \{1, 2, \ldots, k\} \). We consider the function \( f : [1, \infty) \to \mathbb{R}, f(x) = a^x - x - 2, a \geq 2, \) fixed. It is derivable on \([1, \infty)\) and \( f(x) = a^x \ln a - 1 \). Because \( a \geq 2 \), and \( x \geq 1 \) it results that \( a^x \geq 2 \), so \( a^x \ln a \geq 2 \ln a = \ln a^2 \geq \ln 4 > \ln e = 1 \), \( f(x) > 0 \) for each \( x \in [1, \infty) \) and \( a \geq 2 \), fixed. But \( f(1) = a-3 \). It results that for \( a \geq 3 \) we have \( f(x) \geq 0 \) means \( a^x \geq x + 2 \).

Particularly, for \( a = p_i, i \in \{1, 2, \ldots, k\} \), we obtain \( p_i^{a_i} \geq a_i + 2 \) for each \( p_i \geq 3 \).

If \( n = 2^s, s \in \mathbb{N}^* \), then \( d(n) = s + 1 < 2^s - 2 = n-2 \) for \( s \geq 3 \).

So we can assume \( k \geq 2 \), i.e. \( p_2 \geq 3 \). The following inequalities result:
\[ p_1^{a_1} \geq a_1 + 1 \]
\[ p_2^{a_2} \geq a_2 + 1 \]
\[ \ldots \ldots \ldots \ldots \]
\[ p_k^{a_k} \geq a_k + 1 , \]
equivalent with
\[ p_1^{a_1} \geq a_1 + 1, p_2^{a_2} - 1 \geq a_2 + 1, \ldots, p_k^{a_k} - 1 \geq a_k + 1 . \]
By multiplying, member with member, of the inequalities (13) we obtain:
\[ p_1^{a_1}(p_2^{a_2} - 1) \cdots (p_k^{a_k} - 1) \geq (a_1 + 1)(a_2 + 1) \cdots (a_k + 1) = d(n). \]  
(14)

Considering the obvious inequality:
\[ n - 2 \geq p_1^{a_1}(p_2^{a_2} - 1) \cdots (p_k^{a_k} - 1) \]  
(15)

and using (14) it results that:
\[ n-2 \geq d(n) \text{ for each } n \geq 7. \]

**Lemma 2.** \( d(n) < (n-2)! \) for each \( n \in \mathbb{N}, \) \( n \geq 7. \)

**Proof.** We carry out an induction after \( n. \) So, for \( n=7, \)
\[ d(7!) = d(2^4 \cdot 3^2 \cdot 5 \cdot 7) = 60 < 120 = 5!. \]
We assume that \( d(n!) < (n-2)!. \)
\[ d((n+1)!) = d((n!(n+1)) < d(n!) d(n+1) < (n-2)!d(n+1) < (n-2)!(n-1) = (n-1)!, \]

because in according to Lemma 1, \( d(n+1) < n-1. \)

**Proposition 5.** The series \( \sum_{n=2}^{\infty} \frac{1}{S(n)!} \) is convergent to a number \( s \in (0.717, 1.253). \)

**Proof.** From Lemma 2 it results that \( a(n) < (n-2)!, \) so \( \frac{a(n)}{n!} < \frac{1}{n(n-1)} \) for every \( n \in \mathbb{N}, \)
\[ n \geq 7 \text{ and } \sum_{n=2}^{\infty} \frac{1}{S(n)!} = \sum_{n=2}^{\infty} \frac{a(n)}{n!} + \sum_{n=7}^{\infty} \frac{1}{(n-1)!}. \]
Therefore
\[ \sum_{n=2}^{\infty} \frac{a(n)}{n!} < \frac{1}{2} + \frac{2}{3} + \frac{4}{4} + \frac{8}{5} + \frac{14}{6} + \sum_{n=7}^{\infty} \frac{1}{n^2 - 1}. \]
Because \( \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = 1 \) we have there is a number \( s > 0, \)
\[ s = \sum_{n=2}^{\infty} \frac{1}{S(n)!}. \]

From (17) we obtain:
\[ \sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{391}{360} + 1 - \frac{1}{2^2 - 2} - \frac{1}{3^2 - 3} - \frac{1}{4^2 - 4} + \frac{1}{5^2 - 5} + \frac{1}{6^2 - 6} = 751 - 5 = 451 < 1,253. \]
But, because \( S(n) \leq n \) for every \( n \in \mathbb{N}^*, \) it results:
\[ \sum_{n=2}^{\infty} \frac{1}{S(n)!} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2. \]
Consequently, for the number \( s \) we obtain the range \( e - 2 < s < 1,253, \) i.e., \( 0.717 < s < 1,253. \)

Because \( S(n) \leq n, \) it results \( \sum_{n=2}^{\infty} \frac{S(n)}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{(n-1)!}. \) Therefore the series \( \sum_{n=2}^{\infty} \frac{S(n)}{n!} \) is convergent to a number \( f. \)

**Proposition 6.** The sum \( f \) of the series \( \sum_{n=2}^{\infty} \frac{S(n)}{n!} \) is an irrational number.

**Proof.** From the above results that \( \lim_{n \to \infty} \sum_{n=2}^{\infty} \frac{S(n)}{n!} = 1. \) Under these circumstances that \( f \in \mathbb{Q}, f > 0. \) Therefore it exists \( a, b \in \mathbb{N}, (a,b)=1, \) so that \( f = \frac{a}{b}. \)

Let \( p \) be a fixed prime number, \( p > b, p \geq 3. \) Obviously, \( \frac{a}{b} = \sum_{i=2}^{p-1} \frac{S(i)}{i!} + \sum_{i=2p}^{\infty} \frac{S(i)}{i!} \) which leads to: 163
\[
\frac{(p - 1)!a}{b} = \sum_{i=2}^{p-1} \frac{(p - 1)!S(i)}{i!} + \sum_{i=p}^{\infty} \frac{(p - 1)!S(i)}{i!}.
\]

Because \( p > b \) results that \( \frac{(p - 1)!a}{b} \in \mathbb{N} \) and \( \sum_{i=2}^{p-1} \frac{(p - 1)!S(i)}{i!} \in \mathbb{N} \). Consequently we have
\[
\sum_{i=p}^{\infty} \frac{(p - 1)!S(i)}{i!} \in \mathbb{N} \text{ too.}
\]

Be \( \alpha = \sum_{i=p}^{\infty} \frac{(p - 1)!S(i)}{i!} \in \mathbb{N} \). So we have the relation
\[
\alpha = \frac{(p - 1)!S(p)}{p!} + \frac{(p - 1)!S(p + 1)}{(p + 1)!} + \frac{(p - 1)!S(p + 2)}{(p + 2)!} + \ldots
\]

Because \( p \) is a prime number it results \( S(p) = p \).

So
\[
\alpha = 1 + \frac{S(p + 1)}{p(p + 1)} + \frac{S(p + 2)}{p(p + 1)(p + 2)!} + \ldots > 1 \quad (18)
\]

We know that \( S(p+1) \leq p + 1(\forall) i \geq 1 \), with equality only if the number \( p+i \) is prime.

Consequently, we have
\[
\alpha < 1 + \frac{1}{p} + \frac{1}{p(p+1)} + \frac{1}{p(p+1)(p+2)!} + \ldots < 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots = \frac{p}{p-1} < 2 \quad (19)
\]

From the inequalities (18) and (19) results that \( 1 < \alpha < 2 \), impossible, because \( \alpha \in \mathbb{N} \).

The proposition is proved.

REFERENCES


