ON SOME DIOPHANTINE EQUATIONS

by

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Let $S(n)$ be defined as the smallest integer such that $(S(n))!$ is divisible by $n$ (Smarandache Function). We shall assume that $S : \mathbb{N}^* \rightarrow \mathbb{N}^*$, $S(1) = 1$. Our purpose is to study a variety of Diophantine equations involving the Smarandache function. We shall determine all solutions of the equations (1), (3) and (8).

1. $x^S(x) = S(x)^x$
2. $x^S(y) = S(y)^x$
3. $x^S(x) + S(x) = S(x)^x + x$
4. $x^S(y) + S(y) = S(y)^x + x$
5. $S(x)^x + x^x = x^{S(x)} + S(x)^x$
6. $S(y)^x + x^x = x^{S(y)} + S(y)^x$
7. $S(x)^x + x^x = x^{S(x)} + S(x)^x$
8. $S(y)^x + x^x = x^{S(y)} + S(y)^x$.

For example, let us solve equation (1):

We observe that if $x = S(x)$, then (1) holds.

But $x = S(x)$ if and only if $x \in \{1, 2, 3, 4, 5, 7, \ldots \} = \{x \in \mathbb{N}^* ; x \text{-prime} \} \cup \{1, 4 \}$.

If $x \geq 6$ is not a prime integer, then $S(x) < x$. We can write $x = S(x) + t$, $t \in \mathbb{N}^*$, which implies that $S(x)^{S(x) + t} = (S(x) + t)^{S(x)}$. Thus we have $S(x)^t = (1 + \frac{t}{S(x)})^{S(x)}$.

Applying the well-known result $(1 + \frac{k}{n})^n < 3^k$, for $n, k \in \mathbb{N}^*$, we have $S(x)^t < 3^t$ which implies that $S(x) < 3$ and consequently $x < 3$. This contradicts our choice of $x$.

Thus, the solutions of (1) are $A_1 = \{x \in \mathbb{N}^* ; x \text{-prime} \} \cup \{1, 4 \}$.

Let us denote by $A_k$ the set of all solutions of the equation (k).

To find $A_k$, for example, we see that $(S(n), n) \in A_k$ for $n \in \mathbb{N}^*$.

Now suppose that $x = S(y)$. We can show that $(x, y)$ does not belong to $A_k$ as follows: $1 < S(y) < x \Rightarrow S(y) \geq 2$ and $x \geq 3$. On the other hand, $S(y)^x - x^{S(y)} > S(y)^x - x^x = (S(y) - x)(S(y)^{x-1} + xS(y)^{x-2} + \ldots + x^{x-1}) \geq (S(y) - x)(S(y)^{x-1} + xS(y)^{x-2} + x^{x-1}) = S(y)^x - x^x$.

Thus, $A_k = \{(x, y) ; y = n, x = S(n), n \in \mathbb{N}^* \}$.

To find $A_2$, we see that $x = 1$ implies $S(x) = 1$ and (3) holds.

If $S(x) = x$, (3) also holds.

If $x \geq 6$ is not a prime number, then $x > S(x)$.

Write $x = S(x) + t$, $t \in \mathbb{N}^* = \{1, 2, 3, \ldots \}$.

Combining this with (3) yields

$S(x)^{S(x) + t} + S(x) + t = (S(x) + t)^{S(x) + t} = S(x)^t + t/S(x)^{S(x)} = (1 + t/S(x))^{S(x)} < 3^t$

which implies $S(x) < 3$. This contradicts our choice of $x$. 


Thus $A_3 = \{ x \in \mathbb{N}^* ; x = \text{prime} \} \cup \{ 1, 4 \}$.

Now, we suppose that the reader is able to find $A_2$, $A_4$, ..., $A_6$.

We next determine all positive integers $x$ such that $x = \sum_{k^2 \leq x} k^2$.

Write $1^2 + 2^2 + \ldots + s^2 = x$ (1)

$s^2 < x$ (2)

$(s + 1)^2 \geq x$ (3)

(1) implies $x = s(s+1)(2s+1)/6$. Combining this with (2) and (3) we have $6s^2 < s(s+1)(2s+1)$ and $6(s+1)^2 \geq s(s+1)(2s+1)$. This implies that $s \in \{ 2, 3 \}$. $s = 2 \Rightarrow x = 5$ and $s = 3 \Rightarrow x = 14$.

Thus $x \in \{ 5, 14 \}$.

In a similar way we can solve the equation $x = \sum_{k^2 \leq x} k^2$.

We find $x \in \{ 9, 36, 100 \}$.

We next show that the set $M_p = \{ n \in \mathbb{N}^* ; n = \sum_{k^2 \leq n} k \}$ has at least $[p/\ln 2] - 2$ elements.

Let $m \in \mathbb{N}^*$ such that $m - 1 < p/\ln 2$ (4)

and $p/\ln 2 < m$ (5)

Write (4) and (5) as:

$2 < e^{p/\ln 2}$ (6)

$e^{p/\ln 2} < 2$ (7)

Write $x_k = (1 + 1/k)^k$, $y_k = (1 + 1/k)^{k+1}$.

It is known that $x_s < e < y_t$ for every $s, t \in \mathbb{N}^*$.

Combining this with (6) and (7) we have $x_{s/\ln 2} < e^{s/\ln 2} < 2 < e^{s/\ln 2} < y_{t/\ln 2}$ for every $s, t \in \mathbb{N}^*$.

We have $2 < y_{t/\ln 2} = ((t+1)/(t+1))^{t+1/(t+1)} \leq ((t+1)/(t+1))^t \leq ((t+1)/(t+1))^t \leq \ldots \leq (t+1)/(t+1)$ if $(t+1)/(t+1) < 1$.

So, if $t \leq m - 2$ we have $2 < ((t+1)/(t+1))^t \iff 2 t^t < (t+1)^t \iff (t+1)^t - t^t > t^t$ (8).

Let $A_p(s)$ denote the sum $1^t + 2^t + \ldots + s^t$.

Proposition 1. $(t+1)^t > A_p(t)$ for every $t \leq m-2, t \in \mathbb{N}^*$.

Proof. Suppose that $A_p(t) \geq (t+1)^t \iff A_p(t-1) > (t+1)^t - t^t \iff A_p(t-2) > t^t - (t-1)^t \iff \ldots \iff A_p(1) > 2^t$ which is not true.

It is obvious that $A_p(t) > t^t$ if $t \in \mathbb{N}^*$, $2 \leq t \leq m-2$ which implies $A_p(t) \in M_p$ for every $t \in \mathbb{N}^*$ and $2 \leq t \leq m-2$.

Therefore $\text{card } M_p > m-3 = (m-1) - 2 = [p/\ln 2] - 2$.

REFERENCES:


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