On an inequality for the Smarandache function

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1. In paper [2] the author proved among others the inequality \( S(ab) \leq aS(b) \) for all \( a, b \) positive integers. This was refined to

\[ S(ab) \leq S(a) + S(b) \] (1)

in [1]. Our aim is to show that certain results from our recent paper [3] can be obtained in a simpler way from a generalization of relation (1). On the other hand, by the method of Le [1] we can deduce similar, more complicated inequalities of type (1).

2. By mathematical induction we have from (1) immediately:

\[ S(a_1a_2\ldots a_n) \leq S(a_1) + S(a_2) + \ldots + S(a_n) \] (2)

for all integers \( a_i \geq 1 \) \((i = 1, \ldots, n)\). When \( a_1 = \ldots = a_n = n \) we obtain

\[ S(a^n) \leq nS(a). \] (3)

For three applications of this inequality, remark that

\[ S((m!)^n) \leq nS(m!) = nm \] (4)

since \( S(m!) = m \). This is inequality 3) part 1. from [3]. By the same way, \( S((n!)^{(n-1)!}) \leq (n - 1)!S(n!) = (n - 1)!n = n! \), i.e.

\[ S((n!)^{(n-1)!}) \leq n! \] (5)
Inequality (5) has been obtained in [3] by other arguments (see 4) part 1).

Finally, by \( S(n^2) \leq 2S(n) \leq n \) for \( n \) even (see [3], inequality 1), \( n > 4 \), we have obtained a refinement of \( S(n^2) \leq n \):

\[
S(n^2) \leq 2S(n) \leq n \tag{6}
\]

for \( n > 4 \), even.

3. Let \( m \) be a divisor of \( n \), i.e. \( n = km \). Then (1) gives \( S(n) = S(km) \leq S(m) + S(k) \), so we obtain:

If \( m|n \), then

\[
S(n) - S(m) \leq S\left(\frac{n}{m}\right). \tag{7}
\]

As an application of (7), let \( d(n) \) be the number of divisors of \( n \). Since \( \prod_{k|n} k = n^{d(n)/2} \), and \( \prod_{k \leq n} k = n! \) (see [3]), and by \( \prod_{k|n} k \prod_{k \leq n} k \), from (7) we can deduce that

\[
S(n^{d(n)/2}) + S(n!/n^{d(n)/2}) \geq n. \tag{8}
\]

This improves our relation (10) from [3].

4. Let \( S(a) = u \), \( S(b) = v \). Then \( b|v! \) and \( u!|x(x-1)\ldots(x-u+1) \) for all integers \( x \geq u \). But from \( a|u! \) we have \( a|x(x-1)\ldots(x-u+1) \) for all \( x \geq u \). Let \( x = u + v + k \) (\( k \geq 1 \)). Then, clearly \( ab(v+1)\ldots(v+k)!(u+v+k)! \), so we have \( S[ab(v+1)\ldots(v+k)] \leq u + v + k \). Here \( v = S(b) \), so we have obtained that

\[
S[ab(S(b)+1)\ldots(S(b)+k)] \leq S(a) + S(b) + k. \tag{9}
\]

For example, for \( k = 1 \) one has

\[
S[ab(S(b)+1)] \leq S(a) + S(b) + 1. \tag{10}
\]

This is not a consequence of (2) for \( n = 3 \), since \( S[S(b)+1] \) may be much larger than 1.
References

